Time Dilation of accelerated Clocks

The definitions of proper time and proper length in General Relativity Theory are presented. Time dilation and length contraction are explicitly computed for the example of clocks and rulers which are at rest in a rotating reference frame, and hence accelerated versus inertial reference frames. Experimental proofs of this time dilation, in particular the observation of the decay rates of accelerated muons, are discussed. As an illustration of the equivalence principle, we show that the general relativistic equation of motion of objects, which are at rest in a rotating reference frame, reduces to Newtons equation of motion of the same objects at rest in an inertial reference system and subject to a gravitational field. We close with some remarks on real versus ideal clocks, and the “clock hypothesis”.

1. Coordinate Diffeomorphisms

In flat Minkowski space, we place a rectangular three-dimensional grid of fiducial marks in three-dimensional position space, and assign cartesian coordinate values x^1, x^2, x^3 to each fiducial mark. The values of the x^i are constants for each fiducial mark, i. e. the fiducial marks are at rest in the coordinate frame, which they define.

Now we stretch and/or compress and/or skew and/or rotate the grid of fiducial marks locally and/or globally, and/or re-name the fiducial marks, e. g. change from cartesian coordinates to spherical coordinates or whatever other coordinates. All movements and renamings of the fiducial marks are subject to the constraint that the map from the initial grid to the deformed and/or renamed grid must be differentiable and invertible (i. e. bijective), and that the inverse map as well must be differentiable. If the shifts and renamings of the fiducial marks are subject to that constraint, then the map from the primary to the final coordinate system is called a *diffeomorphism*. The condition that the maps must be invertible and differentiable makes sure that the new coordinate grid is "smooth" and free of singularities.

We also allow for time-dependent diffeomorphisms. Such time-dependent diffeomorphisms are parametrized by the time t displayed by a clock, which is at rest in the primary cartesian coordinate system with flat Minkowski metric, from which the diffeomorphism starts.

2. Proper Time and Proper Length

To all fiducial marks equally built standard clocks are fixed, which by definition read the *proper time* τ at these points of space. We will specify in section 6, what qualifies a clock to be a *standard clock*.

In the primary flat Minkowski space, proper time is identical at all fiducial marks, and equal to the *coordinate time* $t \equiv x^0/c$ of the inertial cartesian coordinate system, with c being the speed of light in vacuum. After the diffeomorphism, the rate at which proper time is passing is determined by the two conditions (2) and (4) stated below. Proper time is still displayed at any point in space by the standard clock fixed to this point of space. But now proper time may be different at different points in space. And proper time τ may differ from coordinate time $t \equiv x^0/c$. Furthermore the run-rates of coordinate time may be different for different coordinate systems. The relation between coordinate time t and proper time τ will be clarified in (3) below. c is assumed to have the same value everywhere in space at any time, and to be independent of the coordinate system, i. e. a constant of nature.

For the following, we stipulate that greek space-time indices $\mu, \nu, \rho, \dots$ have to be summed over $0, 1, 2, 3$ automatically, whenever they show up twice in a product. Latin space indices $i, j, k, \dots$ have to be summed over $1, 2, 3$ automatically, whenever they show up twice in a product.

Before the diffeomorphism (i. e. in flat Minkowski-space), the metric at any fiducial mark is

$$(g_{\mu\nu}) = (\eta_{\mu\nu}) = \text{diagonal}(1, -1, -1, -1) \ . \tag{1}$$

After the diffeomorphism, the metric $g_{\mu\nu}(x)$ is defined such, that for the line element

$$\underbrace{\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}_{\text{after the diffeomorphism}} = \underbrace{\mathrm{d}s^2}_{\text{before the diffeomorphism}} \quad \text{holds at any fiducial mark.} \tag{2}$$

