Fourier Series

This is a collection of formulas with minimal explanatory text, intended to assist in the application of the complex Fourier series, the real Fourier series, and the discrete Fourier series.

1. The complex Fourier series

Any piecewise continuously integrable[1] complex-valued function $A(t) \in \mathbb{C}$ can be decomposed into a Fourier series if it is periodic with period T, or if it is defined only on the finite interval $[t_0, t_0 + T] \in \mathbb{R}$:

$$A(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+ik\frac{2\pi}{T}\cdot t} \quad \text{with } k = 0, \pm 1, \pm 2, \ldots \tag{1a}$$

The complex Fourier coefficients $c_k \in \mathbb{C}$ are

$$c_k = \frac{1}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\, t \; A(t) e^{-ik\frac{2\pi}{T}\cdot t} \quad . \tag{1b}$$

Renaming $c_{-k} \leftrightarrow c_k$ results in these formulas:

$$\begin{aligned} A(t) &\overset{(1a)}{=} \sum_{k=-\infty}^{+\infty} c_k e^{-ik\frac{2\pi}{T}\cdot t} \quad \text{with } k = 0, \pm 1, \pm 2, \ldots \\ c_k &\overset{(1b)}{=} \frac{1}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\, t \; A(t) e^{+ik\frac{2\pi}{T}\cdot t} \end{aligned}$$

[1] All functions that occur in physics satisfy this criterion.

So it doesn't matter whether you write $+i$ in the first exponent and $-i$ in the second exponent, or vice versa. Both conventions appear frequently in the literature. You just have to make sure that you always stick to one convention consistently.

2. The real Fourier series

If $A(t)$ is real-valued, then

$$c_0 \overset{(1b)}{=} \frac{1}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\,t\ A(t)$$

is clearly real. All other Fourier coefficients $c_k = $ (1b), however, are generally complex even for $A(t) \in \mathbb{R}$. In order to formulate the series expansion with real Fourier coefficients in this case, we use the fact that the Fourier coefficient c_k is obviously complex conjugate to the coefficient c_{-k} in the case $A(t) \in \mathbb{R}$:

$$\frac{1}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\,t\ A(t) e^{-ik\frac{2\pi}{T}\cdot t} \overset{(1b)}{=} c_k = \overline{c_{-k}} = \frac{1}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\,t\ A(t) e^{+ik\frac{2\pi}{T}\cdot t} \tag{2}$$

So, if $A(t) \in \mathbb{R}$, the series (1a) can be written as follows:

$$\begin{aligned} A(t) &\overset{(1a)}{=} \sum_{k=-\infty}^{+\infty} c_k e^{+ik\frac{2\pi}{T}\cdot t} \quad \text{with } k = 0,\ \pm 1,\ \pm 2,\ \dots \\ &= c_0 + \sum_{k=+1}^{+\infty} \Big(c_k e^{+ik\frac{2\pi}{T}\cdot t} + \overline{c_{-k}} e^{-ik\frac{2\pi}{T}\cdot t} \Big) \quad \text{with } k = 1,\ 2,\ 3,\ \dots \end{aligned} \tag{3}$$

Using the definitions

$$\frac{a_k - ib_k}{2} := c_k \quad \text{with } a_k,\, b_k \in \mathbb{R} \tag{4a}$$

$$x := k\frac{2\pi}{T} \cdot t \tag{4b}$$

and Euler's formula

$$e^{\pm ix} = \cos x \pm i \sin x \quad , \tag{4c}$$

the Fourier series becomes

$$\begin{aligned} A(t) &\overset{(3)}{=} \frac{a_0}{2} + \sum_{k=1}^{+\infty} \Big[\frac{a_k - ib_k}{2} (\cos x + i \sin x) + \\ &\qquad\qquad\qquad + \frac{a_k + ib_k}{2} (\cos x - i \sin x) \Big] \\ &= \frac{a_0}{2} + \sum_{k=1}^{+\infty} \Big[a_k \cos(k\frac{2\pi}{T} \cdot t) + b_k \sin(k\frac{2\pi}{T} \cdot t) \Big] \end{aligned} \tag{5a}$$

