Radiation and Radiation-Backreaction

The electromagnetic fields of charged point-particles according to Maxwell's electrodynamics, and the radiation-backreaction according to the theory of Abraham and Lorentz

Abstract

The four propagators of the classical electromagnetic field are derived. On this basis, in section 2 the Lienard-Wiechert potentials are computed, and from these again the retarded and advanced fields are derived, which are radiated by point-particle charges. The properties of these fields are investigated, and Larmor's radiation law is computed. In section 3, radiation-backreaction is considered from the point of view of energy conservation. Subsequently the same quantity is derived again in the last section, based on the classical model of an extended electron, which has been proposed by Abraham and Lorentz.

Contents

1. Propagators

The Lagrangian of the classical electromagnetic field is [1, (4.121)]

$$\mathcal{L} = -j^\mu A_\mu - \frac{1}{4\mu_0}\, F_{\sigma\tau} F^{\sigma\tau} \tag{1}$$
$$F_{\sigma\tau} \equiv \mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma \quad , \quad (A^\mu) \equiv (\Phi/c\, ,\, \boldsymbol{A})\ .$$

By variation of the Lagrangian with respect to the field component A_μ one gets the field equation [1, (4.126)]

$$\begin{aligned} 0 &= \mathrm{d}_\nu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu A_\mu)} - \frac{\partial \mathcal{L}}{\partial A_\mu} \\ &= -\frac{1}{\mu_0}\, \mathrm{d}_\nu(\mathrm{d}^\nu A^\mu - \mathrm{d}^\mu A^\nu) + j^\mu\ . \end{aligned} \tag{2}$$

By means of the

$$\text{Lorentz gauge:} \quad \mathrm{d}_\nu A^\nu = 0 \tag{3}$$

the field equation simplifies to

$$\Box A^\mu(x) \equiv \mathrm{d}_\nu \mathrm{d}^\nu A^\mu(x) = \mu_0 j^\mu(x)\ . \tag{4}$$

According to Huygens' principle, the field $A(x)$ can be considered as the superposition of waves, which propagate from the sources $\mu_0 j(y)$ to the space-time point x, and there superpose to a field with the components

$$A^\rho(x) = \int \mathrm{d}^4y\, D^{\rho\sigma}(x,y)\, g_{\sigma\tau}\, \mu_0 j^\tau(y)\ . \tag{5}$$

The Greens-function $D(x,y)$, which has 4×4 space-time components, often is called the propagator of the field $A(x)$. (5) is inserted into the field-equation (4):

$$\begin{aligned}
\Box A^\rho(x) &= \int \mathrm{d}^4y\, \Box D^{\rho\sigma}(x,y)\, g_{\sigma\tau}\, \mu_0 j^\tau(y) \;= \\
&= \mu_0 j^\rho(x) = \int \mathrm{d}^4y\, \delta^{(4)}(x-y)\, \underbrace{g^{\rho\sigma} g_{\sigma\tau}}_{g^\rho{}_\tau}\, \mu_0\, j^\tau(y) \\
&\Longrightarrow \quad \Box D^{\rho\sigma}(x,y) = g^{\rho\sigma}\delta^{(4)}(x-y)
\end{aligned} \tag{6}$$

Note, that the operator $\Box \equiv \mathrm{d}_\nu \mathrm{d}^\nu$ acts only onto the space-time coordinate x, but not onto the space-time coordinate y. The propagator's dimension is

$$\Big[D^{\rho\sigma}(x,y)\Big] = \frac{1}{\text{length}^2}\ . \tag{7}$$

If the considered system is invariant under translations in space and time, then $D^{\rho\sigma}(x,y)$ depends only on the difference $(x-y)$. Then the Fourier-transformations

$$D^{\rho\sigma}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \widetilde{D}^{\rho\sigma}(k)\, \exp\{-ik(x-y)\} \tag{8a}$$

$$\delta^{(4)}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \exp\{-ik(x-y)\} \tag{8b}$$

can be performed. Thus one gets in the four-dimensional space of the wavenumbers k the equation

$$\begin{aligned}
\Box\widetilde{D}^{\rho\sigma}(k) &= -k^2\widetilde{D}^{\rho\sigma}(k) = g^{\rho\sigma} \\
\widetilde{D}^{\rho\sigma}(k) &= -\frac{g^{\rho\sigma}}{k^2} \quad , \quad \Big[\widetilde{D}^{\rho\sigma}(k)\Big] = \text{length}^2\ .
\end{aligned} \tag{9}$$

Backtransformation to time-position space results into

$$D^{\rho\sigma}(x-y) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{g^{\rho\sigma}}{k^2}\,\exp\{-ik(x-y)\}$$
$$= -\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,g^{\rho\sigma}\exp\{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot F \tag{10a}$$
$$F \equiv \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\,\frac{\exp\{-ick^0(t-t_y)\}}{(k^0)^2-\boldsymbol{k}^2} \tag{10b}$$
$$t \equiv x^0/c \quad , \quad t_y \equiv y^0/c\ . \tag{10c}$$

F has two poles, because "on mass-shell"

$$k^2 = (k^0)^2 - \boldsymbol{k}^2 = (k^0+\omega/c)(k^0-\omega/c) = 0 \tag{11}$$
$$\omega \equiv +c\sqrt{\boldsymbol{k}^2} \geq 0\ .$$

We stipulate that ω/c shall always be interpreted as the positive root of $\boldsymbol{k}^2$, i. e. that a negative value shall be written as $-\omega/c \leq 0$. But in (10b), k^0 is *not* fixed to mass-shell. Instead, being the variable of integration, it assumes — independent of $\boldsymbol{k}$ — all values in the interval $-\infty \leq k^0 \leq +\infty$. To avoid the divergence of the integral F at the two poles (11), infinitesimal small terms $\pm i\epsilon$ with $0 < \epsilon \in \mathbb{R}$ are inserted into the denominator. There are four different alternatives to do this:

$$F_r \equiv \lim_{\epsilon\to 0}\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\,\frac{\exp\{-ick^0(t-t_y)\}}{(k^0+\omega/c+i\epsilon)(k^0-\omega/c+i\epsilon)} \tag{12a}$$
$$F_a \equiv \lim_{\epsilon\to 0}\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\,\frac{\exp\{-ick^0(t-t_y)\}}{(k^0+\omega/c-i\epsilon)(k^0-\omega/c-i\epsilon)} \tag{12b}$$
$$F_{\mathrm{f}} \equiv \lim_{\epsilon\to 0}\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\,\frac{\exp\{-ick^0(t-t_y)\}}{(k^0+\omega/c-i\epsilon)(k^0-\omega/c+i\epsilon)} \tag{12c}$$

$$F_{\mathrm{af}} \equiv \lim_{\epsilon\to 0} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \, \frac{\exp\{-ick^0(t-t_y)\}}{(k^0+\omega/c+i\epsilon)(k^0-\omega/c-i\epsilon)} \tag{12d}$$

* F_r has poles at $k^0 = -\omega/c - i\epsilon$ and at $k^0 = +\omega/c - i\epsilon$. This shift of poles results into the retarded propagator.
* F_a has poles at $k^0 = -\omega/c + i\epsilon$ and at $k^0 = +\omega/c + i\epsilon$. This shift of poles results into the advanced propagator.
* F_{f} has poles at $k^0 = -\omega/c + i\epsilon$ and at $k^0 = +\omega/c - i\epsilon$. This shift of poles, which is depicted in figure 1, results into the Feynman-propagator.
* F_{af} has poles at $k^0 = -\omega/c - i\epsilon$ and at $k^0 = +\omega/c + i\epsilon$. This shift of poles results into a propagator, for which no common name exists in the literature. We will call it anti-Feynman-propagator.

Dependent on the value of $t - t_y$, the integrals (12) can be closed — as sketched in figure 1 — in the upper or lower complex plane. The values of the integrals are not changed, because for very large $|k^0|$ their value is negligible due to the $(k^0)^2$ in the denominator.

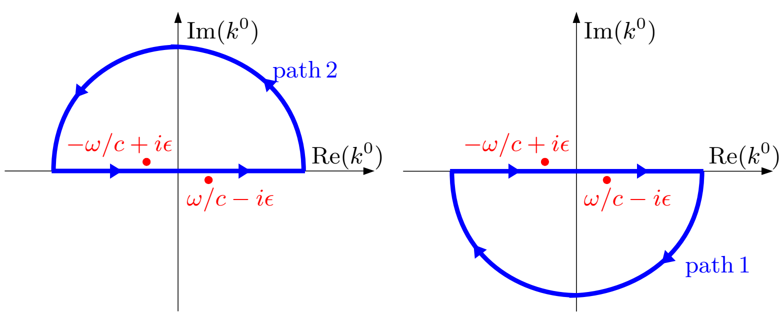

Fig. 1 : Integration paths (blue) and poles (red) for the Feynman-propagator

And due to the factor $\exp\{-ick^0(t-t_y)\}$ in the numerator, the integrals along the lower half circle (path 1) are negligible for very large negative imaginary parts of k^0 in case of $t > t_y$. On the other hand, in case of $t < t_y$ the integrals along the upper half circle (path 2) are negligible for very large positive imaginary parts of k^0. Applying Cauchy's integral theorem and the residue theorem[1], the integrals along the closed paths 1 and 2 can be solved. By means of the step function

$$\Theta(t-t_y) = \begin{cases} 1 \text{ if } t > t_y \\ 0 \text{ if } t < t_y \end{cases} \tag{13}$$

the results can be clearly represented. No Greens-function is defined for $t = t_y$.

$$F_r = i\Theta(t-t_y)\Big(\frac{\exp\{+i\omega(t-t_y)\}}{+2\omega/c} + \frac{\exp\{-i\omega(t-t_y)\}}{-2\omega/c}\Big) \tag{14a}$$

$$F_a = i\Theta(t_y-t)\Big(\frac{\exp\{+i\omega(t-t_y)\}}{-2\omega/c} + \frac{\exp\{-i\omega(t-t_y)\}}{+2\omega/c}\Big) \tag{14b}$$

$$F_{\mathrm{f}} = i\Theta(t_y-t)\,\frac{\exp\{+i\omega(t-t_y)\}}{-2\omega/c} + \\ + i\Theta(t-t_y)\,\frac{\exp\{-i\omega(t-t_y)\}}{-2\omega/c} \tag{14c}$$

$$F_{\mathrm{af}} = i\Theta(t-t_y)\,\frac{\exp\{+i\omega(t-t_y)\}}{+2\omega/c} + \\ + i\Theta(t_y-t)\,\frac{\exp\{-i\omega(t-t_y)\}}{+2\omega/c} \tag{14d}$$

The colors are indicating, that only four of the eight terms are different. With (10a) therefore the following relation holds for

[1] A short explication of these valuable mathematical tools, which is tailored to the particular needs of physicists, can be found in [2].

the retarded propagator $D_r^{\rho\sigma}(x-y)$, the advanced propagator $D_a^{\rho\sigma}(x-y)$, the Feynman-propagator $D_{\text{f}}^{\rho\sigma}(x-y)$, and the anti-Feynman-propagator $D_{\text{af}}^{\rho\sigma}(x-y)$:

$$D_r^{\rho\sigma} + D_a^{\rho\sigma} = D_{\text{f}}^{\rho\sigma} + D_{\text{af}}^{\rho\sigma} \tag{15}$$

We now are going to compute a generic propagator $D_{\text{s}}^{\rho\sigma}(x-y)$ by means of

$$\begin{aligned} F_{\text{s}} &\equiv i\Theta\Big(s_1(t-t_y)\Big)\,\frac{\exp\{s_2 i\omega(t-t_y)\}}{s_3 2\omega/c} \\ s_n &= +1 \text{ or } -1 \quad , \quad n=1,2,3\ . \end{aligned} \tag{16}$$

The generic propagator is

$$\begin{aligned} D_{\text{s}}^{\rho\sigma}(x-y) &\overset{(10a)}{=} -g^{\rho\sigma}\,\frac{i\Theta\Big(s_1(t-t_y)\Big)}{s_3 2(2\pi)^3}\,\cdot \\ &\qquad \cdot\int \mathrm{d}\boldsymbol{k}\ \exp\{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\ \frac{\exp\{s_2 i\omega(t-t_y)\}}{\omega/c}\ . \end{aligned} \tag{17}$$

Once we have found this propagator, the four propagators for which we are looking can easily be constructed by inserting the actual values of s_1, s_2, s_3 according to (14).