$\mathrm{d}s$ is the (under diffeomorphisms invariant) differential of a four-dimensional length in space-time. A standard clock with constant spatial coordinates (i. e. at rest at some point of space) by definition displays the proper time at this point of space. Its line element is

$$\begin{aligned} &\text{if } \mathrm{d}x^1 = \mathrm{d}x^2 = \mathrm{d}x^3 = 0 \ : \\ &\mathrm{d}s^2 \overset{(2)}{=} g_{00}(\mathrm{d}x^0)^2 \equiv g_{00}c^2\,\mathrm{d}t^2 = c^2\,\mathrm{d}\tau^2 \ . \end{aligned} \tag{3a}$$

In another coordinate system, in which the same clock is moving, the relation between proper time τ of the moving clock (i. e. the time displayed by this clock) and coordinate time t becomes

$$\begin{aligned} \mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2 \overset{(2)}{=} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu &= \\ &= g_{00}c^2\,\mathrm{d}t^2 + 2g_{0i}c\,\mathrm{d}t\,\mathrm{d}x^i + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \ . \end{aligned} \tag{3b}$$

(2) does not completely determine the metric. As a further constraint (which still does not completely determine the metric),

we require that for a light signal in vacuum the invariant line element is zero:

$$\text{light signal}: \quad \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \\ = g_{00}c^2\,\mathrm{d}t^2 + 2g_{0i}c\,\mathrm{d}t\,\mathrm{d}x^i + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = 0 \tag{4}$$

Consequently we have $\mathrm{d}\tau \overset{(3b)}{=} 0$ for a clock moving at the speed of light in vacuum. This is a purely theoretical result, of course, because no clock can be accelerated to that speed.

Following Cook [1], we define the proper length differential $\mathrm{d}\ell$ by means of the proper time differential and the constant c:

$$\mathrm{d}\ell \equiv c\,\mathrm{d}\tau \tag{5a}$$

To implement $\mathrm{d}\ell$ practically at a certain fiducial mark F, a mirror M is placed in arbitrary direction from F. Then we let a light signal run from F to M and back to F. The mirror is shifted, until the time between emission and absorption of the light signal at point F, as measured by the standard clock fixed at F, is

$$2\,\mathrm{d}\tau = \frac{2\times(\text{distance } F \text{ to } M)}{c} = \frac{2\,\mathrm{d}\ell}{c}\ . \tag{5b}$$

$\mathrm{d}\ell$ is determined due to (5) by means of a light signal, whose invariant space-time line element is zero:

$$\text{light signal:} \quad \mathrm{d}s^2 \overset{(2)}{=} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \overset{(4)}{=} 0 \overset{(5)}{=} c^2\,\mathrm{d}\tau^2 - \mathrm{d}\ell^{\,2} \tag{6}$$

This is a quadratic equation for $\mathrm{d}t = \mathrm{d}x^0/c$, see (4). Its solutions are

$$\mathrm{d}x^0 = -\frac{g_{0i}}{g_{00}}\,\mathrm{d}x^i \pm \sqrt{\frac{g_{0i}g_{0j}}{g_{00}^2}\,\mathrm{d}x^i\,\mathrm{d}x^j - \frac{g_{ij}}{g_{00}}\,\mathrm{d}x^i\,\mathrm{d}x^j}\ . \tag{7}$$

Inserting this expression into (5b), we get the same expression for the signal running from F to M, and for the reflected signal running from M to F, but the signs of the $\mathrm{d}x^i$ are changed. Thus the total time for the signal running from F to M and back to F, measured by the standard clock fixed at F, is

$$2\,\mathrm{d}\tau = \text{total run-time of the light signal} =$$
$$\overset{(3a)}{=} \frac{\sqrt{g_{00}}}{c}\,(\mathrm{d}x^0_{FM} + \mathrm{d}x^0_{MF}) \overset{(7)}{=} +\frac{2}{c}\sqrt{\Big(\frac{g_{0i}g_{0j}}{g_{00}} - g_{ij}\Big)\,\mathrm{d}x^i\,\mathrm{d}x^j}\ . \tag{8}$$

We decided for the positive square root for the physical reason, that $\mathrm{d}\tau$ shall be ≥ 0. Note that our signs differ from Cook's [1], because Cook is using triple-plus convention for Minkowski metric, while we are using triple-minus convention. Thus we reasonably get in case of Minkowski metric $\mathrm{d}\tau \in \mathbb{R}$.

