Sudoku

How long have Sudoku puzzles been around? Who invented them? How many different Sudoku puzzles are there? How are Sudoku puzzles created? This article contains some (not very profound) thoughts on this popular type of brain teaser.

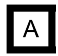
1. Who invented the Sudoku?

Special arrangements of numbers in "magic" squares have fascinated people for centuries. Albrecht Dürer's copperplate engraving "Melencholia I" from 1514 is a famous example. At the top right, under the bell, you can see a magic square of 4x4 numbers. (Because the square is difficult to see in this print, it is reproduced enlarged below.) The numbers in each row, in each column, in each diagonal, in the 2x2 quadrants of all four corners,

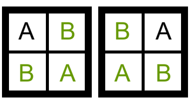
and in the four fields in the corners always add up to the same number: 34

More direct precursors to Sudoku were the “Latin squares”, which Leonhard Euler (1707–1783) studied intensively in the 18th century. They got their name because Euler did not enter numbers into the squares, but rather letters from the Latin alphabet. A square with n rows and n columns is called a square of order n.

Euler sought the answer to the following question: How many different Latin squares of order n can be formed if each letter from a group of n different letters must appear exactly once in each row and in eacch column of the square?

A Latin square of order 1 is pretty boring: in its 1 row and 1 column, you can arrange a letter in exactly one way: 

Latin squares of order 2 are only slightly more interesting:



When filling in the square, you are free to choose whether you want to place the letter A in the first row on the left (as was done in the left square) or on the right (as was done in the right square). Once this decision has been made, the positions of the green letters are determined. There are therefore exactly 2 different Latin squares of order 2 .

Things gradually become more interesting with Latin squares of order 3 :

A	B	C	A	B	C	B	A	C	B	A	C	B	C	A	B	C	A
C	A	B	B	C	A	A	C	B	C	B	A	A	B	C	C	A	B
B	C	A	C	A	B	C	B	A	A	C	B	C	A	B	A	B	C
A	C	B	A	C	B	C	A	B	C	A	B	C	B	A	C	B	A
B	A	C	C	B	A	A	B	C	B	C	A	A	C	B	B	A	C
C	B	A	B	A	C	B	C	A	A	B	C	B	A	C	A	C	B

The A can be placed on the left, in the middle, or on the right in the first row: 3 possibilities. Then there are 2 different ways to arrange the A in the second row, and finally 2 different ways to arrange the B in the first row. Once these decisions have been made, the positions of the green letters are fixed. So there are exactly

$$3 \times 2 \times 2 = 12$$

different Latin squares of order 3 .

With larger squares, it quickly becomes very tedious and time-consuming to determine the number of different possibilities by writing down all possible variants explicitly. What Euler was actually looking for was a general formula with which the number of different Latin squares of any order n could be calculated without actually having to write down these squares. Euler did not find such a formula, and to this day, no one else has succeeded in doing so either. The best mathematicians have been able to come up with are formulas for the lower and upper bounds of this number. They were able to prove that the number $L(n)$ of different Latin squares of order n must lie in this range:

$$\frac{(n!)^{2n}}{n^{n\cdot n}} \leq L(n) \leq \prod_{k=1}^{n} (k!)^{n/k}$$

Let's try this formula for the nine smallest Latin squares:

$$1 \leq L(1) \leq 1$$
$$1 \leq L(2) \leq 2$$
$$3 \leq L(3) \leq 16$$
$$26 \leq L(4) \leq 1\,046$$
$$2\,078 \leq L(5) \leq 714\,396$$
$$1\,881\,677 \leq L(6) \leq 7\,621\,831\,476$$
$$2.656 \cdot 10^{10} \leq L(7) \leq 1.70412 \cdot 10^{15}$$
$$7.7727 \cdot 10^{15} \leq L(8) \leq 1.03026 \cdot 10^{22}$$
$$6.0547 \cdot 10^{22} \leq L(9) \leq 2.11023 \cdot 10^{30}$$

Due to explicit construction, we found above

$$L(1) = 1 \ , \ L(2) = 2 \ , \ L(3) = 12 \ ,$$

which is compatible with the general formula. It is clear that determining $L(n)$ for $n > 3$ by explicit construction is a task for masochists, and without the help of computers, it becomes quite hopeless for $n > 5$ at the latest. With the help of considerable computer power, Bammel and Rothstein [1] calculated

$$L(9) = 5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \approx 5.525 \cdot 10^{27} \ .$$

It was Howard Garns (1905 – 1989), an US-American from Indiana, who published in a puzzle-journal a number puzzle called "number place", which was modeled on Euler's Latin squares: A Latin square of order 9 was only partially filled with the numbers 1 to 9; readers of the magazine were asked to fill in the missing

1				2			7	
9			5					3
		4		8				6
		3	1			5		
		6			2		4	
	9		7				8	
	4				6			2
	8			7		1		
5					3	9		

numbers. An important innovation in Garns' square was that not only did each number from 1 to 9 have to appear exactly once in each row and in each column of the 9×9 square, but also in each of the nine 3 × 3 sub-blocks. A (not so easy) example of the number puzzle invented by Garns is printed here. Below, I will describe how this puzzle was constructed.