The computation of the integral (17) is demonstrated in [3, chap. 20]. We define spherical coordinates in wavenumber-space with azimuthal angle φ, polar angle ϑ, and radial coordinate $\omega/c = |\boldsymbol{k}|$:

$$\int\limits_{\varphi=0}^{2\pi} \mathrm{d}\varphi \int\limits_{\vartheta=0}^{\pi} \mathrm{d}\vartheta \int\limits_{\omega=0}^{\infty} \frac{\mathrm{d}\omega}{c}\,\frac{\omega^2}{c^2}\,\sin\vartheta \equiv \int \mathrm{d}\boldsymbol{k} \tag{18}$$

We choose the k^3-axis of the spherical coordinates parallel to $(\boldsymbol{x}-\boldsymbol{y})$. Thus

$$\begin{aligned}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k} &= R\,\frac{\omega}{c}\,\cos\vartheta \\ R &\equiv |\boldsymbol{x}-\boldsymbol{y}|\end{aligned} \tag{19}$$

holds. These definitions are inserted into the generic propagator (17). Integration over φ gives 2π. Thus we get

$$\begin{aligned}D_{\mathrm{s}}^{\rho\sigma}(x-y) &= -g^{\rho\sigma}\,\frac{i\Theta\Big(s_1(t-t_y)\Big)}{s_3 2(2\pi)^2}\cdot \\ &\quad\cdot\int\limits_{\omega=0}^{\infty}\frac{\mathrm{d}\omega}{c}\,\frac{\omega^2}{c^2}\,\frac{\exp\{s_2 i\omega(t-t_y)\}}{\omega/c}\cdot K \\ K &\equiv \int\limits_{\vartheta=0}^{\pi}\mathrm{d}\vartheta\,\sin\vartheta\,\exp\{iR\,\frac{\omega}{c}\,\cos\vartheta\}\ .\end{aligned} \tag{20}$$

For the computation of K, we substitute

$$u\equiv iR\,\frac{\omega}{c}\,\cos\vartheta \quad , \quad \frac{ic}{R\omega}\int\limits_{+iR\omega/c}^{-iR\omega/c}\mathrm{d}u = \int\limits_{\vartheta=0}^{\pi}\mathrm{d}\vartheta\,\sin\vartheta \tag{21}$$

and get

$$\begin{aligned}K &= \frac{ic}{R\omega}\int\limits_{+iR\omega/c}^{-iR\omega/c}\mathrm{d}u\,\exp\{u\} \\ &= \frac{ic}{R\omega}\Big(\exp\{-iR\omega/c\}-\exp\{+iR\omega/c\}\Big)\ .\end{aligned} \tag{22}$$

Thereby the generic propagator becomes

$$D_{\mathrm{s}}^{\rho\sigma}(x-y) = +g^{\rho\sigma}\frac{\Theta\Big(s_1(t-t_y)\Big)}{s_3 2(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big($$
$$\exp\{+i\omega(s_2t-s_2t_y-R/c)\}-\exp\{+i\omega(s_2t-s_2t_y+R/c)\}\Big)\,.$$

By means of this formula, the four propagators can immediately be constructed due to comparison of (16) with (14a) to (14d):

$$\begin{aligned}D_{\mathrm{r}}^{\rho\sigma}(x-y) = +g^{\rho\sigma}\frac{\Theta(t-t_y)}{2(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(&\\ \exp\{+i\omega(t-t_y-R/c)\}-\exp\{+i\omega(t-t_y+R/c)\}&\\ -\exp\{+i\omega(t_y-t-R/c)\}+\exp\{+i\omega(t_y-t+R/c)\}\Big)&\end{aligned}\tag{23a}$$

$$\begin{aligned}D_{\mathrm{a}}^{\rho\sigma}(x-y) = -g^{\rho\sigma}\frac{\Theta(t_y-t)}{(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(&\\ \exp\{+i\omega(t-t_y-R/c)\}-\exp\{+i\omega(t-t_y+R/c)\}&\\ -\exp\{+i\omega(t_y-t-R/c)\}+\exp\{+i\omega(t_y-t+R/c)\}\Big)&\end{aligned}\tag{23b}$$

$$\begin{aligned}D_{\mathrm{f}}^{\rho\sigma}(x-y) = +g^{\rho\sigma}\frac{\Theta(t_y-t)}{-2(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(&\\ \exp\{+i\omega(t-t_y-R/c)\}-\exp\{+i\omega(t-t_y+R/c)\}\Big)&\\ +g^{\rho\sigma}\frac{\Theta(t-t_y)}{-2(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(&\\ \exp\{+i\omega(t_y-t-R/c)\}-\exp\{+i\omega(t_y-t+R/c)\}\Big)&\end{aligned}\tag{23c}$$

$$D^{\rho\sigma}_{\text{af}}(x-y) = +g^{\rho\sigma}\frac{\Theta(t-t_y)}{2(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(\exp\{+i\omega(t-t_y-R/c)\}-\exp\{+i\omega(t-t_y+R/c)\}\Big)$$
$$+\,g^{\rho\sigma}\frac{\Theta(t_y-t)}{2(2\pi)^2 Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(\exp\{+i\omega(t_y-t-R/c)\}-\exp\{+i\omega(t_y-t+R/c)\}\Big) \tag{23d}$$

For half of the terms we used the fact, that the sign of $\boldsymbol{k}$ in the exponent of (10a) may be inverted, because the integration is running symmetrically over all positive and negative wavenumbers $\boldsymbol{k}$.

Two terms $+\exp\{\ldots\}$ and $-\exp\{\ldots\}$ each in the retarded and in the advanced propagator can be combined due to extension of the integration range from $0\to+\infty$ to $-\infty\to+\infty$. Thereby the delta-functions

$$\int\limits_{-\infty}^{+\infty}\mathrm{d}\omega\,\exp\{\pm i\omega(t-t_y-R/c)\} = 2\pi\,\delta(t-t_y-R/c) \tag{24a}$$
$$\int\limits_{-\infty}^{+\infty}\mathrm{d}\omega\,\exp\{\pm i\omega(t-t_y+R/c)\} = 2\pi\,\delta(t-t_y+R/c) \tag{24b}$$

become visible. Thus one gets the following retarded and advanced propagators:

$$D^{\rho\sigma}_{r}(x-y) = +\frac{g^{\rho\sigma}\Theta(t-t_y)}{4\pi Rc}\Big(\delta(t-t_y-R/c)-\delta(t-t_y+R/c)\Big) \tag{25a}$$

$$D^{\rho\sigma}_{a}(x-y) = -\frac{g^{\rho\sigma}\Theta(t_y-t)}{4\pi Rc}\Big(\delta(t-t_y-R/c)-\delta(t-t_y+R/c)\Big) \tag{25b}$$

As $R/c > 0$ always holds, the second delta-function in D_r and the first delta-function in D_a are always zero due to the step functions, and therefore may be skipped. As we exclude $R = 0$, the remaining delta-functions enforce $t > t_y$ in D_r and $t < t_y$ in D_a. Therefore the step functions may be skipped as well:

$$D_r^{\rho\sigma}(x-y) = +\frac{g^{\rho\sigma}}{4\pi Rc}\,\delta(t - t_y - R/c) \tag{26a}$$

$$D_a^{\rho\sigma}(x-y) = +\frac{g^{\rho\sigma}}{4\pi Rc}\,\delta(t - t_y + R/c) \tag{26b}$$

An explicitly covariant formulation of these propagators, in which only the four-vector $x - y$ shows up, but not it's space components $\boldsymbol{R}$, will turn out useful in the sequel. To derive that formulation, we invert in (25) the signs of the effectless delta-functions.

$$D_r^{\rho\sigma}(x-y) = +\frac{g^{\rho\sigma}\Theta(t-t_y)}{4\pi Rc}\Big(\delta(t - t_y - R/c) + \delta(t - t_y + R/c)\Big) \tag{27a}$$

$$D_a^{\rho\sigma}(x-y) = +\frac{g^{\rho\sigma}\Theta(t_y-t)}{4\pi Rc}\Big(\delta(t - t_y - R/c) + \delta(t - t_y + R/c)\Big) \tag{27b}$$

Now the retarded and the advanced propagator differ by nothing than the step function. We make use of the formula

$$\delta\Big(f(a)\Big) = \sum_i \frac{\delta(a - a_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}a}\Big|_{a_i}\right|} \quad \text{with } f(a_i) = 0\ ,\ \frac{\mathrm{d}f}{\mathrm{d}a}\Big|_{a_i} \neq 0 \tag{28}$$

$$f(a) \equiv (a + a_i)(a - a_i) = a^2 - a_i^2$$

$$\Longrightarrow \quad \delta(a^2 - a_i^2) = \frac{\delta(a - a_i) + \delta(a + a_i)}{|2a_i|}\ ,$$

insert $a \equiv t - t_y$ and $a_i \equiv R/c$, and get

$$\frac{\Big(\delta(t-t_y-R/c)+\delta(t-t_y+R/c)\Big)}{2R/c} = \delta\Big((t-t_y)^2 - R^2/c^2\Big) \ . \tag{29}$$

Using $c^2(t-t_y)^2 - R^2 = x^2 - y^2$ and $\delta(bx) = \delta(x)/|b|$, we get the formulation

$$D_r^{\rho\sigma}(x-y) = \frac{g^{\rho\sigma}\Theta(t-t_y)}{2\pi}\,\delta\Big((x-y)^2\Big) \tag{30a}$$

$$D_a^{\rho\sigma}(x-y) = \frac{g^{\rho\sigma}\Theta(t_y-t)}{2\pi}\,\delta\Big((x-y)^2\Big) \ , \tag{30b}$$

which is equivalent to (26). The step function $\Theta(t-t_y)$ does not compromise the covariance of this formulation, because the combined delta-functions and step functions enforce that y is on the backwards-lightcone of x in case of the retarded propagator, and on the forward-lightcone of x in case of the advanced propagator. If this holds true in one coordinate system, then it will hold true in any other coordinate system which can be reached from the first system by a proper Lorentztransformation. Thus in the context of (30) the step functions may be regarded as Lorentz-scalars.

In the Feynman-propagator and in the anti-Feynman-propagator there are no delta-functions. Instead they are usually quoted in the formulations (10a+12) or (23). For later reference, we here compile a list of the four propagators:

$$D_r^{\rho\sigma}(x-y) \overset{(26)}{=} \frac{g^{\rho\sigma}}{4\pi Rc}\,\delta(t-t_y-R/c) \tag{31a}$$

$$\overset{(30)}{=} \frac{g^{\rho\sigma}\Theta(t-t_y)}{2\pi}\,\delta\Big((x-y)^2\Big) \tag{31b}$$

$$D_a^{\rho\sigma}(x-y) \overset{(26)}{=} \frac{g^{\rho\sigma}}{4\pi Rc}\,\delta(t-t_y+R/c) \tag{31c}$$

$$\overset{(30)}{=} \frac{g^{\rho\sigma}\Theta(t_y-t)}{2\pi}\,\delta\Big((x-y)^2\Big) \tag{31d}$$

$$D_{\mathrm{f}}^{\rho\sigma}(x-y) \overset{(10a),(14)}{=} -ig^{\rho\sigma}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\cdot$$
$$\cdot\ \frac{\Theta(t-t_y)\exp\{-ik(x-y)\}-\Theta(t_y-t)\exp\{+ik(x-y)\}}{-2\omega/c} \tag{31e}$$

$$\overset{(23c)}{=} -\frac{g^{\rho\sigma}}{8\pi^2Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(\Theta(t-t_y)\exp\{-i\omega(t-t_y+R/c)\}-$$
$$-\Theta(t-t_y)\exp\{-i\omega(t-t_y-R/c)\}+$$
$$+\Theta(t_y-t)\exp\{+i\omega(t-t_y-R/c)\}-$$
$$-\Theta(t_y-t)\exp\{+i\omega(t-t_y+R/c)\}\Big) \tag{31f}$$

$$D_{\mathrm{af}}^{\rho\sigma}(x-y) \overset{(10a),(14)}{=} -ig^{\rho\sigma}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\cdot$$
$$\cdot\ \frac{\Theta(t_y-t)\exp\{-ik(x-y)\}-\Theta(t-t_y)\exp\{+ik(x-y)\}}{+2\omega/c} \tag{31g}$$

$$\overset{(23d)}{=} -\frac{g^{\rho\sigma}}{8\pi^2Rc}\int\limits_{\omega=0}^{\infty}\mathrm{d}\omega\,\Big(\Theta(t_y-t)\exp\{-i\omega(t-t_y-R/c)\}-$$
$$-\Theta(t_y-t)\exp\{-i\omega(t-t_y+R/c)\}+$$
$$+\Theta(t-t_y)\exp\{+i\omega(t-t_y+R/c)\}+$$
$$-\Theta(t-t_y)\exp\{+i\omega(t-t_y-R/c)\}\Big) \tag{31h}$$

Comparing (31a) with (31c) resp. comparing (31f) with (31h), the following transformation property of the propagators can be discerned:

$$\begin{aligned} &\text{Inversion of time direction} \quad \Longrightarrow \\ &\Longrightarrow \quad D_r^{\rho\sigma} \longleftrightarrow D_a^{\rho\sigma} \quad \text{and} \quad D_{\mathrm{f}}^{\rho\sigma} \longleftrightarrow D_{\mathrm{af}}^{\rho\sigma} \ . \end{aligned} \tag{32}$$

Note that upon time inversion also the sign of the frequency ω changes, while R and c (the modulus of the speed of light) remain unchanged.