Inserting (8) into (6), we get the proper length differential

$$\boxed{\mathrm{d}\ell = \sqrt{-\Big(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\Big)\,\mathrm{d}x^i\,\mathrm{d}x^j}}\ . \tag{9a}$$

(8) is merely the runtime of the light signal. The general expression for the proper time differential is

$$\boxed{\begin{aligned}\mathrm{d}\tau &\overset{(3b)}{=} \sqrt{g_{00}\,\mathrm{d}t^2 + \frac{2g_{0i}}{c}\,\mathrm{d}t\,\mathrm{d}x^i + \frac{g_{ij}}{c^2}\,\mathrm{d}x^i\,\mathrm{d}x^j} \\ &\overset{(3a)}{=} \sqrt{g_{00}}\,\mathrm{d}t \quad \text{if } \mathrm{d}x^1 = \mathrm{d}x^2 = \mathrm{d}x^3 = 0\ .\end{aligned}} \tag{9b}$$

These formulas for proper length and proper time are valid in arbitrary coordinate systems with arbitrarily curved space-time, provided that the map from cartesian coordinates in flat Minkowski space-time to the new coordinates in possibly curved space-time is a diffeomorphism.

If two events A and B happen on the worldline of a clock, then the proper time interval $\tau(B) - \tau(A)$, measured by this clock, is given by the line integral

$$\tau(B) - \tau(A) = \int\limits_{\text{path}\,AB} \mathrm{d}\tau \tag{10}$$

along the clocks worldline, with $\mathrm{d}\tau$ according to (9b). Note that the proper time interval does depend on the worldline of the clock. Both events A and B may also be on the worldline of another clock, but in-between the two events the worldlines of the two clocks may differ. Then the proper time interval $\tau(B) - \tau(A)$ may be different for these two clocks. (Remember the well-known "twin paradox".)

The proper length interval between two space points F and M is given by the line integral

$$\ell_{FM} = \frac{1}{2}\Big(\int\limits_{\text{path}\,FM} \mathrm{d}\ell + \int\limits_{\text{path}\,MF} \mathrm{d}\ell \Big) = \int\limits_{\text{path}\,FM} \mathrm{d}\ell \tag{11}$$

along the worldline of a light signal sent from F to G and mirrored back to F, with $\mathrm{d}\ell$ according to (9a). Note that the definition (5) assumes that the local metric $g_{\mu\nu}(x)$ at any point on the path of the light signal must still be the same when the mirrored signal is moving from M to F as it was when the signal moved from F to M. Therefore the two line integrals in (11) could be reduced to one. For the same reason, the notion "proper length" is restricted to space intervals in which the metric does not vary appreciable while the light signal is moving forth and back. No such limitation exists for the notion "proper time".

Note furthermore that the notion of proper length is ambiguous, if there exist different possible paths for the light signal, e. g. if there are "gravitational lenses" between F and M. In that case the notion “proper length interval between F and M” is only defined, if the world line of the light signal between F and M is specified, like the notion “proper time interval between A and B” is only defined, if the world line of the clock moving from A to B is specified.

3. A rotating Reference Frame

As an application of (9), we define in flat Minkowski space an inertial reference frame with cylinder coordinates $(ct_I, \rho_I, \theta_I, z_I)$. Furthermore we define a rotating (hence not inertial) reference frame with cylinder coordinates $(ct_R, \rho_R, \theta_R, z_R)$ such that

$$\begin{aligned} t \equiv t_I &= t_R \\ \rho \equiv \rho_I &= \rho_R \\ \theta_I &= \theta_R + \omega t \\ z \equiv z_I &= z_R \ . \end{aligned} \tag{12}$$

Thus the only difference between the two systems is, that the $\theta_R = 0$ axis is rotating versus the $\theta_I = 0$ axis with angular velocity ω in the $z = 0$ plane. The invariant line element becomes in these two reference systems