The Fourier coefficients are

$$\frac{a_k - ib_k}{2} \overset{(4a)}{=} c_k \overset{(1b)}{=} \frac{1}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\, t \ A(t) e^{-ik\frac{2\pi}{T} \cdot t} \quad .$$

With (4c), this results in the real Fourier coefficients

$$a_k = \frac{2}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\, t \ A(t) \cos(k\frac{2\pi}{T} \cdot t) \tag{5b}$$

$$b_k = \frac{2}{T} \int\limits_{t_0}^{t_0+T} \mathrm{d}\, t \ A(t) \sin(k\frac{2\pi}{T} \cdot t) \quad . \tag{5c}$$

3. The discrete Fourier series

In many technical applications, $A(t)$ is not a continuous function, but rather a sequence of N discrete values A_n with $n = 0, 1, 2, \ldots, (N-1)$.[2] Then the Fourier coefficients are not determined by the integral (1b), but by the sum

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} A_n e^{-ik\frac{2\pi}{N}\cdot n} \quad , \tag{6a}$$

and the Fourier series (1a) becomes

$$A_n = \sum_{k=0}^{N-1} c_k e^{+ik\frac{2\pi}{N}\cdot n} \quad . \tag{6b}$$

For the continuous series (1) and (5), one must calculate an infinite number of Fourier coefficients (i. e. , calculate the sequences analytically, but not numerically) in order to represent the function exactly. For an exact representation of the N elements of the sequence $\{A_n\}$, on the other hand, exactly N Fourier coefficients are required. This is reasonable because the sequences $\{A_n\}$ and $\{c_k\}$ contain exactly the same amount of information, namely $2N$ real numbers each, N for the real parts and N for the imaginary parts of each sequence.

If all $A_n \in \mathbb{R}$ are real, the information content of the sequence $\{A_n\}$ consists of only N real numbers. In the sequence of Fourier coefficients $\{c_k\}$, two imaginary parts vanish due to

$$c_0 \overset{(6a)}{=} \frac{1}{N} \sum_{n=0}^{N-1} A_n \in \mathbb{R} \quad ; \quad c_{N/2} \overset{(6a)}{=} \frac{1}{N} \sum_{n=0}^{N-1} A_n \cdot (-1)^n \in \mathbb{R} \ . \tag{7}$$

[2] This is obviously always the case if the series is not calculated analytically, but on a computer using a finite number of sampling points.

The imaginary parts of all other $N-2$ Fourier coefficients are generally different from zero, however, even in the case of real A_n. But due to $c_{N-k} = \overline{c_k}$ these $N-2$ coefficients contain only $N/2-1$ independent real parts and $N/2-1$ independent imaginary parts:

$$\begin{aligned} c_{N-k} &\overset{(6a)}{=} \frac{1}{N}\sum_{n=0}^{N-1} A_n e^{-i(N-k)\frac{2\pi}{N}\cdot n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} A_n \underbrace{e^{-iN\frac{2\pi}{N}\cdot n}}_{=1} e^{+ik\frac{2\pi}{N}\cdot n} = \overline{c_k} \end{aligned} \tag{8}$$

Therefore, in the case of real A_n, in addition to c_0 and $c_{N/2}$, only $N/2-1$ further Fourier coefficients c_k need to be calculated; the rest results from (8).

The information content of the sequence $\{c_k\}$ in the case of a purely real sequence $\{A_n\}$ is 2 real parts according to (7) plus $N/2-1$ real parts plus $N/2-1$ imaginary parts according to (8), making a total of N real numbers. Thus the sequences $\{c_k\}$ and $\{A_n\}$ again have the same information content also in this case.