The propagators $D(x-y)$ are functions of the four-vector

$$x - y = \Big(c(t-t_y), \boldsymbol{x} - \boldsymbol{y}\Big) \ . \tag{33}$$

The space-time-components of this four-vector are not mutually independent, but are subject to restrictions, which can most clearly be discerned in the delta-functions of (31a) and (31c). It's not immediately obvious, which components should be regarded as independent and which as dependent variables. These three definitions seem possible:

$$t_y(t,R) \equiv t \mp R/c \tag{34a}$$

$$t(t_y,R) \stackrel{?}{\equiv} t_y \pm R/c \quad \text{is wrong!} \tag{34b}$$

$$R(t,t_y) \stackrel{?}{\equiv} \pm c(t-t_y) \quad \text{is wrong!} \tag{34c}$$

We must take care that our formalism stays consistent. The point of departure was the potential in the notation (5). There $x = (t, \boldsymbol{x})$ was a fixed "outside" quantity, while $y = (t_y, \boldsymbol{y})$ was introduced as variable of integration. Thus t and $\boldsymbol{x}$ clearly do not depend on whatever else variables. Therefore definition (34b) must be discarded. Only t_y and $\boldsymbol{y}$ may possibly be dependent variables.

We try the following definitions:

$$t_y(t,\boldsymbol{x},\boldsymbol{y}) \equiv t \mp |\boldsymbol{x}-\boldsymbol{y}|/c = t \mp R/c \quad \text{is correct} \tag{34d}$$

$$R(\boldsymbol{x},\boldsymbol{y}) \equiv |\boldsymbol{x}-\boldsymbol{y}| \quad \text{is correct} \tag{34e}$$

$$\boldsymbol{y}(t,\boldsymbol{x},t_y) \overset{\boldsymbol{?}}{\equiv} \boldsymbol{??} \quad \text{is not viable} \tag{34f}$$

The relation $|\boldsymbol{x}-\boldsymbol{y}| = \pm c(t-t_y)$ is not sufficient to define $\boldsymbol{y}$ as a function of t, $\boldsymbol{x}$, and t_y. Consequently only (34d) and (34e) remain as consistent and viable definitions of the dependencies inbetween the variables. t and $\boldsymbol{x}$ are fixed "from outside". The parameter $\boldsymbol{y}$ as well can and must be chosen arbitrary, it does not depend on t nor on $\boldsymbol{x}$ nor on t_y. Only after that, $t_y \equiv$ (34d) can be computed as dependent variable, while R is merely a shorthand notation for $|\boldsymbol{x}-\boldsymbol{y}|$. Therefore R as well does not depend on t nor on t_y:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t_y} = 0 \tag{35}$$

2. The electromagnetic field of a charged point-particle

We are in particular interested in the case that the source of the field is a charged point-particle. Let it's charge be q, it's position at time t_y be $\boldsymbol{r}(t_y)$, and it's velocity be $\boldsymbol{v}(t_y) = \mathrm{d}\boldsymbol{r}(t_y)/\mathrm{d}t$. The derivation of the electromagnetic field emanating from that charge is following by and large Jackson [4, chap. 14].

We will derive the retarded and advanced fields from the retarded and advanced potentials. To compute the potentials, we insert the retarded and advanced propagators (31) into equation (5). Using $g^{\rho\sigma}g_{\sigma\tau} = g^{\rho}{}_{\tau} = \delta_{\rho\tau}$ and $\delta(x/c^2) = c^2\delta(x)$, one finds the two notations

$$A^\rho_r(x) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, \frac{j^\rho(y)}{R} \, \delta(t - t_y - R/c) \tag{36a}$$

$$= \frac{\mu_0 c}{2\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, j^\rho(y) \, \Theta(t - t_y) \, \delta\Big([x-y]^2\Big) \tag{36b}$$

$$A^\rho_a(x) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, \frac{j^\rho(y)}{R} \, \delta(t - t_y + R/c) \tag{36c}$$

$$= \frac{\mu_0 c}{2\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, j^\rho(y) \, \Theta(t_y - t) \, \delta\Big([x-y]^2\Big) \ . \tag{36d}$$

As these potentials differ only by some few signs, we combine them to a generic potential $A^\rho_s(x)$, in which the index $_s$ codes for $_r$ or $_a$. Double signs $\pm$ or $\mp$ shall be interpreted as $^{\text{retardet}}_{\text{advanced}}$, i. e. the upper sign always holds for retarded fields, the lower sign for advanced fields. In this generic notation, the potentials become

$$A^\rho_s(x) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, \frac{j^\rho(y)}{R} \, \delta(t - t_y \mp R/c) \tag{36e}$$

$$= \frac{\mu_0 c}{2\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, j^\rho(y) \, \Theta(\pm t \mp t_y) \, \delta\Big([x-y]^2\Big) \ . \tag{36f}$$

We adapt the definitions (19) for the case of a charged point-particle by

$$\boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{r} \quad , \quad R \equiv |\boldsymbol{R}| \quad , \quad \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R} \ . \tag{37}$$

The current density

$$j^\rho(y) = j^\rho(t_y, \boldsymbol{y}) \overset{?}{=} q\,\Big(c, \boldsymbol{v}(t_y)\Big)^\rho\,\delta^{(3)}\Big(\boldsymbol{y} - \boldsymbol{r}(t_y)\Big)$$

would be appropriate only for a non-relativistic description, because $(c, \boldsymbol{v})$ is no four-vector (it's square $c^2 - \boldsymbol{v}^2$ is no Lorentz-scalar). We use instead the four-velocity

$$\begin{aligned} V^\rho(t) &\equiv \frac{\mathrm{d}\,r^\rho(t)}{\mathrm{d}\,\tau} \equiv \frac{\mathrm{d}\Big(ct, \boldsymbol{r}(t)\Big)^\rho}{\mathrm{d}\,\tau} = \\ &= \gamma(t)\,\frac{\mathrm{d}\Big(ct, \boldsymbol{r}(t)\Big)^\rho}{\mathrm{d}\,t} = \gamma(t)\,\Big(c, \boldsymbol{v}(t)\Big)^\rho \\ \gamma(t) &\equiv \frac{\mathrm{d}\,t}{\mathrm{d}\,\tau} = \Big(1 - \frac{v^2(t)}{c^2}\Big)^{-1/2}\;, \end{aligned} \tag{38}$$

in which τ is the time in the coordinate system with $v = 0$, i. e. the proper time of the source. As (r^ρ) is a Lorentz-vector and τ is a Lorentz-scalar, (V^ρ) is a Lorentz-vector as well. Thereby the current density[2]

$$j^\rho(y) \equiv c\int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\,qV^\rho(\tau)\,\delta^{(4)}\Big(y - r(\tau)\Big) \tag{39}$$

can be defined, which — being the integral over the proper time of the source — is again a Lorentz-vector.

Inserting this current density into (36f), the integral over d^4y

[2] This formula has been proposed by Dirac [5, equation (5)].

can be solved:

$$A_s^\rho(x) = \frac{\mu_0 c}{2\pi} \int \mathrm{d}^3 y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, c \int\limits_{-\infty}^{+\infty} \mathrm{d}\tau \, q V^\rho(\tau) \, \delta^{(4)}\Big(y - r(\tau)\Big) \cdot$$
$$\cdot \, \Theta(\pm t \mp t_y) \, \delta\Big([x-y]^2\Big)$$
$$= \frac{\mu_0 c}{2\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}\tau \, q V^\rho(\tau) \, \Theta(\pm t \mp \underbrace{r^0(\tau)/c}_{\tau}) \, \delta\Big([x - r(\tau)]^2\Big) \ . \quad (40)$$

Comparing (31a) with (31b) resp. (31c) with (31d), one can discern:

$$\frac{1}{2Rc} \, \delta(t - \tau \mp R/c) = \Theta(\pm t \mp \tau) \, \delta\Big([x - r(\tau)]^2\Big) \quad (41)$$

Thereby the integral over τ becomes easy:

$$A_s^\rho(x) = \frac{\mu_0}{4\pi} \frac{q V^\rho(\tau_s)}{R} \quad \text{with } \tau_s = t \mp R/c \quad (42)$$

In (41) we transformed the delta function of $[x - r(\tau)]^2$ back into the delta function of $t - \tau \mp R/c$ (starting upfront with (36e) instead of (36f) would clearly be possible as well.) Now we want to transform it into the delta function of $\tau - \tau_s$. For that purpose we apply again formula (28) with $f(\tau) \equiv [x - r(\tau)]^2$. f is zero at

$$[x - r(\tau)]^2 = c^2(t-\tau)^2 - R^2 = 0$$
$$\tau = t \mp r/c \ ,$$

and the derivatives of f at these two points are

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = -2(x - r) \frac{\mathrm{d}r}{\mathrm{d}\tau} \overset{(38)}{=} -2(x-r)V \ .$$

The step function excludes in each case one of the two terms:

$$2\Theta(\pm t \mp \tau)\,\delta\Big([x - r(\tau)]^2\Big) = \Theta(\pm t \mp \tau)\,\frac{\delta(\tau - \tau_s)}{|(x - r)V|}\ . \tag{43}$$

The product in the denominator is

$$[x - r(\tau_s)]V \overset{(37)}{=} c[t - \tau_s]V^0 - \boldsymbol{R}\cdot\boldsymbol{V} = \pm R\,c\gamma - \boldsymbol{R}\cdot\boldsymbol{v}\gamma\ =$$

$$= \pm R\,c\gamma(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c) \quad \text{with } \boldsymbol{n} \equiv \boldsymbol{R}/R \tag{44a}$$

$$\Big|[x - r(\tau_s)]V\Big| = \pm[x - r(\tau_r)]V = Rc\gamma\,(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c) > 0\ , \tag{44b}$$

because the velocity v of the source is smaller than the velocity c of light in any reference system. The retarded and advanced potentials become, using $c^{-2} = \epsilon_0\mu_0$, in four- and three-dimensional notation

$$A^\rho_s(t,\boldsymbol{x}) \overset{(40)}{=} \frac{\mu_0 c}{4\pi}\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\, qV^\rho(\tau)\,\Theta(\pm t \mp \tau)\,\frac{\delta(\tau - \tau_s)}{|(x - r)V|}$$

$$= \pm\,\frac{\mu_0 c}{4\pi}\,\frac{qV^\rho(\tau_s)}{[x - r(\tau_s)]\,V(\tau_s)} \tag{45a}$$

$$\Phi_s(t,\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{R(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)}\Bigg|_{\tau_s} \tag{45b}$$

$$\boldsymbol{A}_s(t,\boldsymbol{x}) = \frac{\mu_0}{4\pi}\,\frac{q\boldsymbol{v}}{R(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)}\Bigg|_{\tau_s} \tag{45c}$$

$$\text{with } t = \tau_s \pm R/c = \tau_r + R/c = \tau_a - R/c\ .$$

The step function $\Theta(\pm\tau_s \mp \tau)$ can be skipped, if the values of τ_r and τ_a are indicated explicitly. These equations for the potentials generated by point charges have been found by Liénard and Wiechert.