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\rho^2 - \rho^2\,\mathrm{d}\theta_I^2 - \mathrm{d}z^2 \tag{13a}$$

$$\begin{aligned} &= c^2\,\mathrm{d}t^2 - \mathrm{d}\rho^2 - (\rho^2\,\mathrm{d}\theta_R^2 + 2\rho\omega\,\mathrm{d}t\,\rho\,\mathrm{d}\theta_R + \rho^2\omega^2\,\mathrm{d}t^2) - \mathrm{d}z^2 \\ &= \Big(1 - \frac{\rho^2\omega^2}{c^2}\Big)\,c^2\,\mathrm{d}t^2 - \frac{2\rho\omega}{c}\,c\,\mathrm{d}t\,\rho\,\mathrm{d}\theta_R - \\ &\quad - \mathrm{d}\rho^2 - \rho^2\,\mathrm{d}\theta_R^2 - \mathrm{d}z^2 \ . \end{aligned} \tag{13b}$$

The non-zero elements of the metric tensors are

$$g_{I00} = 1 \ , \ g_{I11} = -1 \ , \ g_{I22} = -1 \ , \ g_{I33} = -1 \tag{14a}$$

$$\begin{aligned} &g_{R00} = 1 - \rho^2\omega^2/c^2 \ , \ g_{R11} = -1 \ , \ g_{R22} = -1 \ , \\ &g_{R33} = -1 \ , \ g_{R02} = g_{R20} = -\rho\omega/c \ . \end{aligned} \tag{14b}$$

Note that we took care to get all components of the metric tensor dimension-less, i. e. we identified $\rho\,\mathrm{d}\theta$, but not $\mathrm{d}\theta$, as a component of $\mathrm{d}s$ with the correct dimension [length].

The metric $(g_{I\mu\nu})$ of the inertial coordinate system is "time-orthogonal", because all g_{I0i} are zero. The metric $(g_{R\mu\nu})$ of the non-inertial rotating system is called "not time-orthogonal" or "asynchronous", because there are some $g_{R0i} \neq 0$. In a time-orthogonal system, a global time can be defined, and for any two events it can be uniquely stated whether they happen simultaneously, or not. Different time-orthogonal systems, however, will in general answer the question of simultaneity of the identical events differently. In an asynchronous system, the question of simultaneity of two events, which happen at different points in space, can not be answered uniquely (not even within this single coordinate system!), because in such systems no global time can be uniquely defined.

Inserting (14) into (9a) and (9b), we get

$$\mathrm{d}\ell_I \overset{(9a)}{=} \Big[\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta_I^2 + \mathrm{d}z^2\Big]^{1/2} \tag{15a}$$

$$\mathrm{d}\ell_R \overset{(9a)}{=} \Big[\mathrm{d}\rho^2 + \Big(1-\frac{\rho^2\omega^2}{c^2}\Big)^{-1}\rho^2\,\mathrm{d}\theta_R^2 + \mathrm{d}z^2\Big]^{1/2} \tag{15b}$$

$$\mathrm{d}\tau_I \overset{(9b)}{=} \Big[\mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}\rho^2 - \frac{\rho^2}{c^2}\,\mathrm{d}\theta^2 - \frac{1}{c^2}\,\mathrm{d}z^2\Big]^{1/2} \tag{15c}$$

$$\begin{aligned}\mathrm{d}\tau_R \overset{(9b)}{=} \Big[\Big(1-\frac{\rho^2\omega^2}{c^2}\Big)\,\mathrm{d}t^2 - \frac{2\rho^2\omega}{c^2}\,\mathrm{d}t\,\mathrm{d}\theta_R^2 - \\ -\frac{1}{c^2}\,\mathrm{d}\rho^2 - \frac{\rho^2}{c^2}\,\mathrm{d}\theta_R^2 - \frac{1}{c^2}\,\mathrm{d}z^2\Big]^{1/2}\,.\end{aligned} \tag{15d}$$