Worldwide popular became Garns' number puzzles via Japan. A Japanese magazine regularly printed these puzzles between 1984 and 1986 under the name "Suji wa dokushin ni kagiru" (roughly translated as "Isolate the numbers; the numbers may only appear once"), abbreviated to "Sudoku". The London Times has been printing Sudoku puzzles since 2004 .[1]

2. How many different completely filled-in Sudoku grids can be constructed?

An obvious idea for automatically solving Sudoku puzzles is as follows: With the help of a computer, a complete catalog of all fully completed Sudoku-grids is calculated. To solve a Sudoku puzzle, you then only need to look up which entry in the catalog matches the numbers entered in that Sudoku-puzzle. But you will realize that this method cannot work, as soon as you consider how large the catalog would have to be.

Due to the additional condition (each of the nine numbers must appear exactly once not only in each row and column, but also

[1] The informations about the origins of Sudoku come from Wikipedia [2].

in each of the nine 3×3 blocks), there are far fewer different Sudoku solutions than Latin squares of order 9. How much fewer? Supported by considerable computer-power, Felgenhauer and Jarvis [3] calculated this in 2005. If their result is correct, then there exist "only"

$$6\,670\,903\,752\,021\,072\,936\,960 \approx 6.671 \cdot 10^{21}$$

different completely filled Sudoku grids. The number of different completely filled Sudokus is therefore more than 800 000 times smaller than the number $L(9)$ of different Latin squares of order 9. But it is still far too large for a catalog. Workman [4] roughly estimated that the total global storage capacity for digital data in 2015 was approximately

$$2.5 \cdot 10^{21} \text{ bytes}.$$

To save a single completed Sudoku puzzle, you need about 80 bytes. Therefore, the complete Sudoku-catalog requires approximately

$$80 \text{ bytes} \cdot 6.671 \cdot 10^{21} \approx 5.3 \cdot 10^{23} \text{ bytes}$$

of storage space. Even if you were to confiscate all the storage space available in 2015 on private PCs, in data centers, server farms, and wherever else around the world for the Sudoku catalog, you would only be able to store about 0.5% of the catalog. Not to mention that, before you could save the catalog, you would first have to calculate it ...

3. The construction of new Sudoku puzzles

An obvious and (in principle) simple method for constructing new Sudoku puzzles is to fill an empty Sudoku grid with individual numbers until the solution is unique. This is a very tedious method, however, because you will often reach a dead end and construct a puzzle that has no solution at all. This method is therefore only suitable if you are supported by powerful computer software.

It is easier to first construct a completely filled Sudoku grid (i. e., an element from the "catalog" discussed in the previous section) and then delete enough numbers so that the Sudoku does not become too easy, but still has a unique solution. The easiest way to find a completely filled grid is by permuting known complete grids.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

For example, we can start with this very simple grid, which obviously satisfies all three conditions (each of the nine numbers appears exactly once in each row, each column, and each of the nine 3×3 blocks): In the three upper 3×3 blocks, the number groups (1 2 3), (4 5 6), and (7 8 9) each appear once, but in a different row in each block. Apparently, new, correct Sudokus can be obtained by permuting the number groups in a suitable way. The (1 2 3) of the first block can be arranged in the first, second, or third row (3 possibilities). This leaves two possible rows for the (1 2 3) in the second block, and two possibilities for placing the (4 5 6) in the first block. Once these have been determined, the positions of the nine horizontal groups of three in the top three blocks are uniquely defined.

Corresponding horizontal groups of three, which can be permuted in the same way, also exist in the three middle and three lower blocks. So by permuting the horizontal groups of three, a total of

$$(3 \cdot 2 \cdot 2)^3 = 12^3 = \mathbf{1728}$$

different correct, completely filled Sudoku grids can be generated. Permutations of complete rows within the upper, middle, or lower blocks would also be permissible, but would not result in any new grids compared to the previously considered permutations of the horizontal groups of three.