We use greek letters $\rho, \sigma, \tau, \ldots$ for the four space-time indices $0, 1, 2, 3$, and latin letters $j, k, l, \ldots$ for the three space indices $1, 2, 3$. To derive the fields

$$F^{\sigma\rho}(t, \boldsymbol{x}) = \mathrm{d}^\sigma A^\rho(t, \boldsymbol{x}) - \mathrm{d}^\rho A^\sigma(t, \boldsymbol{x}) \tag{46a}$$

$$\frac{E^k}{c} \equiv F^{k0} \;, \; B^j \equiv -F^{kl} \;, \; jkl = 123\,\text{cyclic} \tag{46b}$$

$$\boldsymbol{E}(t, \boldsymbol{x}) = -\nabla\Phi(t, \boldsymbol{x}) - \frac{\mathrm{d}\,\boldsymbol{A}(t, \boldsymbol{x})}{\mathrm{d}t} \tag{46c}$$

$$\boldsymbol{B}(t, \boldsymbol{x}) = \boldsymbol{\nabla} \times \boldsymbol{A}(t, \boldsymbol{x}) \tag{46d}$$

from the potentials, we start from the integral form (40) of the potentials. As before, double signs are to be interpreted as $\substack{\text{retardet}\\\text{advanced}}$, i. e. the upper sign holds for retarded fields, the lower sign for advanced fields. The derivative with respect to t does not affect the arguments τ and $r(\tau)$ of the various functions, but only the argument t of the theta-function and the argument x of the delta-function. We are interested only in solutions with $R \neq 0$. Then in the product

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\Theta(\pm t \mp \tau)\,\delta\Big((x - r(\tau))^2 = \\
&= \delta(\pm t \mp \tau)\,\delta\Big((x - r(\tau))^2\Big) + \Theta(\pm t \mp \tau)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\delta\Big((x - r(\tau))^2\Big) \\
&= \delta\Big(\pm R\Big) + \Theta(t - \tau)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\delta\Big((x - r(\tau))^2\Big)
\end{aligned} \tag{47}$$

only the second term gives a contribution. Consequently we have for $\sigma = 0, 1, 2, 3$

$$\mathrm{d}^\sigma A_s^\rho(x) \overset{(40)}{=} \frac{\mu_0 c}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\tau\, q V^\rho(\tau)\,\Theta(\pm t \mp \tau)\,\mathrm{d}^\sigma \delta\Big((x - r(\tau))^2\Big) \;.$$

Into this equation we insert

$$\begin{aligned}
&\mathrm{d}^\sigma\delta\Big([x-r(\tau)]^2\Big) \;= \\
&= \Big(\mathrm{d}^\sigma[x-r(\tau)]^2\Big)\,\frac{\mathrm{d}\,\tau}{\mathrm{d}\,[x-r(\tau)]^2}\,\frac{\mathrm{d}\,\delta\Big([x-r(\tau)]^2\Big)}{\mathrm{d}\,\tau} \\
&= \frac{2[x-r(\tau)]^\sigma}{-2[x-r(\tau)]\,V(\tau)}\,\frac{\mathrm{d}\,\delta\Big([x-r(\tau)]^2\Big)}{\mathrm{d}\,\tau}
\end{aligned}$$

and integrate by parts:

$$\begin{aligned}
&\frac{2\pi}{\mu_0 c}\,\mathrm{d}^\sigma A_s^\rho(x) \;= \\
&= -q\,\Theta(\pm t\mp\tau)\,\frac{[x-r(\tau)]^\sigma V^\rho(\tau)}{[x-r(\tau)]\,V(\tau)}\,\delta\Big([x-r(\tau)]^2\Big)\Big|_{\tau=-\infty}^{\tau=+\infty} + \\
&\quad + \int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\,q\,\Theta(\pm t\mp\tau)\,\delta\Big([x-r(\tau)]^2\Big)\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\frac{[x-r(\tau)]^\sigma V^\rho(\tau)}{[x-r(\tau)]\,V(\tau)}
\end{aligned}$$

Only the last line is different from zero because of $\Theta[\pm t\mp(\pm\infty)]=0$ and $\delta\Big([x-r(\mp\infty)]^2\Big)=0$. For the reason given at (47), the theta-function could be shifted out of the derivative with respect to τ. Insertion of (43) with the correct signs according to (44) results into

$$\begin{aligned}
\mathrm{d}^\sigma A_s^\rho(x) = \pm\,\frac{\mu_0 c}{4\pi}\int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\,q\,\Theta(\pm t\mp\tau)\,\cdot \\
\cdot\,\frac{\delta(\tau-\tau_s)}{[x-r(\tau_s)]V(\tau_s)}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\frac{[x-r(\tau)]^\sigma V^\rho(\tau)}{[x-r(\tau)]\,V(\tau)}\;.
\end{aligned} \tag{48}$$

We integrate over τ and then compute the derivative with respect to τ:

$$\begin{aligned}
\mathrm{d}^\sigma A_s^\rho(x) &= \pm\,\frac{\mu_0 c}{4\pi}\,\frac{q\,\Theta(\pm t\mp\tau_s)}{[x-r(\tau_s)]\,V(\tau_s)}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,\frac{[x-r(\tau_s)]^\sigma V^\rho(\tau_s)}{[x-r(\tau_s)]\,V(\tau_s)}\\
\mathrm{d}^\sigma A_s^\rho(x) &= \pm\,\frac{\mu_0 c}{4\pi}\,\frac{q\,\Theta(\pm t\mp\tau_s)}{\Big([x-r(\tau_s)]\,V(\tau_s)\Big)^3}\cdot\Bigg[\\
&\qquad\Big(-\frac{\mathrm{d}r^\sigma}{\mathrm{d}\tau}\,V^\rho+[x-r]^\sigma\frac{\mathrm{d}V^\rho}{\mathrm{d}\tau}\Big)[x-r]\,V-\\
&\qquad-\Big(-\frac{\mathrm{d}r}{\mathrm{d}\tau}\,V+[x-r]\,\frac{\mathrm{d}V}{\mathrm{d}\tau}\Big)[x-r]^\sigma V^\rho\Bigg]_{\tau_s}
\end{aligned}\tag{49}$$

Using (44a) and using

$$\begin{aligned}
\frac{\mathrm{d}r^\sigma}{\mathrm{d}\tau} &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\,\frac{\mathrm{d}}{\mathrm{d}t}\,(ct,\boldsymbol{r})^\sigma = \gamma\,(c,\boldsymbol{v})^\sigma = V^\sigma\\
V^2 &= \gamma^2(c^2-\boldsymbol{v}^2) = \frac{c^2-v^2}{1-v^2/c^2} = c^2
\end{aligned}$$

one gets

$$\begin{aligned}
F_s^{\sigma\rho}(x) &= \pm\,\frac{\mu_0 c}{4\pi}\,\frac{q}{\Big(\pm Rc\gamma(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)\Big)^3}\cdot\Bigg[\\
&\Big([x-r]^\sigma\frac{\mathrm{d}V^\rho}{\mathrm{d}\tau}-[x-r]^\rho\frac{\mathrm{d}V^\sigma}{\mathrm{d}\tau}\Big)(\pm Rc\gamma)(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)+\\
&+\Big(c^2-[x-r]\,\frac{\mathrm{d}V}{\mathrm{d}\tau}\Big)\Big([x-r]^\sigma V^\rho-[x-r]^\rho V^\sigma\Big)\Bigg]_{\tau_s}\\
&\quad\text{mit } t=\tau_s\pm R/c=\tau_r+R/c=\tau_a-R/c\\
&\quad V=V(\tau_s),\ r=r(\tau_s),\ \boldsymbol{n}=\boldsymbol{n}(\tau_s),\ \boldsymbol{v}=\boldsymbol{v}(\tau_s)\ .
\end{aligned}\tag{50}$$

The step function $\Theta(\pm t\mp\tau_s)$ could be skipped, because the time-arguments $\tau_s=t\mp R/c$ are explicitly indicated.

To compute the three-dimensional fields $\boldsymbol{E}$ and $\boldsymbol{B}$ we need the components F^{j0} and F^{kl}. Insertion of the four-vector

$$x-r(\tau_s)\overset{(37)}{=}\Big(c(t-\tau_s)\,,\boldsymbol{R}\Big)=\Big(\pm R\,,\boldsymbol{R}\Big)\overset{(37)}{=}R\Big(\pm 1\,,\boldsymbol{n}\Big)\ , \tag{51}$$

of the notation

$$\dot{\boldsymbol{v}}\equiv\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\neq\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\tau}\quad,\quad \dot{v}\equiv|\dot{\boldsymbol{v}}|\overset{!}{\neq}\frac{\mathrm{d}v}{\mathrm{d}t}\quad,\quad v\equiv|\boldsymbol{v}|\ , \tag{52}$$

of the four-vector

$$V=\gamma(c,\boldsymbol{v})\ , \tag{53}$$

and of the derivatives

$$\begin{aligned}\frac{\mathrm{d}V^\rho}{\mathrm{d}\tau}&\overset{(38)}{=}\Big[\underbrace{\frac{\mathrm{d}t}{\mathrm{d}\tau}}_{\gamma}\,\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{(1-v^2/c^2)^{-1/2}}_{\gamma}\Big]\Big(c,\boldsymbol{v}\Big)^\rho+\gamma^2\frac{\mathrm{d}}{\mathrm{d}t}\Big(c,\boldsymbol{v}\Big)^\rho\\&=\Big[-\frac{1}{2}\gamma^4\Big(-\frac{\boldsymbol{v}\cdot\dot{\boldsymbol{v}}+\dot{\boldsymbol{v}}\cdot\boldsymbol{v}}{c^2}\Big)\Big](c,\boldsymbol{v})^\rho+\gamma^2(0,\dot{\boldsymbol{v}})^\rho\\&=\Big(\gamma^4\,\frac{\boldsymbol{v}\cdot\dot{\boldsymbol{v}}}{c}\,,\gamma^4\,\frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\,\boldsymbol{v}+\gamma^2\,\dot{\boldsymbol{v}}\Big)^\rho\ ,\end{aligned} \tag{54}$$

results into

$$\begin{aligned}F_s^{j0}(x)=\frac{\mu_0 c}{4\pi}\,&\frac{q}{\Big(Rc\gamma(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)\Big)^3}\cdot\Bigg[\Big(R^j\gamma^4\,\frac{\boldsymbol{v}\cdot\dot{\boldsymbol{v}}}{c}\mp\\&\mp R(\gamma^4\,\frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\,v^j+\gamma^2\,\dot{v}^j)\Big)(\pm Rc\gamma)(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)+\\&+\Big(c^2-(\pm R,\boldsymbol{R})\,(\gamma^4\,\frac{\boldsymbol{v}\cdot\dot{\boldsymbol{v}}}{c}\,,\gamma^4\,\frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\,\boldsymbol{v}+\gamma^2\,\dot{\boldsymbol{v}})\Big)\cdot\\&\cdot\Big(R^j\gamma c\mp R\gamma v^j\Big)\Bigg]_{\tau_s}\end{aligned}$$

$$F_s^{j0}(x) = \frac{\mu_0 c}{4\pi} \frac{qRc\gamma}{\Big(Rc\gamma(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)\Big)^3} \cdot \Big[c^2(n^j \mp v^j/c) + \\ + R\gamma^2\Big((n^j \mp v^j/c)(\boldsymbol{n}\cdot\dot{\boldsymbol{v}}) - (1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)\,\dot{v}^j\Big)\Big]_{\tau_s} \tag{55a}$$

$$F_s^{kl}(x) = \frac{\mu_0 c}{4\pi} \frac{q}{\Big(Rc\gamma(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)\Big)^3} \cdot \Big[\Big(R^k(\gamma^4 \frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\, v^l + \\ + \gamma^2\,\dot{v}^l) - R^l(\gamma^4 \frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\, v^k + \gamma^2\,\dot{v}^k)\Big)(\pm Rc\gamma)(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c) \\ + \Big(c^2 - (\pm R, \boldsymbol{R})\,(\gamma^4 \frac{\boldsymbol{v}\cdot\dot{\boldsymbol{v}}}{c}\,,\, \gamma^4 \frac{(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})}{c^2}\,\boldsymbol{v} + \gamma^2\,\dot{\boldsymbol{v}})\Big) \cdot \\ \cdot \Big(R^k\gamma v^l - R^l\gamma v^k\Big)\Big]_{\tau_s} = \\ = \frac{\mu_0 c}{4\pi} \frac{qRc\gamma}{\Big(Rc\gamma(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)\Big)^3} \cdot \Big[c^2(n^k v^l/c - n^l v^k/c) + \\ + R\gamma^2\Big((n^k v^l/c - n^l v^k/c)\boldsymbol{n}\,\dot{\boldsymbol{v}} \pm \\ \pm (n^k\,\dot{v}^l - n^l\,\dot{v}^k)(1 \mp \boldsymbol{n}\cdot\boldsymbol{v}/c)\Big)\Big]_{\tau_s} \tag{55b}$$