If only lengths with $\rho =$ constant and $z =$ constant are measured, and if the clocks are at rest in the respective coordinate systems, this simplifies to

$$\mathrm{d}\ell_I \overset{(15a)}{=} \rho\,\mathrm{d}\theta_I \qquad \text{if } \mathrm{d}\rho = \mathrm{d}z = 0 \tag{15e}$$

$$\mathrm{d}\ell_R \overset{(15b)}{=} \Big(1-\frac{\rho^2\omega^2}{c^2}\Big)^{-1/2}\rho\,\mathrm{d}\theta_R \qquad \text{if } \mathrm{d}\rho=\mathrm{d}z=0 \tag{15f}$$

$$\mathrm{d}\tau_I \overset{(15c)}{=} \mathrm{d}t \qquad \text{if } \mathrm{d}\rho=\mathrm{d}\theta_I=\mathrm{d}z=0 \tag{15g}$$

$$\mathrm{d}\tau_R \overset{(15d)}{=} \Big[1-\rho^2\omega^2/c^2\Big]^{1/2}\mathrm{d}t \quad \text{if } \mathrm{d}\rho=\mathrm{d}\theta_R=\mathrm{d}z=0\ . \tag{15h}$$

In (15f) and (15h) the well-known phenomena of contraction of a moving ruler and dilation of a moving clock are visible. As $\rho\omega = v$ is just the (position-dependent!) relative velocity v of the two coordinate systems, time dilation and length contraction are determined by the Lorentz factor

$$\gamma \equiv \Big(1-\frac{v^2}{c^2}\Big)^{-1/2}\ . \tag{16}$$

A clock, which is at rest in the rotating coordinate system, is accelerated by

$$a = v^2/\rho = \rho\omega^2 \tag{17}$$

in the inertial reference system, and its velocity in the inertial system is $v = \rho\omega$. Note the remarkable fact, that time dilation (15h) and length contraction (15f) would have exactly the same values, if the second coordinate system would not be rotating, but would be a second inertial system, which is moving linearly with velocity v versus the first inertial system. This result is caused by our assumption that the time is measured by "standard clocks" as specified below in section 6 .

4. Equivalence Principle

Within a sufficiently small laboratory, its impossible to find out by whatever type of measurement or physical experiment inside the laboratory (without looking out of the window), whether a gravitational field observed inside the laboratory is caused by some mass concentration outside the laboratory, or by mechanical acceleration of the laboratory in a region of space with no significant gravitation.

> Einstein's Equivalence Principle (EP) [2] :
> All laws of nature are identical in an inertial reference system in a homogeneous gravitational field with gravitative acceleration $\boldsymbol{g}$, and in a reference system which is mechanically accelerated by $\boldsymbol{a} = -\boldsymbol{g}$ in a region of space which is free of measurable gravitation. (18)

Note: The EP says, that the metric (14b) of the rotating reference system, which is caused by the acceleration $\rho\omega^2 = v^2/\rho$ in a space free of significant gravitation (mid sketch in fig. 1), could *as well* be explained as a gravitational effect in an inertial reference system

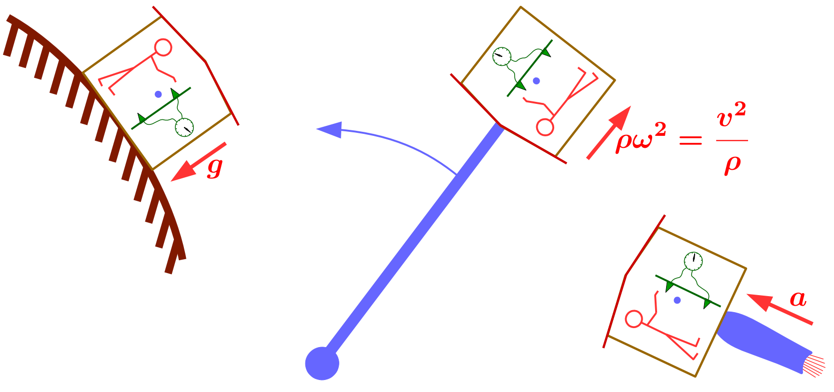

Fig. 1 : Einsteins Equivalence Principle (EP): If the accelerations $g = \rho\omega^2 = a$ are equal, then the tree physicists will find exactly the same results for the free fall, and for whatever other physical experiments they perform in their small laboratories.