The three 3×3 blocks that were initially at the top of the Sudoku grid can alternatively be placed in the middle or at the bottom (3 possibilities), leaving two possibilities for the placement of the three blocks that were initially in the middle. In total, this permutation allows

$$3 \cdot 2 = \mathbf{6}$$

different correct, completely filled Sudoku grids to be generated.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

In the three lower blocks of the initial grid, vertical groups of three are highlighted by color. Apparently, new, correct Soduko grids are obtained by swapping the group $\begin{pmatrix}3\\6\\9\end{pmatrix}$ with the group $\begin{pmatrix}6\\9\\3\end{pmatrix}$ or with the group $\begin{pmatrix}9\\3\\6\end{pmatrix}$, or the group $\begin{pmatrix}6\\9\\3\end{pmatrix}$ with the group $\begin{pmatrix}9\\3\\6\end{pmatrix}$. Further six different grids can be obtained by permuting the red-colored groups of three, and another six different grids by permuting the gray-colored groups of three. This makes a total of 6^3 different lattices by permuting vertical triplets in the lower blocks. The same number of permutations of vertical triplets is possible in the upper and middle blocks. So we get a total of

$$(6^3)^3 = 6^9 = \mathbf{10\,077\,696}$$

different correct, completely filled Sudoku grids by permutations of vertical triplets. Another

$$6^3 = \mathbf{216}$$

different correct, completely filled Sudoku-grids can be obtained by permuting complete columns within the left, middle, or right blocks. Permutations of the complete left, middle, and right blocks would also be permissible, but would not result in any new grids compared to the previously considered permutations of the vertical groups of three.

By mirroring the entire grid along the diagonal marked by the dashed red line, the number of correct Sudoku grids can be increased by a factor of

$$\mathbf{2}\,.$$

Further reflections on the other diagonal, or on the middle row, or on the middle column would be permissible, i. e. they would lead to further correct Sudoku grids, just like rotations of the entire grid by 90°, or by 180°, or by 270°. I suspect, however (though I am not entirely sure), that all of these transformations would lead to grids that have already been generated by the permutations considered above.

There is another important type of permutations that we have not considered yet: renaming the numbers. A correct Sudoku grid remains a correct Sudoku grid if e. g. every 5 is renamed to 7, and at the same time every 7 is renamed to 5. Or if every 4 is renamed to 3, and at the same time every 3 is renamed to 7, and at the same time every 7 is renamed to 4. There are 9 ways to rename the old 1 to a new number (which can also be the old number, i. e., 1). Then there are 8 ways to rename the old 2 to a new number, then 7 ways to rename the old 3 to a new number, and so on. In total,

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 9! = \mathbf{362\,880}$$

different correct, completely filled Sudoku grids can be created by renaming the numbers.

By combining all of the possibilities listed above, a total of

$$\mathbf{1728} \cdot \mathbf{6} \cdot \mathbf{10\,077\,696} \cdot \mathbf{216} \cdot \mathbf{2} \cdot \mathbf{362\,880} = 1.63796 \cdot 10^{19}$$

different correct, completely filled-in Sudoku grids can be created from the simple Sudoku grid that has been printed several times on the previous pages. Although this is only 0.25% of the total

$6,671 \cdot 10^{21}$ grids which are possible according to Felgenhauer and Jarvis [3], it is still a considerable amount: If a puzzle magazine wants to present its readers with 100 new Sudoku puzzles every week and is determined that no two Sudoku puzzles should ever have the same solution, then the supply of $1.63796 \cdot 10^{19}$ different solutions will last for

$$1.63796 \cdot 10^{17} \text{ weeks} \approx 3 \cdot 10^{15} \text{ years} .$$

Considering that in approximately 5 billion years $= 5 \cdot 10^9$ years the Earth will burn up in the dying sun, the supply of $1.63796 \cdot 10^{19}$ different Sudoku grids is undoubtedly more than sufficient.

To make a completely filled Sudoku grid a puzzle, some of the numbers are not printed, but the readers have to search for them. The trick is to omit enough numbers to make the puzzle interesting for the readers, but not so many that there is no longer a unique solution.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

If you only pay attention to the horizontal groups of three, you might think that you can omit the six numbers marked with red dots. After all, the arrangement of the horizontal groups of three (1 2 3), (4 5 6), and (7 8 9) remains unique. In fact, however, the solution would be ambiguous, because swapping the vertical groups of three $\begin{pmatrix}6\\9\\3\end{pmatrix}$ and $\begin{pmatrix}9\\3\\6\end{pmatrix}$ would also result in a correct solution. The ambiguity would be eliminated as soon as even one – no matter which – of the six numbers marked with red dots is *not* omitted.

Entire rows can be swapped if they are within the same three blocks. Therefore, two such columns must never be completely empty if the Sudoku is to remain unique. The same applies to columns.

Finally, with regard to the uniqueness of Sudoku puzzles, one must also keep in mind the interchangeability of the numbers: the puzzle can only have a unique solution if at least eight of the numbers from 1 to 9 are printed at least once.