In each factor of F_s^{kl} the outer vector-product $(\boldsymbol{a} \times \boldsymbol{b})^j = a^k b^l - a^l b^k$ can be discerned. Furthermore we make use of $c^{-2} = \epsilon_0\mu_0$ and of the Grassmann-identity

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\ . \tag{56}$$

Thereby the retarded and advanced three-dimensional fields (46b) can be written in the form

$$\boldsymbol{E}_s(x) = \boldsymbol{E}_s^{(v)}(x) + \boldsymbol{E}_s^{(a)}(x) \quad , \quad \boldsymbol{B}_s(x) = \boldsymbol{B}_s^{(v)}(x) + \boldsymbol{B}_s^{(a)}(x)$$

$$\boldsymbol{E}_s^{(v)}(x) = \frac{1}{4\pi\epsilon_0}\,\frac{q(\boldsymbol{n}\mp\boldsymbol{v}/c)}{R^2\gamma^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\bigg|_{\tau_s} \tag{57a}$$

$$\begin{aligned}\boldsymbol{E}_s^{(a)}(x) &= \frac{1}{4\pi\epsilon_0}\,\frac{q}{Rc^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\cdot\Big[\\ &\qquad (\boldsymbol{n}\mp\boldsymbol{v}/c)(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})-\dot{\boldsymbol{v}}\,(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)\Big]_{\tau_s}\\ &= \frac{1}{4\pi\epsilon_0}\,\frac{q\ \boldsymbol{n}\times\Big((\boldsymbol{n}\mp\boldsymbol{v}/c)\times\dot{\boldsymbol{v}}\Big)}{Rc^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\bigg|_{\tau_s}\end{aligned} \tag{57b}$$

$$\boldsymbol{B}_s^{(v)}(x) = -\frac{\mu_0 c}{4\pi}\,\frac{q\,(\boldsymbol{n}\times\boldsymbol{v}/c)}{R^2\gamma^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\bigg|_{\tau_s} \tag{57c}$$

$$\begin{aligned}\boldsymbol{B}_s^{(a)}(x) &= -\frac{\mu_0 c}{4\pi}\,\frac{q}{Rc^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\cdot\Big[(\boldsymbol{n}\times\boldsymbol{v}/c)(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})+\\ &\qquad +(\boldsymbol{n}\times\dot{\boldsymbol{v}})(\pm 1-\boldsymbol{n}\cdot\boldsymbol{v}/c)\Big]_{\tau_s}\end{aligned} \tag{57d}$$

$$\text{with}\ \begin{cases} {}_s\ :\ \text{retarded}\ {}_r\ ,\ \ \text{advanced}\ {}_a\\ \pm:\ \text{retarded}\ +\ ,\ \text{advanced}\ \text{-}\\ \mp:\ \text{retarded}\ \text{-}\ ,\ \text{advanced}\ +\\ t=\tau_s\pm R/c=\tau_r+R/c=\tau_a-R/c\ . \end{cases}$$

We marked the acceleration fields, which are proportional to $\dot{v}$, by an index$^{(a)}$ (don't confuse this with the lower index $_a$, which codes for $_a$ = advanced), and the velocity fields, which are different from zero also in case $\dot{\boldsymbol{v}} = 0$, by an index$^{(v)}$.

Before we discuss these results in detail, we want to point out a remarkable relation between the electric and magnetic fields. For that purpose we compute the outer vector-products

$$
\begin{aligned}
\pm\boldsymbol{n}(\tau_s)\times\boldsymbol{E}_s^{(v)}(x) &= -\frac{1}{4\pi\epsilon_0}\,\frac{q(\boldsymbol{n}\times\boldsymbol{v}/c)}{R^2\gamma^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\bigg|_{\tau_s} \\
&= c\boldsymbol{B}_s^{(v)}(x) \qquad (58a)\\
\pm\boldsymbol{n}(\tau_s)\times\boldsymbol{E}_s^{(a)}(x) &= -\frac{1}{4\pi\epsilon_0}\,\frac{q}{Rc^2(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)^3}\cdot\Big[\\
&\quad (\boldsymbol{n}\times\boldsymbol{v}/c)(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\pm(\boldsymbol{n}\times\dot{\boldsymbol{v}})\,(1\mp\boldsymbol{n}\cdot\boldsymbol{v}/c)\Big]_{\tau_s} \\
&= c\boldsymbol{B}_s^{(a)}(x)\ . \qquad (58b)
\end{aligned}
$$

Independent of velocity and acceleration of the source, the magnetic field $\boldsymbol{B}_s$ is always vertical to the electric field $\boldsymbol{E}_s$, and the moduli of the amplitudes of the electric and magnetic field differ only by the factor c.

We stated in (32), that under time inversion the retarded propagator becomes the advanced propagator, and vice versa. As all of our derivation of the potentials and fields is based onto these propagators, one might state as well: Retarded electrodynamics become advanced electrodynamics under time inversion, and vice versa. But this is not to say, that each single retarded field component becomes the advanced field component under time inversion. That symmetry only holds for measurable phenomena, caused by some action of the combined fields.

Actually the symmetry of the fields (57) under time inversion is more complicated: The signs of the magnetic fields change, while the signs of the electric fields stay invariant. The velocity $\boldsymbol{v}$ of the source is the only factor, which's sign changes under time inversion. $\dot{\boldsymbol{v}}$, being the second derivative with respect to time of the position operator, is invariant under time inversion. The same holds for the modulus c of the velocity of light.

An observable phenomenon is for example the Lorentz force, which a field $(F_s^{\sigma\tau})$ exerts upon a particle with charge Q, mass m, four-momentum $(p^\alpha) = m(u^\alpha)$, four-velocity $(u^\alpha) = \gamma(c, \boldsymbol{u})$, and proper time τ:

$$\begin{aligned} \frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} &= QF_s^{\alpha\beta}u_\beta = QF_s^{\alpha 0}\gamma c + QF_s^{\alpha j}\gamma u_j \\ mc\,\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= -QF_s^{0j}u^j \\ m\,\frac{\mathrm{d}\gamma u^j}{\mathrm{d}t} &= QF_s^{j0}c - QF_s^{jk}u^k\ . \end{aligned} \tag{59a}$$

Insertion of the three-dimensional fields (46b) gives the relativistic form of the Lorentz force:

$$mc\,\frac{\mathrm{d}\gamma}{\mathrm{d}t} = +Q\,\boldsymbol{u}\cdot\boldsymbol{E}_s/c \tag{59b}$$

$$m\,\frac{\mathrm{d}\gamma\boldsymbol{u}}{\mathrm{d}t} = Q(\boldsymbol{E}_s + \boldsymbol{u}\times\boldsymbol{B}_s) \tag{59c}$$

The first equation describes the particle's gain in energy, to which $\boldsymbol{B}_s$ contributes nothing, and to which $\boldsymbol{E}_s$ contributes only with it's component parallel to $\boldsymbol{u}$. $\mathrm{d}\gamma/\,\mathrm{d}t$ is proportional to $\boldsymbol{v}\cdot\dot{\boldsymbol{v}}$, see (54). Being the first derivative with respect to time of the position vector, $\boldsymbol{v}$ changes sign under time inversion, while $\dot{\boldsymbol{v}}$ is invariant. Thus the left side of (59b) changes sign under time inversion, just like $\boldsymbol{u}$ on the right side. Consequently

$$\boldsymbol{E}_a = \boldsymbol{E}_r \tag{60a}$$

must hold. In contrast, due to the additional factor $\boldsymbol{u}$, the left side of (59c) is invariant under time inversion. On the right side, $\boldsymbol{E}_s$ is invariant according to (60a), while $\boldsymbol{u}$ changes sign. Consequently

$$\boldsymbol{B}_a = -\boldsymbol{B}_r \tag{60b}$$

must hold. The fields (57) comply with these requirements.

In case of $\boldsymbol{v} = \dot{\boldsymbol{v}} = 0$ one gets $\boldsymbol{B}_s^{(v)} = \boldsymbol{B}_s^{(a)} = \boldsymbol{E}_s^{(a)} = 0$ and $\gamma = 1$. Then $\boldsymbol{E}_s^{(v)}$ assumes the well-known form

$$\boldsymbol{E}_s^{(v)}(x) \overset{(57a)}{=} \frac{1}{4\pi\epsilon_0} \, \frac{q\,\boldsymbol{n}}{R^2}\Big|_{\tau_s} \tag{61}$$

of Coulomb's law. Inbetween the charges q and Q the retarded force

$$m\,\frac{\mathrm{d}\gamma\boldsymbol{u}}{\mathrm{d}t} = Q\boldsymbol{E}_r \overset{(61)}{=} \frac{1}{4\pi\epsilon_0} \, \frac{Qq\,\boldsymbol{n}}{R^2}\Big|_{\tau_r} \tag{62a}$$

and the advanced force

$$m\,\frac{\mathrm{d}\gamma\boldsymbol{u}}{\mathrm{d}t} = Q\boldsymbol{E}_a \overset{(61)}{=} \frac{1}{4\pi\epsilon_0} \, \frac{Qq\,\boldsymbol{n}}{R^2}\Big|_{\tau_a} \tag{62b}$$

are acting. Addition of the two equations, and division by 2 gives the result

$$m\,\frac{\mathrm{d}\gamma\boldsymbol{u}}{\mathrm{d}t} = \frac{1}{2}\,Q(\boldsymbol{E}_r + \boldsymbol{E}_a) \overset{(61)}{=} \frac{1}{2}\,\frac{Qq}{4\pi\epsilon_0}\left(\frac{\boldsymbol{n}}{R^2}\Big|_{\tau_r} + \frac{\boldsymbol{n}}{R^2}\Big|_{\tau_a}\right) . \tag{62c}$$

Since the end of the nineteenth century, these equations are interpreted differently. According to one point of view, the advanced fields are nothing but an artifact of the theory, don't exist at all in reality, and the observed force is caused alone by the retarded field, which is described correctly by (62a).

Obviously it's impossible by measurements of the force in the static case, to decide whether (62a) or (62c) is correct; both equations predict the identical force. In contrast, the observations of the fields of moving and accelerated charges seem to be evidence against the existence of advanced fields: If a flash of light is emitted in direction of a mirror, then the reflected flash is always observed after the emission, but never before the emission. Still there is an alternative to Maxwell's electrodynamics, namely action-at-a-distance electrodynamics, which has been proposed and worked out in particular by Gauß, Schwarzschild, and Frenkel (see [6, 7] and the references cited in these articles). This theory claims to be able to describe all observable electrodynamic phenomena as good as Maxwell's theory. In action-at-a-distance theory, fields are no self-contained physical objects. Instead this theory only knows charged particles, which interact due to action-at-a-distance forces. Fields are considered to be only computing aids on the theorist's paper. We will not delve into that theory. But we want to keep the option for both types of electrodynamics. Therefore we will continue to describe retarded fields and advanced fields as well.

In the case $\boldsymbol{v} \neq 0$ and $\dot{\boldsymbol{v}} = 0$, $\boldsymbol{B}_s^{(v)}$ increases proportional to $\boldsymbol{v}$. The position $\boldsymbol{r}$ of the source, it's velocity $\boldsymbol{v}$, and the point of observation $\boldsymbol{x}$ define the plane of the left sketch in Fig. 2. In the sketched example, $\boldsymbol{B}_s^{(v)} \sim -q\boldsymbol{n} \times \boldsymbol{v}$ is directed for positive charge $(q > 0)$ of the source out of the drawing plane vertically up. $\boldsymbol{E}_s^{(v)} \sim \boldsymbol{n} \mp \boldsymbol{v}/c$ lies in the drawing plane. Only in the high-relativistic case $v \to c$ the direction of $\boldsymbol{E}_s^{(v)}$ can be significantly different from $\boldsymbol{n}$.