(left sketch in fig. 1). But it does *not* say that there is an *additional* modification of this metric due to an equivalent gravitational effect.

We now want to check whether the equation of motion of GRT

$$\frac{\mathrm{d}^2 x^\kappa}{\mathrm{d}\tau^2} = -\Gamma^\kappa_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{19}$$
$$\Gamma^\kappa_{\mu\nu} \equiv \frac{g^{\kappa\sigma}}{2}\Big(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\Big)$$

of a small massive object, i. e. an object which does not significantly modify the space-time metric at its position x, can indeed be reduced to Newtons equation of motion

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\frac{\partial\Phi}{\partial x^i} \tag{20}$$

for the same small object in an inertial reference system, if the curved metric $g_{\mu\nu}(\boldsymbol{x})$ of (19) is replaced by the equivalent gravitational field with potential $\Phi(\boldsymbol{x})$ in a flat space with Minkowski metric $\eta_{\mu\nu}$. Thereby we follow by and large the presentation by Fließbach [3, Kap. 11].

For simplicity we assume that the field is static:

$$\frac{\partial\Phi}{\partial t} = 0 \quad\Longleftrightarrow\quad \frac{\partial g_{\nu\sigma}}{\partial x^0} = 0 \tag{21}$$

With this assumption, the general relativistic equation of motion (19) simplifies to

$$\begin{aligned}\frac{\mathrm{d}^2 x^\kappa}{\mathrm{d}\tau^2} \overset{(19)}{=} &+\frac{g^{\kappa j}}{2}\frac{\partial g_{00}}{\partial x^j}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} - \\ &- \Big(g^{\kappa 0}\frac{\partial g_{00}}{\partial x^i} + g^{\kappa j}\frac{\partial g_{0j}}{\partial x^i} - g^{\kappa j}\frac{\partial g_{i0}}{\partial x^j}\Big)\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} - \\ &- \frac{g^{\kappa\sigma}}{2}\Big(\frac{\partial g_{j\sigma}}{\partial x^i} + \frac{\partial g_{i\sigma}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^\sigma}\Big)\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\ .\end{aligned} \tag{22}$$

Newtons theory is a good approximation for weak gravitational fields. Hence we assume that in case of weak fields the metric ($g_{\mu\nu}$) does not deviate very much from Minkowski metric ($\eta_{\mu\nu}$):

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1 \tag{23}$$

Furthermore we use reference frames, in which the velocity of the test object is much smaller than the speed of light:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}x^0}{\mathrm{d}\tau} \approx c \quad \text{if } v \ll c \tag{24}$$

Therefore we may drop all small terms

$$\mathcal{O}\Big(\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\Big) \quad , \quad \mathcal{O}\Big(\frac{\partial h_{\mu\nu}}{\partial x^k}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\Big) \quad , \quad \mathcal{O}\Big(h_{\rho\sigma}\frac{\partial h_{\mu\nu}}{\partial x^k}\Big) \ , \tag{25}$$

and get in this approximation the equation of motion

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}\tau^2} \overset{(22)}{\approx} 0 \tag{26a}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} \overset{(22)}{\approx} -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i} \ . \tag{26b}$$

From this equation we conclude

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} \approx \text{constant} = 1 \tag{27a}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i} \overset{(20)}{=} -\frac{\partial \Phi}{\partial x^i} \ . \tag{27b}$$

Thus we get Newtons result with

$$g_{00}(\boldsymbol{x}) \overset{(23)}{=} \eta_{00} + h_{00}(\boldsymbol{x}) \overset{(27)}{=} 1 + \frac{2\Phi(\boldsymbol{x})}{c^2} \quad \text{if } \frac{|2\Phi|}{c^2} \ll 1 \ . \tag{28}$$

