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Foundations of Relativistic Quantum Field Theory

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Contents

Part 2: Field Quantization

Part 4: Non-Abelian Gauge-Theories

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* At times improved versions are published and posted in the net. A paper print might soon be out-dated.
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* The book is not (yet) equipped with a keyword index. The reader's search function can compensate this deficit to some extend.
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✓ Book-View

or similar. Never select a view, which forces you to scroll horizontally, or which places the upper or lower page borders anywhere else than at the upper or lower window borders!

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Preface

Physics really is much to difficult for physicists!
David Hilbert

This book was written for physics students at the end of their master studies, or at the beginning of their graduate studies. Clearly it also is suited for other readers with similar precognitions, who are looking for an introduction to relativistic quantum field theory. I was motivated to write this book by the quite hard and sometimes really frustrating times I had, when I first time became acquainted with this matter, and often had to struggle heavily to keep my head above water.

When Albert Einstein developed his General Relativity Theory, and had to delve into the secrets of non-euclidean geometry, which at that time was completely unknown to physicists, David Hilbert – in early 20. century Ordinarius for Mathematics at the Göttingen university – became one of his most important and helpful partners. Possibly it were these discussions with Einstein, which led Hilbert to the bonmot printed above.

Today, General Relativity Theory has lost its horror, because a wide spectrum of textbooks on GRT is available to the student on any thinkable level in between mathematical abstraction and pedagogical detailedness.

Regarding relativistic quantum field theory, the situation unfortunately is very different. There are some elementary treatments on "bachelor level" (with this characterisation I really don't want to tread on someone's toes), which can not at all satisfy a reader, who is striving towards a solid comprehension of the basic facts and physical significance of the theory. The book of Harris [1] could be cited as an example of that type of presentations. It's probably the purpose of such books, not to release the reader from university before she/he at least once has marveled at some quantum field theory formulas with their impressive and bewildering multiplicity of indices.

But such books don't lay the foundations for the study of the more ambitious and profound literature.

On the other hand, any well-assorted university library has minimum eight shelf-meters of textbooks available, which treat in physical (and, less often, in mathematical) rigor the faintest details of QFT. Of course there is a price to pay for this: If a textbook wants to treat quantum chromodynamics on page 400 by the latest, wants to discuss all important methods of renormalization before, which clearly must be motivated by an appropriate number of examples of second order perturbation computations, then not much time and space is left for the discussion of the theory's foundations.

If one is looking for textbooks in english or german language, which treat the essential basic features and physical implications of relativistic quantum field theory thoroughly and comprehensively, and at the same time presuppose merely the mathematical skills of a student, who can pass with good grade the master examination in theoretical physics at a german university, then the choice becomes – to put it diplomatically cautious – displeasing scarce.

Even the physically extensive and good textbooks on relativistic quantum field theory, which usually are recommended to newcomers, like e. g. the older one by Bjorken and Drell [2], or the somewhat newer by Peskin and Schroeder [3], put the endurance of the readers to a very tough test. These books take for granted a lot of mathematical und group-theoretical skills, which students on master level in general don't have. Seemingly simpler books of this kind, as e. g. those by Maggiore [4] or by Mandl and Shaw [5], turn out on closer scrutiny as merely more superficial, but not simpler (just the contrary!) than those mentioned before, because they are sweeping many most difficult issues hand-waving under the rug. The books by Greiner and Reinhard [6–8] deserve special mention. These authors demonstrate almost all computations in remarkable detail. However these books have the character of lecture scripts (actually they are declared as such). They can not really be recommended for self-study without the explanations, which the authors certainly add verbally in their lectures. Unfortunately I discovered the book of Gross [9], which in many respects comes nearer to the book which I have missed than most others, only after I had already

almost completed my book.

The physicist, who first time became acquainted with relativistic quantum field theory, in this situation had no other choice than to cumbersomely search and collect the necessary informations and tools from a large variety of physical and mathematical textbooks. It was my intention to write a book which fills this gap in the textbook literature. All necessary tools, which are exceeding the standard skills of a physics student on master level, are provided within the book itself.

The books, which turned out as the (relatively) most useful sources, are listed in the bibliography [2–30].

Clearly the extensive treatment of the foundations necessitates reductions in other places. Applications are rarely mentioned, and non-abelian gauge theories are treated only rather scantily in part IV. This book was not written to replace the rightly recommended textbooks on the subject, but shall facilitate the access to them. The present author is convinced, that the student will make faster progress by first working through this book, and subsequently continuing with the textbooks mentioned above, than by jumping in at the deep end immediately, and trying somehow to survive.