The power, which is transported by the fields $\boldsymbol{E}_s$ and $\boldsymbol{B}_s$ through

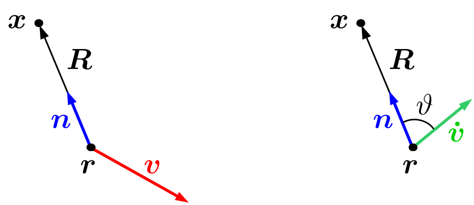

Fig. 2 : Velocity and acceleration of the source

a surface $\boldsymbol{f}$, is equal to $\boldsymbol{S}\cdot\boldsymbol{f}$. Thereby

$$\boldsymbol{S} = \frac{1}{\mu_0}\,\boldsymbol{E_s}\times\boldsymbol{B_s} \overset{(58)}{=} \pm\frac{1}{c\mu_0}\,\boldsymbol{E_s}\times(\boldsymbol{n}\times\boldsymbol{E_s}) \tag{63}$$

is the Poynting-vector. It is zero in the static case $\boldsymbol{v} = \dot{\boldsymbol{v}} = 0$ because of $\boldsymbol{B}_s = 0$. But also in case $\boldsymbol{v}\neq 0$ and $\dot{\boldsymbol{v}} = 0$, no power is radiated by the source, because due to $\boldsymbol{n}\times\boldsymbol{n} = 0$ we have in this case

$$\begin{aligned}\boldsymbol{S} &\sim (\pm)(\mp)(\boldsymbol{n}\mp\boldsymbol{v}/c)\times(\boldsymbol{n}\times\boldsymbol{v}/c) = \\ &= -\,\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{v}/c)\pm\boldsymbol{v}\times(\boldsymbol{n}\times\boldsymbol{v}/c^2)\ .\end{aligned}$$

The first term is vertical to $\boldsymbol{n}$, i. e. this part of the power flows around the source, but not away from the source nor towards the source. The second term, which only in the hight-relativistic case is of same order of magnitude as the first, has a component different from zero, which is parallel to $\boldsymbol{n}$. But if we integrate over a surface of a sphere with the source in the center, then the contribution in direction of $\boldsymbol{n}$ is compensated by an inversely directed stream of energy of same value, but direction $-\boldsymbol{n}$. The net stream of energy through the surface is zero. The second term describes as well only the field-energy which is carried along by the source. No energy is radiated or absorbed.

Alternative explication: We start from the fact, that a charge at rest does not radiate energy. To describe a source with constant velocity $\boldsymbol{v}$ (no acceleration), we transform ourselves into a coordinate system moving with constant velocity $-\boldsymbol{v}$ relative to the source. This transformation cannot change the physical fact, that the source is not radiating energy.

The issue can be made plausible by still another argument: Let's assume, that a charge in constant motion (no acceleration) would radiate energy. If the theorem of energy conservation holds, then the radiating charge must loose kinetic energy, and come to rest eventually. But that would mean, that one of the infinitely many inertial systems with constant relative velocity is special, because this one would be the particular inertial system in which radiating charges eventually come to rest. That scenario cannot be reconciled with special relativity theory.

In case $\dot{\boldsymbol{v}} \neq 0$, in addition to the velocity fields $\boldsymbol{E}_s^{(v)}$ and $\boldsymbol{B}_s^{(v)}$ the acceleration fields $\boldsymbol{E}_s^{(a)}$ and $\boldsymbol{B}_s^{(a)}$ appear. They are proportional to R^{-1}, while the velocity fields are proportional to R^{-2}. Above we have stated, that velocity fields do not contribute to the radiation of energy. We could have arrived at that result by a much simpler argument: If a source radiates energy, then the stream of energy per area will decrease at large distance R from the source like R^{-2} (because the surface of a sphere with radius R is $4\pi R^2$). But from (57a) and from (57c) one can immediately conclude:

$$\begin{aligned} \boldsymbol{S} \sim \boldsymbol{E}_s \times \boldsymbol{B}_s &= \Big(\boldsymbol{E}_s^{(v)} + \boldsymbol{E}_s^{(a)}\Big) \times \Big(\boldsymbol{B}_s^{(v)} + \boldsymbol{B}_s^{(a)}\Big) = \\ &= \underbrace{\boldsymbol{E}_s^{(v)} \times \boldsymbol{B}_s^{(v)}}_{\sim R^{-4}} + \underbrace{\boldsymbol{E}_s^{(v)} \times \boldsymbol{B}_s^{(a)}}_{\sim R^{-3}} + \underbrace{\boldsymbol{E}_s^{(a)} \times \boldsymbol{B}_s^{(v)}}_{\sim R^{-3}} + \underbrace{\boldsymbol{E}_s^{(a)} \times \boldsymbol{B}_s^{(a)}}_{\sim R^{-2}} \end{aligned} \tag{64}$$

If we want to investigate the radiation of energy, we only need to consider the acceleration fields, which for this reason also are called radiation fields. The Poynting-vector built from them at position $\boldsymbol{x}$ and time t is

$$\begin{gathered} \boldsymbol{S} = \boldsymbol{E}_s^{(a)} \times \boldsymbol{H}_s^{(a)} \overset{(58)}{=} \pm \frac{1}{c\mu_0}\, \boldsymbol{E}_s^{(a)} \times \Big(\boldsymbol{n}(\tau_s) \times \boldsymbol{E}_s^{(a)}\Big) \\ |\boldsymbol{S}| = \frac{1}{c\mu_0}\, |\boldsymbol{E}_s^{(a)}|^2 \quad , \quad \boldsymbol{n} \cdot \boldsymbol{S} \neq 0 \; . \end{gathered} \tag{65}$$

Note the negative sign of the advanced Poynting-vector: The advanced field propagates against the "usual" direction of time from the future into the past. Viewed through the glasses of the usual direction of time, the advanced radiation fields form a spherical wave, which — coming from infinity — is collapsing onto the position of the source.

All formulas stated thus far are valid for arbitrary velocities, including relativistic velocities, of the source q. The non-relativistic approximations of the radiation fields can easily be extracted from (57) and (58):

$$\boldsymbol{E}_s^{(a)}(x) = \frac{1}{4\pi\epsilon_0}\, \frac{q\ \boldsymbol{n}\times(\boldsymbol{n}\times\dot{\boldsymbol{v}})}{Rc^2}\Big|_{\tau_s} \tag{66a}$$

$$\Big|\boldsymbol{E}_s^{(a)}(x)\Big| = \frac{1}{4\pi\epsilon_0}\, \frac{|q\ \sin(\boldsymbol{n},\dot{\boldsymbol{v}})|}{Rc^2}\Big|_{\tau_s} \tag{66b}$$

$$c\boldsymbol{B}_s^{(a)}(x) = \pm\boldsymbol{n}(\tau_r)\times\boldsymbol{E}_r^{(a)}(x) \tag{66c}$$

$$\text{if } v/c \ll 1 \quad , \quad \text{with } t = \tau_s \pm R/c = \tau_r + R/c = \tau_a - R/c$$

$\boldsymbol{E}_s^{(a)}$ lies in the non-relativistic case in the plane which is defined by $\boldsymbol{n}$ and $\dot{\boldsymbol{v}}$ (see the right sketch in fig. 2), is vertical to $\boldsymbol{n}$, and the angle $(\boldsymbol{E}_s^{(a)},\dot{\boldsymbol{v}})$ is always $\geq \pi/2$. In the example of fig. 2 the direction of $\boldsymbol{E}_s^{(a)}$ is in case of $q > 0$ approximately 8 o'clock. The retarded vector $\boldsymbol{B}_r^{(a)}$ is in this case directed vertically up out of the drawing plane, and the advanced vector $\boldsymbol{B}_a^{(a)}$ is directed vertically down into the drawing plane.

To compute the radiated power $\mathrm{d}P$, which is going through an infinitesimal area $\mathrm{d}\vartheta\ \mathrm{d}\varphi$ in the distance R from the source, we define a system of spherical coordinates such, that the source is at the origing $(\boldsymbol{r} = 0)$, and the polar axis $\vartheta = 0$ is parallel to $\dot{\boldsymbol{v}}$, see the right sketch of figure 2. (Thus this is an accelerated coordinate system. But that's of no relevance for our non-relativistic considerations.) We have

$$\frac{\mathrm{d}P}{R^2 \sin\vartheta\ \mathrm{d}\vartheta\ \mathrm{d}\varphi} = \frac{|\boldsymbol{E}_s^{(a)}|^2}{c\mu_0} \overset{(66a)}{=} \frac{1}{c\mu_0}\left(\frac{q\,\dot{\boldsymbol{v}}}{4\pi\epsilon_0 Rc^2}\right)^2 \sin^2\vartheta\ . \tag{67}$$

Note the dependence of the radiated power on the direction of acceleration, described by $\sin^2\vartheta$. The total radiated power is equal to the integral

$$P = \int\limits_0^{\pi} \mathrm{d}\vartheta\, R^2 \sin\vartheta \int\limits_0^{2\pi} \mathrm{d}\varphi\, \frac{1}{c\mu_0} \left(\frac{q\,\dot{\boldsymbol{v}}}{4\pi\epsilon_0 R c^2}\right)^2 \sin^2\vartheta$$
$$= \frac{q^2\dot{v}^2}{8\pi\epsilon_0 c^3} \int\limits_0^{\pi} \mathrm{d}\vartheta\, \sin^3\vartheta = \frac{2q^2\dot{v}^2}{3c^3(4\pi\epsilon_0)}\ . \qquad (68a)$$

This formula was found in 1897 by Larmor. To derive the result for the relativistic case $v \to c$, one could try to repeat the computation with the relativistic fields (57). But there is a much simpler alternative. We are looking for a relativistically covariant equation (i. e. an equation which is composed exclusively of Lorentz-tensors), which reduces in the limit $v \ll c$ to (68a), and which shall furthermore — with regard to (57) — shall only depend on $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$. If the non-relativistic formula (68a) is written in in the form

$$P = \frac{2q^2}{3c^3(4\pi\epsilon_0)m^2} \left(\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right)^2 ,$$

with m being the mass of the charged particle and $\boldsymbol{p}$ it's momentum, then it seems plausible that the relativistic generalization of (68a) is

$$P = -\frac{2q^2}{3c^3(4\pi\epsilon_0)m^2} \left(\frac{\mathrm{d}p}{\mathrm{d}\tau}\right)^2 . \qquad (68b)$$

$\mathrm{d}p/\mathrm{d}\tau$ is the derivative of the particle's four-momentum with respect to it's proper time τ. To check the result, we insert $p =$

$(E/c, \boldsymbol{p}) = \gamma(mc, m\boldsymbol{v})$ and get

$$\begin{aligned} P &= -\frac{2q^2}{3c^3(4\pi\epsilon_0)}\left[\left(\frac{mc}{mc}\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)^2 - \frac{m}{m}\left(\frac{\mathrm{d}\gamma\boldsymbol{v}}{\mathrm{d}\tau}\right)^2\right] \\ &\approx \frac{2q^2}{3c^3(4\pi\epsilon_0)}\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)^2 \quad \text{for } \frac{v^2}{c^2} \ll 1 \; . \end{aligned}$$

3. Radiation-backreaction: energy conservation

In the previous section we computed the retarded and advanced fields radiated by an accelerated point-charge. Where does the radiated energy come from? It can come only from the radiating particle's kinetic energy, i. e. the radiating particle must be decelerated. This decelerating action is called radiation back-reaction.

Our discussion of radiation back-reaction will follow by and large Jackson [4, chap. 16]. If a charged particle is accelerated by an external force $\boldsymbol{F}_{\text{ext}}$, then it's kinetic energy is increased. At the same time, it looses energy due to radiation. This part of the energy must be supplied by the external force as well. Therefore the force $\boldsymbol{F}_{\text{ext}}$ can formally be considered to consist of two parts: One part $\boldsymbol{F}_{\text{acc}}$, which is accelerating the particle, and one part $\boldsymbol{F}_{\text{rad}}$, which is supplying the radiated energy.