For the rotating frame with no gravitational field we had

$$g_{00}(\boldsymbol{x}) \overset{(14b)}{=} 1 - \rho^2\omega^2/c^2 \ . \tag{29}$$

Hence both systems indeed are equivalent, as stated by the EP, if

$$-\Phi(\boldsymbol{x}) \equiv -\Phi(\rho) = +\frac{\rho^2\omega^2}{2} = +\frac{v^2}{2} \ll c^2 \ . \tag{30}$$

The time dilation of a clock at rest in the rotating system with no gravitational field is

$$\mathrm{d}\tau_R \overset{(9b)}{=} \sqrt{g_{00}}\ \mathrm{d}t \overset{(29)}{=} \sqrt{1 - \frac{\rho^2\omega^2}{c^2}}\ \mathrm{d}t = \sqrt{1 - \frac{v^2}{c^2}}\ \mathrm{d}t \ . \tag{31}$$

Consequently the proper time differential, measured by a clock which is in the equivalent gravitational field at rest in an inertial reference system, is

$$\mathrm{d}\tau \overset{(9b)}{=} \sqrt{g_{00}}\ \mathrm{d}t \overset{(28)}{=} \sqrt{1 + \frac{2\Phi}{c^2}}\ \mathrm{d}t \tag{32}$$
$$\text{if } \mathrm{d}x^1 = \mathrm{d}x^2 = \mathrm{d}x^3 = 0 \text{ and } \frac{|2\Phi|}{c^2} \ll 1 \ .$$

As $\Phi \leq 0$, clocks slow down in gravitational fields.

These are the values of $|2\Phi|/c^2$ at the surfaces of some typical celestial bodies:

$$\frac{|2\Phi|}{c^2} \approx \begin{cases} 1.4 \cdot 10^{-9} & \text{earth} \\ 4 \cdot 10^{-6} & \text{sun} \\ 3 \cdot 10^{-4} & \text{white dwarf} \\ 3 \cdot 10^{-1} & \text{neutron star} \end{cases} \tag{33}$$

Hence (28) is an excellent approximation in our sun system, a quite good approximation at the surface of white dwarfs, and it is still a useful rough approximation at the surface of neutron stars. It is of course not at all capable to describe black holes appropriately.

5. Experimental Results

The decay rate of muons can be considered a clock. Indeed the observation of the significantly extended lifetime of muons at high linear speed versus muons at rest [4] has been interpreted as an experimental confirmation of the special-relativistic effect of time dilation.

The decay rate of muons has also been used to check experimentally the time dilation of clocks at rest in accelerated reference frames: In the CERN [5] and Brookhaven [6] muon storage rings, muons with

$$\text{energy} = 3.1\,\text{GeV} \tag{34a}$$

were stored in 14 m diameter rings. Thus the Lorentz factor of these muons in the laboratory coordinate system, which may be considered an approximate inertial system, was

$$\gamma = \frac{3.1\,\text{GeV}}{105.7\,\text{MeV}} = 29.3 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} , \tag{34b}$$

their velocity was

$$v = c\sqrt{\frac{29.3^2 - 1}{29.3^2}} = 0.9994\,c , \tag{34c}$$

and the Lorentz force due to the ring's magnetic field accelerated them radially by

$$a = \frac{v^2}{7\,\text{m}} \approx 1.3 \cdot 10^{16}\,\frac{\text{m}}{\text{s}^2} . \tag{34d}$$

The time dilation as predicted by GRT, extending the muon lifetime from $2.198\,\mu\text{s}$ to $64.44\,\mu\text{s} = \gamma \cdot 2.198\,\mu\text{s}$, was confirmed with an accuracy of 0.1 %:

$$\underbrace{\mathrm{d}\tau}_{2.198\,\mu\mathrm{s}} \overset{(15h)}{=} \underbrace{\sqrt{1-\frac{\rho^2\omega^2}{c^2}}}_{1/29.3} \underbrace{\mathrm{d}t}_{64.44\,\mu\mathrm{s}} = \sqrt{1-\frac{v^2}{c^2}}\,\mathrm{d}t \tag{35}$$

According to GRT, clocks are slowed down due to gravitational fields, see [2, §3] and our result (32). This result has been confirmed experimentally since the seventies [7], and by today it is confirmed every day by the Global Positioning System [8].