In the end, only the readers can decide whether this book is meeting their needs, and opens a path to relativistic quantum field theory. The author is looking forward to feedback from the readers, to applause, and to criticism. And he will be thankful for hints on typos and other errors.

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Neunhof, in spring 2012 Gerold Gründler

Conventions

K.1 Matrices

M^j, N^{jk} etc. are elements of the matrices M, N etc. The complete matrices are indicated by the matrix name without indices, or by the matrix name with indices and brackets:

$$M = (M^j) = \begin{pmatrix} M^1 \\ M^2 \\ M^3 \end{pmatrix} \; ; \; N = (N^{jk}) = \begin{pmatrix} N^{11} & N^{12} & N^{13} \\ N^{21} & N^{22} & N^{23} \\ N^{31} & N^{32} & N^{33} \end{pmatrix} \tag{K.1}$$

With the sign * for the complex conjugate, and with the sign $^\sim$ for the transposed matrix,

$$N^\dagger = N^{*\sim} = \begin{pmatrix} N^{11\,*} & N^{21\,*} & N^{31\,*} \\ N^{12\,*} & N^{22\,*} & N^{32\,*} \\ N^{13\,*} & N^{23\,*} & N^{33\,*} \end{pmatrix} \tag{K.2}$$

is the matrix which is adjoint to N.
N is called self-adjoint oder hermitean, if $N^\dagger = N$.
N is called orthogonal, if $N^\sim = N^{-1}$.
N is called unitary, if $N^\dagger = N^{-1}$.

K.2 Brackets and Differential Operators

The differential operator $\mathrm{d}_\mu = \frac{\mathrm{d}}{\mathrm{d}x^\mu}$ affects all factors to its right, until it is stopped by a closing) bracket, whose opening (counterpart is placed left of the operator:

$$\begin{aligned}(A\mathrm{d}_\mu(B+C)DE)F = &(A(\mathrm{d}_\mu B+\mathrm{d}_\mu C)DE)F\\ &+(A(B+C)(\mathrm{d}_\mu D)E)F\\ &+(A(B+C)D\mathrm{d}_\mu E)F\end{aligned} \tag{K.3}$$

Important exception: A differential operator is *not* stopped by a closing bracket, which is following the operator *immediately*. An example is Euler's equation

$$\rho\Big(\mathrm{d}_t\boldsymbol{v}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\Big)=-\nabla P\ ,$$

in which the operator $\boldsymbol{v}\cdot\boldsymbol{\nabla}$ is acting onto the vector $\boldsymbol{v}$.

K.3 Kronecker Symbol

$$\delta_{ab}\equiv\begin{cases}1 & \text{if } a=b\\ 0 & \text{if } a\neq b\end{cases} \tag{K.4}$$

We will write *all* indices of the Kronecker symbol subscript (not superscript), i.e. we don't distinguish covariant from contra-variant index position. If the indices happen to be the four space-time-indices, then the Kronecker symbol often is replaced by the elements of the four-dimensional unit tensor:

$$g^\mu{}_\nu=g_\mu{}^\nu=\begin{cases}1 & \text{if } \mu=\nu\\ 0 & \text{if } \mu\neq\nu\end{cases} \tag{K.5}$$

With the metric tensor, we carefully distinguish sub- and superscript indices. Only with diagonal position of the indices,

$$(g^\mu{}_\nu)=(g_\mu{}^\nu)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{K.6}$$

is the four-dimensional unit tensor. Find more informations on the metric tensor in chapter 2.

K.4 Einstein's sum-convention

If a space-time-index or a spinor-index is showing up two times in a product, it shall always be summed up, even if no $\sum$ sign is explicitly written:

$$G_a F^a \equiv \sum_a G_a F^a \tag{K.7}$$

Usually for spinor indices, we use the letters $a, b, c, \ldots$ For space-time-indices we use greek letters $\lambda, \mu, \nu, \ldots$, if the sum of all four space-time-coordinates is to be taken, and latin indices $j, k, l, \ldots$, if only the sum of the three space-coordinates is to be taken:

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu = A_0 B^0 + \sum_{k=1}^{3} A_k B^k = A_0 B^0 + A_k B^k \tag{K.8}$$

Other indices than space-time-indices and spinor-indices are *not* to be summed automatically.