Under certain circumstances, a Sudoku puzzle can already be ambiguous if only 4 fields are left undefined. The two Sudoku grids(taken from [5])

9	2	6	5	7	1	4	8	3
3	5	1	4	8	6	2	7	9
8	7	4	9	2	3	5	1	6
5	8	2	3	6	7	1	9	4
1	4	9	2	5	8	3	6	7
7	6	3	1	9	4	8	2	5
2	3	8	7	4	9	6	5	1
6	1	7	8	3	5	9	4	2
4	9	5	6	1	2	7	3	8

9	2	6	5	7	1	4	8	3
3	5	1	4	8	6	2	7	9
8	7	4	9	2	3	5	1	6
5	8	2	3	6	7	1	9	4
1	4	9	2	5	8	3	6	7
7	6	3	1	4	9	8	2	5
2	3	8	7	9	4	6	5	1
6	1	7	8	3	5	9	4	2
4	9	5	6	1	2	7	3	8

differ only in the four fields printed in red. A Sudoku puzzle in which all 77 black-printed fields were given and only the four red-printed fields were empty would therefore be ambiguous.

With more favorably constructed grids, you need to specify far fewer entries and still obtain a unique solution. Sudoku puzzles published in magazines usually have between 22 and 35 specified entries. How many specifications are required in the best case scenario to ensure that the solution is unique?

According to [5], Australian Gordon Royle found 36 628 “truly different” unique Sudokus with only 17 clues, but not a single unique one with only 16 clues. By “truly different” is meant that these 36 628 Sudokus differ in more ways than just renaming the numbers, or mirroring them along an axis, or permuting rows and/ or columns, or rotating the entire grid.

							1	
4								
	2							
				5		4		7
		8				3		
		1		9				
3			4			2		
	5		1					
			8		6			

This is one of the Sudokus from Royle's collection. Herzberg and Murty [5] published it with the comment: "We leave it to the reader that the puzzle [...] has a unique solution." I pass this invitation on unfiltered to the readers of this circular.

McGuire, Tugemann, and Civario [6] searched for a uniquely solvable Sudoku puzzle with only 16 given numbers using enormous computing power, but found none. The authors were certain: "had one existed, we would have found it," they wrote in their article.

To conclude these considerations, we will construct a Sudoku. We will start with our simple and clear "basic-Sudoku-grid", because it is very easy to see which numbers can be omitted without compromising the uniqueness of the Sudoku puzzle, and which cannot. The numbers we will omit are marked with red dots. The red dots are concentrated in the vertical groups of three. As many of these numbers as possible are omitted so that the Sudoku remains unique.

1•	2•	3	4•	5•	6	7	8•	9•
4•	5	6•	7•	8•	9	1•	2	3•
7•	8	9•	1	2•	3•	4	5•	6•
2	3•	4•	5	6•	7•	8•	9	1•
5•	6•	7	8	9•	1•	2•	3•	4
8•	9•	1	2•	3	4•	5•	6	7•
3•	4	5•	6•	7	8•	9•	1•	2
6	7•	8•	9•	1	2•	3	4•	5•
9	1•	2•	3•	4•	5	6•	7•	8

First we swap the rows 1–3 with the rows 4–6, then the new rows 1–3 with the rows 7–9, then the columns 1–3 with the columns 4 − 6, then the new columns 1 − 3 with the columns 7 − 9, then column 1 with column 2, column 5 with column 6, and column 7 with column 9 . Finally we make these renaming of numbers:

1	2	3	4	5	6	7	8	9
↓	↓	↓	↓	↓	↓	↓	↓	↓
9	2	5	7	4	3	1	8	6

Thereby we arrive at this Sudoku grid. If all red marked numbers are skipped, it is identical to the Sudoku printed on page 5 .

1	3•	8•	6•	2	9•	4•	7	5•
9	6•	2•	5	4•	7•	8•	1•	3
7•	5•	4	3•	8	1•	2•	9•	6
4•	7•	3	1	6•	8•	5	2•	9•
8•	1•	6	9•	5•	2	3•	4	7•
2•	9	5•	7	3•	4•	6•	8	1•
3•	4	1•	8•	9•	6	7•	5•	2
6•	8	9•	2•	7	5•	1	3•	4•
5	2•	7•	4•	1•	3	9	6•	8•

4. A "Sudoku Assistant"

The Sudoku Assistant is an Excel macro that reliably warns the user if they make a mistake while solving the puzzle. It can also automatically enter the solution to the Sudoku puzzle. Above all, however, the assistant checks within a few seconds whether a particular Sudoku puzzle has a unique solution, i.e. whether it is a correctly set task or not. You can download the Excel file with the macro here. (Excel, 205KB):
https://www.astrophys-neunhof.de/utls/Sudoku-Assistant.xls

References

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[6] G. McGuire, B. Tugemann, G. Civario: *There is no 16-Clue Sudoku: Solving the Sudoku Minimum Number of Clues Problem*, arXiv: 1201.0749 (2012) https://arxiv.org/abs/1201.0749