The equivalence principle (18) says, that the time dilation (35) can locally as well be explained as the effect (32) of a gravitational field, i. e. this is a possible alternative explanation for the observed time dilation. The equivalence principle does *not* say, however, that there should be any time dilation *in addition* to (35).

6. Ideal versus Real Clocks: The "Clock Hypothesis"

The time dilation of clocks at rest in a rotating system, as predicted by GRT, has been confirmed by the muon experiment, see (35). But there is a problem: How can we be sure that the acceleration does not impair the functionality of the accelerated clock? To understand the concern, lets replace the muons by a pendulum clock, consisting of a mass M suspended by a thin wire of length L and mass m_w. The other end of the wire is fixed at rest in the rotating system. The oscillation period of this clock, as seen by a co-moving observer who is at rest in the rotating system, is

$$T = 2\pi\sqrt{\frac{L}{a}} \quad \text{if } m_w \ll M \text{ , with } a = \rho\omega^2 \text{ .} \tag{36}$$

The co-moving observer, enclosed in a laboratory of size $\ll \rho$, can interpret the acceleration a as a gravitational acceleration g.

Another observer, at rest in the inertial system, should observe the time dilation (15h) of the rotating clock.

If we now try to check (15h) systematically due to variation of ω, we face a problem: With increased rotation frequency, the length L of the wire will increase as well (as the modulus of elasticity of any real material is finite), thus changing the proper frequency of the clock. And further increased rotation frequency will eventually break the wire and thus destroy the clock. Hence (15h) can only be checked approximately at low rotation speed, if a pendulum clock is used for that purpose.

There are other types of clocks available, of course, which are better suited to check (15h). But all of them are somehow influenced by accelerations, hence no ideal clocks. Even the decay rate of muons, discussed in section 5, is not an ideal clock. As pointed out by Lorek et. al. [9], field-theoretical effects like the creation of particles out of the vacuum in accelerated reference frames (this is the Unruh-effect [10]), or pair production of particles and antiparticles at sufficiently high energies, will impact the rate at which time is passing according to any type of real clock.

All our considerations in the previous sections rest on the tacitly implied condition, that the function of standard clocks is not affected by their acceleration, i. e. that they are ideal clocks with regard to accelerations. Have these considerations at all been meaningful, if such clocks don't exist in reality?

The "clock hypothesis", often encountered in the literature, says that the standard clocks, which display the proper time at their positions, are not affected by their accelerations, but behave as ideal clocks. Brown and Read [12] suggested to replace the misleading notion "clock hypothesis" by "clock condition". This wording indeed is much better, emphasizing that we must not expect that any arbitrary type of clock is an appropriate standard clock in any arbitrary reference frame, but that for each reference frame an appropriate type of standard clock must be carefully chosen, to make sure that this standard clock will not be significantly affected by the acceleration of the reference system, in which it is at rest. This interpretation presupposed, the clock hypothesis says that appropriate real standard clocks can be found for any accelerated reference frame.

Considering quantum field theoretical effects, the decay rate of accelerated muons has been computed by Eisele [11]. He confirmed that indeed the decay rate of muons is no ideal clock. But he also found that this type of real clock comes remarkably close to an ideal clock: In the muon storage ring experiments [5, 6] mentioned in section 5, the relation (15h) was confirmed with an accuracy of 10^{-3}. According to Eiseles findings, this experiment would need an accuracy better than 10^{-25}, to see the deviation from (15h).

Hence by today the precision, with which the GRT prediction (15h) of time dilation in rotating systems can be tested, is not limited by lack of an ideal clock, but by the insufficient precision of other parts of the experiments. It seems unlikely that this situation will change in foreseeable future.

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