K.5 Unit Vectors

We define four unit vectors in direction of the four coordinate axes of space and time:

$$e^{(0)} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad e^{(1)} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad e^{(2)} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad e^{(3)} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{K.9}$$

These contra-variant unit vectors fulfill with arbitrary contra-variant Lorentz vectors V the relations

$$e^{(\sigma)} \cdot V = e^{(\sigma)\rho} g_{\rho\tau} V^\tau = g_{\sigma\tau} V^\tau \tag{K.10a}$$

$$V^\lambda = g^\lambda{}_\tau V^\tau = g^{\lambda\sigma} g_{\sigma\tau} V^\tau = \sum_{\sigma=0}^{3} g^{\lambda\sigma} e^{(\sigma)} \cdot V \tag{K.10b}$$

$$V_\rho = g_{\rho\lambda} V^\lambda = \sum_{\sigma=0}^{3} g_{\rho\lambda} g^{\lambda\sigma} e^{(\sigma)} \cdot V = e^{(\rho)} \cdot V \tag{K.10c}$$

$$V = \sum_{\lambda=0}^{3} V^\lambda e^{(\lambda)} = \sum_{\lambda=0}^{3}\sum_{\sigma=0}^{3} g^{\lambda\sigma}(e^{(\sigma)}\cdot V)e^{(\lambda)} \tag{K.10d}$$

$$e^{(\sigma)}\cdot e^{(\tau)} = e^{(\sigma)\rho} g_{\rho\lambda} e^{(\tau)\lambda} = g_{\sigma\tau} \tag{K.10e}$$

$$\text{with } \lambda,\rho,\sigma,\tau = 0,1,2,3\ .$$

Sometimes we will use the space-like components

$$e^{(1)} \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}^{(1)} \end{pmatrix} \quad , \quad e^{(2)} \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}^{(2)} \end{pmatrix} \quad , \quad e^{(3)} \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}^{(3)} \end{pmatrix} \tag{K.11}$$

of these unit vector as 3-vectors. As usual, 4-vectors are printed thin, and 3-vectors are printed boldfaced:

$$\boldsymbol{e}^{(1)} \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{e}^{(2)} \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{e}^{(3)} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{K.12}$$

For 3-vectors, usually (but not always), Euclidean metric is used. Arbitrary vectors $\boldsymbol{V}$ of three-dimensional space fulfill the relations

$$V^j = \boldsymbol{e}^{(j)} \cdot \boldsymbol{V} \tag{K.13a}$$

$$\boldsymbol{V} = \sum_{j=1}^{3} V^j \boldsymbol{e}^{(j)} = \sum_{j=1}^{3} (\boldsymbol{e}^{(j)}\cdot\boldsymbol{V})\boldsymbol{e}^{(j)} \tag{K.13b}$$

$$\boldsymbol{e}^{(j)} \cdot \boldsymbol{e}^{(l)} = g_j{}^l \tag{K.13c}$$

$$\text{for } j,l = 1,2,3\ .$$

The letters in brackets (σ), (j) etc. do not indicate the unit vectors components, but are their "names". They are put into brackets to make clearly visible, that these are not component indices. Therefore no summation shall be performed, if a name is showing up two times in a product.

K.6 The Partial Derivative

For the derivative, we use the convention established in physics, which is different from the convention established in the mathematical literature. With the physicists convention, functions like $L\big(t, q(a(t)), \dot{q}(t)\big)$ are admissible, in

which identical variables (in this example t) are showing up explicitly *and* implicitly (in this example as implicit variable of L and q, and as explicit variable of L and $\dot{q}$ and a).

The chain rule has to be applied in the total derivative (indicated by the letter d). If the chain rule shall *not* be applied, but only the derivative with respect to the explicit variable shall be taken, this is indicated by the sign ∂:

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t, q(a(t)), \dot{q}(t)) = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\frac{\partial q}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\partial L}{\partial \dot{q}}\frac{\mathrm{d}\dot{q}}{\mathrm{d}t} \tag{K.14}$$

$$\frac{\partial L}{\partial a} = 0 \tag{K.15}$$

The derivative $\frac{\partial L}{\partial q}$ to the explicit variable is called "partial derivative". This notion is conventional but misleading, because the total derivative $\frac{\mathrm{d}L}{\mathrm{d}q}$ is as "partial" as the derivative $\frac{\partial L}{\partial q}$ to the explicit variable. An extensive discussion can be found in [31].