In the following non-relativistic investigation we assume, that the velocity $\boldsymbol{v} \neq 0$ of the charged particle is different from zero. This does not reduce the general validity of our considerations, because we can always perform the investigation in an inertial system, in which $\boldsymbol{v} \neq 0$ holds. The work done by $\boldsymbol{F}_{\text{ext}} = \boldsymbol{F}_{\text{acc}} + \boldsymbol{F}_{\text{rad}}$ in the time interval $t_2 - t_1$ is

$$\int_{t_1}^{t_2} \mathrm{d}t\,(\boldsymbol{F}_{\text{acc}} + \boldsymbol{F}_{\text{rad}})\,\boldsymbol{v} \overset{(68a)}{=} \int_{t_1}^{t_2} \mathrm{d}t\left(m\dot{\boldsymbol{v}}\boldsymbol{v} + \frac{2q^2\dot{v}^2}{3c^3 4\pi\epsilon_0}\right) . \tag{69}$$

To derive an explicit expression for $\boldsymbol{F}_{\text{rad}}$, we integrate the second term by parts:

$$\int_{t_1}^{t_2} \mathrm{d}t\, \boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{v} = \frac{2q^2 \dot{\boldsymbol{v}} \cdot \boldsymbol{v}}{3c^3 4\pi\epsilon_0} \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t\, \frac{2q^2 \ddot{\boldsymbol{v}} \cdot \boldsymbol{v}}{3c^3 4\pi\epsilon_0} \tag{70}$$

In the sequel we will concentrate on those cases, where the first term on the right side is zero. This will be the case, if either the external force is acting only during a limited time interval $t'_1 \ldots t'_2$ (then one can choose $t_1 < t'_1$ and $t_2 > t'_2$, and consequently $\dot{\boldsymbol{v}}(t_1) = \dot{\boldsymbol{v}}(t_2) = 0$), or if the observed process is periodic (then the points of time t_1 and t_2 can always be chosen such, that the first term on the right side again is zero). If for example an electron is accelerated up and down in an antenna, then $\boldsymbol{v} = 0$ holds at the points of return. If the particle is accelerated on an elliptic orbit, then there are two points of time with $\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = 0$. In all of these cases, the integrands of the remaining terms must be equal:

$$\begin{aligned} \boldsymbol{F}_{\text{rad}} \cdot \boldsymbol{v} &= -\frac{2q^2 \ddot{\boldsymbol{v}} \cdot \boldsymbol{v}}{3c^3 4\pi\epsilon_0} \\ |\boldsymbol{F}_{\text{rad}}| \cos(\boldsymbol{F}_{\text{rad}}, \boldsymbol{v}) &= -\frac{2q^2\, |\ddot{\boldsymbol{v}}|}{3c^3 4\pi\epsilon_0} \cos(\ddot{\boldsymbol{v}}, \boldsymbol{v}) \end{aligned} \tag{71}$$

This must hold for arbitrary angles $(\boldsymbol{F}_{\text{rad}}, \boldsymbol{v})$. Consequently the angles $(\boldsymbol{F}_{\text{rad}}, \boldsymbol{v})$ and $(\ddot{\boldsymbol{v}}, \boldsymbol{v})$ must always be equal. If the direction of $\ddot{\boldsymbol{v}}$ would differ from the direction of $\boldsymbol{F}_{\text{rad}}$ then the question would arise: why just this direction? Regarding the symmetry of the model, that question can not be answered. Thus the both vectors must be parallel, resulting into the equation

$$\boldsymbol{F}_{\text{rad}} = -\frac{2q^2 \ddot{\boldsymbol{v}}}{3c^3 4\pi\epsilon_0} \ . \tag{72}$$

With this expression, the particle's equation of motion becomes

$$\boldsymbol{F} = \boldsymbol{F}_{\text{acc}} + \boldsymbol{F}_{\text{rad}} = m\dot{\boldsymbol{v}} - \frac{2q^2}{3c^3 4\pi\epsilon_0}\,\ddot{\boldsymbol{v}}\ . \tag{73}$$

$\boldsymbol{F}_{\text{rad}}$ depends only on $\ddot{\boldsymbol{v}}$ and on the modulus of the accelerated particle's charge.

In this equation, the limitations of the classicle model of radiation and radiation-backreaction become clearly visible. First, it has for $\boldsymbol{F} = 0$ besides the reasonable solution $\dot{\boldsymbol{v}} = \ddot{\boldsymbol{v}} = 0$ also the senseless "run-away" solution $\dot{\boldsymbol{v}} = \exp\{+t\, 3c^3 4\pi\epsilon_0 m/(2q^2)\}$. The senseless solution can be avoided due to fixing in addition to position and velocity of the radiating charge at time t_0 also $\dot{\boldsymbol{v}}(t \to \infty) = 0$ as the third boundary condition. With regard to that, Dirac [5, page 158] remarked in his discussion of equation (73): "We now have a striking departure from the usual ideas of mechanics. We must obtain solutions of our equations of motion for which the initial position and velocity of the electron are prescribed, together with its final acceleration, instead of solutions with all the initial conditions prescribed."

But (73) has even worse consequences: In the article just cited, Dirac considered the acceleration of a charge due to a force which is acting only for a very short moment (i. e. a pulse). Thereby he found [5, equation (35)] that the electron is already accelerated *before* the pulse arrives at the particle's position. This is often called "pre-acceleration" in the literature. A discussion worth reading on the strange results and paradoxa, which are following from the classical treatment of radiation and radiation-backreaction, has been published by K. Brown [8].

Given these absurd results, one may try to evade backwards or forwards. The backwards evasion goes along these lines: Our derivation of (73) is based on Larmor's formula (68a), which again has been derived from the formulas (57) for the electromagnetic field of a charged point-particle q. Medina [9] gives good arguments for the assertion, that the assumption of charged point-particles (with the notion point used in the strict mathematical sense) is irreconcilable with classical electrodynamics. Under the premise that in a classical treatment the radius of a charged particle must never assumed to be smaller than the classical radius (95), Medina [9, (96)] finds instead of (72) in non-relativistic approximation the following radiation-backreaction force:

$$\boldsymbol{F}_{\text{rad}} = -\frac{2q^2}{3c^3 4\pi\epsilon_0}\left(\ddot{\boldsymbol{v}} - \boldsymbol{v}\,\frac{(\dot{\boldsymbol{v}}\cdot\boldsymbol{F}_{\text{acc}})}{c^2}\right) .$$

The forwards evasion consists in challenging any attempt of a classical description of radiation and radiation-reaction from the outset. We know that electromagnetic energy is not emitted in form of continuous waves, but in form of photons. If an accelerated charge is emitting photons in irregular time intervals, then it will feel some recoil at each photon emission, and hence it's acceleration is ill defined. If this point of view is assumed, then we should for the sake of consistency abandon the notion of the "orbit" or "path" $\boldsymbol{r}(t)$ of a particle completely, like Heisenberg [10] did in his seminal work on quantum mechanics. Then consequently the notions of the particle's velocity $\boldsymbol{v}(t)$, acceleration $\dot{\boldsymbol{v}}(t)$, and the derivative of acceleration $\ddot{\boldsymbol{v}}(t)$ become obsolete.

Equation (73) has been discovered by Abraham and Lorentz. These scientists did not content themselves with that formula. Instead they looked for an explanation of radiation-backreaction, which should be derived straight forward from Maxwell's equations of electrodynamics. The theory, at which they eventually arrived, will be discussed in the next section.

4. Radiation-backreaction of an extended charge

Abraham and Lorentz considered an accelerated electron, radiating electromagnetic energy. They did not treat the electron as a charged point-particle; instead they assumed that it's charge is distributed over an extended range of space:

$$-e = \int \mathrm{d}^3x\, \rho(\boldsymbol{x}) = -1.6 \cdot 10^{-19}\mathrm{C} \tag{74}$$

The integral must be extended over all space, in which the electron's charge density $\rho(\boldsymbol{x})$ differs significantly from zero.

The radiation back-reaction is in the model of Abraham and Lorentz identical to the Lorentz force, which the accelerated electron exerts onto itself due to the retarded fields $\boldsymbol{E}_r^{(a)}$ and $\boldsymbol{B}_r^{(a)}$ which it is radiating. It is characteristic for the work of Abraham and Lorentz, that they considered the advanced fields $\boldsymbol{E}_a^{(a)}$ and $\boldsymbol{B}_a^{(a)}$ as unphysical artifacts of the theory, and discarded them from the outset. If no other charged particles are within the volume over which the integral is extended, then that force can be written as

$$\boldsymbol{F}^{(a)} = \int \mathrm{d}^3x\, (\rho \boldsymbol{E}_r^{(a)} + \boldsymbol{j} \times \boldsymbol{B}_r^{(a)}) \ . \tag{75}$$

The solution of this integral is described in Jackson [4, chap. 16]. In the sequel we content ourselves with a non-relativistic description ($v \ll c$), and apply an inertial system, in which the electron's velocity is so small, that the second term in (75) is negligible versus the first ($\boldsymbol{j} \times \boldsymbol{B}_r^{(a)} \ll \rho \boldsymbol{E}_r^{(a)}$). Such a reference system can always be defined, at least for a short time interval. If $\boldsymbol{E}_r^{(a)}$ is written as a function of the scalar potential $\Phi_r^{(a)}$ and the vector potential $\boldsymbol{A}_r^{(a)}$, then under these premises

$$\boldsymbol{F}^{(a)} = \int \mathrm{d}^3x\, \rho \boldsymbol{E}_r^{(a)} = -\int \mathrm{d}^3x\, \rho(\boldsymbol{\nabla}\Phi_r^{(a)} + \dot{\boldsymbol{A}}_r^{(a)}) \tag{76}$$

holds. Into this equation we must insert the retarded potentials

$$A_r^\rho(x) \overset{(36e)}{=} \frac{\mu_0}{4\pi} \int \mathrm{d}^3y \int\limits_{-\infty}^{+\infty} \mathrm{d}t_y \, \frac{j^\rho(y)}{R} \, \delta(t - t_y - R/c) \tag{77}$$

$$\text{with } \boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{y} \quad , \qquad R \equiv |\boldsymbol{R}| \; .$$

These potentials allow for the finite time $t - t_y > 0$ needed for the propagation of the fields from the point $\boldsymbol{y}$ to the point $\boldsymbol{x}$. Due to the delta function, the integral over t_y can easily be solved. In a non-relativistic description with $(j^\sigma) \approx (\rho c, \rho \boldsymbol{v})$, the three-dimensional potentials can be written as follows:

$$\left.\begin{aligned} \Phi_r^{(a)}(t, \boldsymbol{x}) &= \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3y \, \frac{\rho(t_r, \boldsymbol{y})}{R} \\ \boldsymbol{A}_r^{(a)}(t, \boldsymbol{x}) &= \frac{\mu_0}{4\pi} \int \mathrm{d}^3y \, \frac{\boldsymbol{j}(t_r, \boldsymbol{y})}{R} \end{aligned}\right\} \text{ with } t_r = t - \frac{R}{c} \tag{78}$$

In the model of Abraham and Lorentz the electron is extended over a finite, but very small volume. In (95) we will compute the "classical radius" of the electron as $\approx 3 \cdot 10^{-15}$m. Therefore $t - t_r = R/c$ is a very short time interval of about 10^{-23}s. Therefore it is reasonable to expand the retarded integrand of (78) in a Taylor series around t (to be precise: before $t > t_r$):

$$j^\alpha(t_r, \boldsymbol{y}) = j^\alpha(t - R/c, \boldsymbol{y}) = \sum_{n=0}^{\infty} \frac{(-R/c)^n}{n!} \, \frac{\mathrm{d}^n}{\mathrm{d}t^n} \, j^\alpha(t, \boldsymbol{y}) \tag{79}$$

We insert the retarded potentials (78) with the expansion (79) into (76), and make use of $c^{-2} = \epsilon_0 \mu_0$:

$$\begin{aligned} \boldsymbol{F}^{(a)} = -\frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, c^n} \int \mathrm{d}^3x \int \mathrm{d}^3y \, \rho(t, \boldsymbol{x}) \Big[\\ \boldsymbol{\nabla} R^{n-1} \, \frac{\mathrm{d}^n \rho(t, \boldsymbol{y})}{\mathrm{d}t^n} + \frac{1}{c^2} \, \frac{\mathrm{d}}{\mathrm{d}t} \, R^{n-1} \, \frac{\mathrm{d}^n \boldsymbol{j}(t, \boldsymbol{y})}{\mathrm{d}t^n} \Big] \end{aligned} \tag{80}$$

For the moment being we ignore the second term in the square brackets. Assuming that $\rho(\boldsymbol{x})$ is a rigid structure with spherical symmetry, for $n = 0$ the first term results into

$$-\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x \int \mathrm{d}^3y\, \rho(t,\boldsymbol{x})\, \rho(t,\boldsymbol{y})\, \boldsymbol{\nabla} R^{-1} = 0\ , \tag{81}$$

because there is — due to the spherical symmetry — for each positive contribution to the integral a negative contribution of same value. For $n = 1$ the first term in in the square brackets of (80) again is zero due to the factor $\boldsymbol{\nabla} R^{1-1} = 0$. Using

$$\boldsymbol{\nabla} R^{n-1} = (n-1)R^{n-2}\,\frac{\boldsymbol{R}}{R} = (n-1)R^{n-3}\boldsymbol{R}$$

we get

$$\begin{aligned}\boldsymbol{F}^{(a)} = -\frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}\int \mathrm{d}^3x \int \mathrm{d}^3y\, \rho(t,\boldsymbol{x}) \Big[\frac{n!}{(n+2)!\,c^2}\,(n+1)\,\cdot \\ \cdot\, R^{n-1}\boldsymbol{R}\,\frac{\mathrm{d}^{n+2}\rho(t,\boldsymbol{y})}{\mathrm{d}t^{n+2}} + \frac{1}{c^2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,R^{n-1}\,\frac{\mathrm{d}^n\boldsymbol{j}(t,\boldsymbol{y})}{\mathrm{d}t^n}\Big]\ .\end{aligned} \tag{82}$$