Part 1:
Classical Field Theory

1 Why Classical Field Theory?

The relationship of quantum theory to classical physics is fundamentally different for example from the relationship of special relativity theory to non-relativistic theory. Two aspects deserve special mention in this context:

* The mathematical description of a process by special relativity theory changes seamlessly over to the description by non-relativistic theory if $v^2/c^2 \ll 1$.
* The description of the experimental set-up, the description of the observed events, the description of the applied measurement devices, and the description of the measurements results, all happen in a homogeneous language. Namely in the language and with the notions of (relativistic or non-relativistic) classical physics.

Both is completely different in the case of quantum phenomena. Quantum field theory can only be called a physical theory, if conclusions can be drawn from it, which can be checked by experiments. Experimental tests are possible only, if physicists can discuss and agree on the experimental set-up, the observed facts, and the functionality of the instruments of measure[1], which produce the results. The inventors of the "Copenhagen interpretation" of quantum theory were convinced, that this is possible only in the language of classical physics. While not all physicists want to agree to this credo, until today nobody was able to express the result of any measurement in the language of quantum theory.

Quantum phenomena invariably are described by first fixing the boundary conditions in the language of classical physics. Then an evolution takes place, for which the language of quantum theory is appropriate. To achieve results, which can be discussed and compared to the predictions of the theory, the

[1] The instrument could be for example a tool as simple as the eye of a human being.

evolution finally must be terminated by an interaction with measurement devices, whose modes of function and measured values imperatively must be described again in the language of classical physics.

Thus we can on no account dispense with classical physics. Relativistic quantum field theory is not a replacement for classical physics, but an addendum and extension of classical physics. Following the historical order of events, we will reproduce quantum field theory in this book in close analogy to classical field theory. There are alternative, axiomatic delineations of QFT, but these are even more abstract and mathematically even more difficult than our approach. As classical physics are indispensable anyway, its reasonable to prefer the simpler way and choose classical field theory as starting point for the development of quantum field theory.

In this part of the book, we will derive a rich variety of classical field theory concepts, which will turn out most useful for quantum field theory as well. And we will first describe all fields, which we study in this book, as classical fields, and evaluate their characteristics, before we then will quantize them in the second part of this book.

2 Relativity Theory

2.1 Invariant Length

In General Relativity Theory, all 10 linearly independent elements of the metrical tensor

$$(g_{\mu\nu}) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix} \tag{2.1}$$

can be different from zero. In Special Relativity Theory, the metrical tensor always and everywhere has[1] the simple form

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.2}$$

In this and the following sections, we allow for non-zero off-diagonal elements of the metrical tensor. Only starting from section 2.4, we constrain to the special case (2.2) for the rest of this book.

The four-dimensional position vector's differential is

[1] Actually the form (2.2) is not unique. In literature on General Relativity and cosmology, often $(g_{\mu\nu}) = \text{diag.}(-1, +1, +1, +1)$ is encountered. And in *very* old literature (before about 1960) even $(g_{\mu\nu}) = \text{diag.}(+1, +1, +1, -1)$ can be found, with the time-like component shifted to the 44-position. In this book we decide for convention (2.2), which is commonly adopted in QFT texts.

$$\mathrm{d}s \equiv \begin{pmatrix} c\mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \mathrm{d}x^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix} \quad . \tag{2.3}$$

Cardinal point of Relativity Theory is the assertion, that the square of this differential is independent of the coordinate system:

$$\boxed{\begin{array}{l} \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ \text{has the same value in all coordinate systems.} \end{array}} \tag{2.4}$$

This is a statement with regard to the geometry of four-dimensional space-time, which is specified by the metrical tensor $g_{\mu\nu}$. Generally, the metrical tensor is a function of space and time. In any reference system the geometry of space and time is deformed such, that the identical value of $\mathrm{d}s^2$ is being measured. Since the differentials $\mathrm{d}x^\mu$ and $\mathrm{d}x^\nu$ commute, the metrical tensor is symmetric:

$$\begin{aligned} \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\mathrm{d}x^\nu \mathrm{d}x^\mu = g_{\nu\mu}\mathrm{d}x^\nu \mathrm{d}x^\mu \\ \Longrightarrow \quad g_{\mu\nu} = g_{\nu\mu} \end{aligned} \tag{2.5}$$

Thus only 10 of the metrical tensor's 16 components are independent.