The derivatives with respect to t can be shifted out of the square brackets because of (35):

$$\begin{aligned}\boldsymbol{F}^{(a)} = -\frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^{n+2}}\int \mathrm{d}^3x \int \mathrm{d}^3y\, \rho(t,\boldsymbol{x})\,\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\Big[\frac{1}{(n+2)}\,\cdot \\ \cdot\, R^{n-1}\boldsymbol{R}\,\frac{\mathrm{d}\rho(t,\boldsymbol{y})}{\mathrm{d}t} + R^{n-1}\,\boldsymbol{j}(t,\boldsymbol{y})\Big]\end{aligned} \tag{83}$$

Using the equation of continuity

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t,\boldsymbol{y}) = -\boldsymbol{\nabla}_y \cdot \boldsymbol{j}\,(t,\boldsymbol{y})\ ,$$

which is describing the conservation of charge, we find

$$\boldsymbol{F}^{(a)} = -\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x\,\rho(t,\boldsymbol{x})\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^{n+2}}\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\int \mathrm{d}^3y\,R^{n-1}\Big[- \frac{\boldsymbol{R}}{(n+2)}\boldsymbol{\nabla}_y\cdot\boldsymbol{j}\,(t,\boldsymbol{y}) + \boldsymbol{j}(t,\boldsymbol{y})\Big]\,. \tag{84}$$

At the boundaries of the volume, over which the integral over $\boldsymbol{y}$ is extended, ρ resp. $\boldsymbol{j}$ is zero (resp. negligible). Partial integration of the first term over $\boldsymbol{y}$ thus gives besides some constant factors

$$\begin{aligned}
&-\int \mathrm{d}^3y\,R^{n-1}\frac{\boldsymbol{R}}{(n+2)}\boldsymbol{\nabla}_y\cdot\boldsymbol{j}\,(t,\boldsymbol{y}) = \\
&= +\int \mathrm{d}^3y\,(\boldsymbol{j}\cdot\boldsymbol{\nabla}_y)\,R^{n-1}\frac{\boldsymbol{R}}{(n+2)} \\
&= \int \mathrm{d}^3y\,\Big(- (n-1)R^{n-3}(\boldsymbol{j}\cdot\boldsymbol{R})\frac{\boldsymbol{R}}{(n+2)} - R^{n-1}\frac{\boldsymbol{j}}{(n+2)}\Big)\,.
\end{aligned}$$

Therefore the $\boldsymbol{y}$-integral over the square brackets of (84) is

$$\begin{aligned}
&\int \mathrm{d}^3y\,R^{n-1}\Big[\ldots\Big] = \\
&= \int \mathrm{d}^3y\,R^{n-1}\Big[- (n-1)R^{-2}(\boldsymbol{j}\cdot\boldsymbol{R})\frac{\boldsymbol{R}}{(n+2)} + \frac{(n+1)\boldsymbol{j}}{(n+2)}\Big]\,.
\end{aligned}$$

As we are assuming a rigid charge distribution,

$$\boldsymbol{j}(t,\boldsymbol{y}) = \rho(t,\boldsymbol{y})\boldsymbol{v}(t) \tag{85}$$

holds. As we furthermore are assuming a spherical charge distribution, all components of the integrand, which are perpendicular to $\boldsymbol{v}$, mutually compensate upon integration over $\boldsymbol{x}$ and $\boldsymbol{y}$. Therefore the integral's value remains unchanged, if the factor $\boldsymbol{R}$ in the square brackets is multiplied by the unit vector $\boldsymbol{v}/v$:

$$\int \mathrm{d}^3y\, R^{n-1} \Big[\dots \Big] = \int \mathrm{d}^3y\, R^{n-1} \rho(t,\boldsymbol{y})\boldsymbol{v}(t) \cdot \\ \cdot \Big[-\frac{(n-1)}{(n+2)} \frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{(vR)^2} + \frac{(n+1)}{(n+2)} \Big] \tag{86}$$

As $\boldsymbol{R}$ assumes under the integral all directions with equal frequency, $(\boldsymbol{v}\cdot\boldsymbol{R})^2/(vR)^2$ may be replaced by it's mean value

$$\left\langle \frac{(\boldsymbol{v}\cdot\boldsymbol{R})^2}{(vR)^2} \right\rangle = \frac{1}{4\pi} \int\limits_0^\pi \sin\vartheta\ \mathrm{d}\vartheta \int\limits_0^{2\pi} \mathrm{d}\varphi\ \cos^2(\boldsymbol{v},\boldsymbol{R}) = \\ = \frac{1}{2} \Big[-\frac{\cos^3\vartheta}{3} \Big]_0^\pi = \frac{1}{3}\ .$$

This results into

$$(86) = \int \mathrm{d}^3y\, R^{n-1} \rho(t,\boldsymbol{y})\boldsymbol{v}(t)\, \frac{2}{3} \quad .$$

Consequently the field radiated by the electron exerts onto the electron itself the force

$$\boldsymbol{F}^{(a)} = \sum_{n=0}^{\infty} \boldsymbol{F}_n^{(a)} \tag{87a}$$

$$\boldsymbol{F}_n^{(a)} \overset{(84)}{=} -\frac{4}{3} U_n^{(a)} \frac{(-1)^n}{n!\, c^{n+2}} \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} \boldsymbol{v}(t) \tag{87b}$$

$$U_n^{(a)} \equiv \frac{1}{2} \int \mathrm{d}^3x \int \mathrm{d}^3y\, \frac{\rho(t,\boldsymbol{x})\,\rho(t,\boldsymbol{y})}{4\pi\epsilon_0\, R^{1-n}}\ . \tag{87c}$$

$U_n^{(a)}$ could be shifted before the differential quotient, because we are assuming a rigid form and charge distribution of the electron. Therefore $U_n^{(a)}$ does not depend on t. The force is zero, if all derivatives $\mathrm{d}^{n+1}\boldsymbol{v}/\,\mathrm{d}t^{n+1}$ are zero, i. e. if the electron is at rest or is moving with constant velocity. In the limit of a point-particle electron ($R \rightarrow 0$), $U_0^{(a)}$ would diverge, $U_1^{(a)}$ is independent of R, and $U_n^{(a)}$ with $n \geq 2$ would converge to zero. As we are assuming an extended, but quite small electron with radius of about $3 \cdot 10^{-15}$m (reasons for this assumption will be given in a moment), the terms with $n \geq 2$ are different from zero, but they are quite small. The result is dominated by the two terms with $n = 0$ and $n = 1$. This becomes obvious, if the orders of magnitude of the terms n and $n - 1$ (with $n \geq 2$) are compared:

$$\frac{|\boldsymbol{F}_n^{(a)}|}{|\boldsymbol{F}_{n-1}^{(a)}|} \approx \frac{R\left|\dfrac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\,\boldsymbol{v}(t)\right|}{c\left|\dfrac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\,\boldsymbol{v}(t)\right|} \approx$$
$$\approx 10^{-23} \quad \text{if } \left|\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\,\boldsymbol{v}(t)\right| \Big/ \left|\frac{\mathrm{d}^{n}}{\mathrm{s}\,\mathrm{d}t^{n}}\,\boldsymbol{v}(t)\right| \approx 1 \qquad (88)$$

Due to the small value of R, all terms with $n \geq 2$ are negligible in most cases. Thus we find the radiation back-reaction

$$\boldsymbol{F}^{(a)} = \boldsymbol{F}_0^{(a)} + \boldsymbol{F}_1^{(a)} = -\frac{4}{3}\frac{U_{\mathrm{s}\,0}}{c^2}\,\dot{\boldsymbol{v}} + \frac{4}{3}\frac{U_{\mathrm{s}\,1}}{c^3}\,\ddot{\boldsymbol{v}}\ . \qquad (89)$$

The term $\boldsymbol{F}_0^{(a)}$ essentially is the electron's electrostatic self-energy

$$U_0^{(a)} \overset{(87)}{=} \frac{1}{2}\int \mathrm{d}^3x \int \mathrm{d}^3y\, \frac{\rho(t,\boldsymbol{x})\,\rho(t,\boldsymbol{y})}{4\pi\epsilon_0\, R}\ . \qquad (90)$$

This is the work required to bring together the charge of the electron from infinite distance against the electrostatic repulsion to it's position within the electron.

Using

$$U_1^{(a)} \overset{(87)}{=} \frac{1}{2}\int \mathrm{d}^3x \int \mathrm{d}^3y\, \frac{\rho(t,\boldsymbol{x})\,\rho(t,\boldsymbol{y})}{4\pi\epsilon_0\, R^{1-1}} = \frac{e^2}{8\pi\epsilon_0}\ , \qquad (91)$$

the radiation back-reaction may be written as

$$\boldsymbol{F}^{(a)} \approx -\frac{4}{3}\frac{U_0^{(a)}}{c^2}\,\dot{\boldsymbol{v}} + \frac{2e^2}{3c^3 4\pi\epsilon_0}\,\ddot{\boldsymbol{v}}\ . \tag{92a}$$

This should be compared with

$$\boldsymbol{F} \overset{(73)}{=} \boldsymbol{F}_{\mathrm{acc}} + \boldsymbol{F}_{\mathrm{rad}} = m\dot{\boldsymbol{v}} - \frac{2e^2}{3c^3 4\pi\epsilon_0}\,\ddot{\boldsymbol{v}} \tag{92b}$$

The signs are different, because $\boldsymbol{F}$ was defined in (73) as an external force acting on the electron, and thus is acting in opposite direction to the back-reaction force $\boldsymbol{F}^{(a)}$ exerted by the electron upon itself. The both last terms in (92) are identical. Thus it seems reasonable to mutually identify as well the both terms which are proportional to $\dot{\boldsymbol{v}}$. The "electrostatic mass" of the electron is defined by

$$m_{\mathrm{e}} \equiv \frac{4}{3}\frac{U_0^{(a)}}{c^2}\ , \tag{93}$$

resulting into the equation

$$\boldsymbol{F}_0^{(a)} = -m_e\dot{\boldsymbol{v}}\ . \tag{94}$$

Some years later (namely in 1905) Einstein discovered the relation $E = mc^2$ of energy and inertial mass. The inertial mass m_{e} found by Abraham and Lorentz differs from Einstein's result by the strange factor 4/3. Regarding the explanation and elimination of this factor we again refer to the work of Medina [9]. Besides that issue, the theory of Abraham and Lorentz does not need a separate mass parameter, but can explain the electron's inert mass by the self-interaction of it's charge! The "classical radius" of the electron

r_e is defined as weighted average of $2R$ in the integral (90):

$$\frac{e^2}{4\pi\epsilon_0 r_e} \equiv \int \mathrm{d}^3x \int \mathrm{d}^3y \, \frac{\rho(t,\boldsymbol{x})\,\rho(t,\boldsymbol{y})}{4\pi\epsilon_0\, 2R} \overset{(90)}{=} U_0^{(a)}$$
$$\Longrightarrow \quad r_e = \frac{4}{3}\,\frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx \frac{4}{3}\, 2.8\cdot 10^{-15}\mathrm{m} \approx 3.8\cdot 10^{-15}\mathrm{m}\ . \qquad (95)$$

This value is found, if the experimentally observed value $m_e = 9.1\cdot 10^{-31}$kg is inserted. If instead of m_e the mass $m = U_0^{(a)}/c^2 = 9.1\cdot 10^{-31}$kg according to Einstein is inserted, one gets $r_e \approx 2.8\cdot 10^{-15}$m.

At the end of the nineteenth century, the electron model of Abraham and Lorentz was a remarkable progress of the theory, because it supplied plausible explanations for

* the inert mass of the electron, and for
* the radiation back-reaction.

The close connection of both explications enforced their persuasive power. Still there are several severe flaws. The two most obvious and most important are:

* Until today (2013) no force is known, which could hold together the electron against the electrostatic repulsion of it's constituents within the small volume of $4\pi r_e^3/3 \approx 10^{-43}\mathrm{m}^3$.
* There is by today convincing experimental evidence, that the radius of the electron (if it should have finite size) must be smaller than $2\cdot 10^{-22}$m [11]. (Remarkably, this limit does not result from high-energy lepton collisions, which can restrict the electron radius to about $< 10^{-18}$m only, but from evaluation of the electron's anomalous magnetic moment.) The classical radius of the electron is larger by more than a factor 10^7 than compatible with observation.

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