2.2 Transformations

The transformation Λ transforms the differential $\mathrm{d}s$ from one reference system to another, which we mark by a prime$'$. The transformed differential is

$$\mathrm{d}s' = \Lambda \mathrm{d}s \ . \tag{2.6}$$

We write the transformation Λ as a (4×4) - matrix with row index left and column index right. Multiplication of the column vector $\mathrm{d}s$ by Λ results into the column vector $\mathrm{d}s'$.

$$\mathrm{d}x'^{\nu} = \Lambda^{\nu}{}_{\mu}\mathrm{d}x^{\mu} \quad \Longrightarrow \quad \Lambda^{\nu}{}_{\mu} = \frac{\mathrm{d}x'^{\nu}}{\mathrm{d}x^{\mu}} \tag{2.7}$$

holds for the components. The sub- and superscript index positions will be explained immediately. Generally Λ varies with space and time, therefore the transformation is non-linear.

The back-transformation of the differential $\mathrm{d}s'$ from the primed reference system to the unprimed system is carried out by application of Λ^{-1}, the inverse matrix of Λ. The concatenation of transformation and back-transformation clearly must yield the identical transformation, i. e. the unit matrix $\mathbb{1}$:

$$\mathrm{d}s = \Lambda^{-1}\mathrm{d}s' \overset{(2.6)}{=} \underbrace{\Lambda^{-1}\Lambda}_{=\mathbb{1}}\mathrm{d}s \tag{2.8}$$

The individual components are

$$\mathrm{d}x^{\mu} = \Lambda^{-1\,\mu}{}_{\nu}\mathrm{d}x'^{\nu} \quad \Longrightarrow \quad \Lambda^{-1\,\mu}{}_{\nu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}x'^{\nu}} \quad . \tag{2.9}$$

Next we want to find out the transformation of the differential operator d_{μ}. To this end, we apply the chain rule:

$$\mathrm{d}'_{\nu} = \frac{\mathrm{d}}{\mathrm{d}x'^{\nu}} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}x'^{\nu}}\frac{\mathrm{d}}{\mathrm{d}x^{\mu}} \overset{(2.9)}{=} \Lambda^{-1\,\mu}{}_{\nu}\frac{\mathrm{d}}{\mathrm{d}x^{\mu}} = \Lambda^{-1\,\mu}{}_{\nu}\mathrm{d}_{\mu} \tag{2.10}$$

If we want to write this equation as a matrix equation, we obviously must interprete (d'_{ν}) and (d_{μ}) as row vectors, because the summation is running over the row index of $(\Lambda^{-1\,\mu}{}_{\nu})$. The matrix equation therefore is

$$\mathrm{d}' = \mathrm{d}\Lambda^{-1} \quad . \tag{2.11}$$

The back-transformation is

$$\mathrm{d}_{\mu} = \frac{\mathrm{d}}{\mathrm{d}x^{\mu}} = \frac{\mathrm{d}x'^{\nu}}{\mathrm{d}x^{\mu}}\frac{\mathrm{d}}{\mathrm{d}x'^{\nu}} \overset{(2.7)}{=} \Lambda^{\nu}{}_{\mu}\frac{\mathrm{d}}{\mathrm{d}x'^{\nu}} = \Lambda^{\nu}{}_{\mu}\mathrm{d}'_{\nu} \tag{2.12}$$

with the matrix equation

$$\mathrm{d} = \mathrm{d}'\Lambda \quad . \tag{2.13}$$

Summary:

$$\Lambda^{\mu}{}_{\nu} \overset{(2.7)}{=} \frac{\mathrm{d}x'^{\mu}}{\mathrm{d}x^{\nu}} \overset{(2.12)}{=} \frac{\mathrm{d}_{\nu}}{\mathrm{d}'_{\mu}} \tag{2.14a}$$

$$\Lambda^{-1\,\mu}{}_{\nu} \overset{(2.9)}{=} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}x'^{\nu}} \overset{(2.10)}{=} \frac{\mathrm{d}'_{\nu}}{\mathrm{d}_{\mu}} \tag{2.14b}$$

2.3 Vectors and Tensors

We wrote the indices of the differential $\mathrm{d}x^{\nu}$ superscript, but the indices of the differential operator d_{ν} subscript. We turn this into a general rule:

If a four-component quantity A is transformed from one reference system to another by the same transformation matrix as the differential operator, then A is a *co-variant vector*, and is being written with subscript index $A = (A_{\nu})$:

$$\boxed{\mathrm{d}'_{\nu} = \Lambda^{-1\mu}{}_{\nu}\,\mathrm{d}_{\mu}} \qquad \boxed{\mathrm{d}_{\nu} = \Lambda^{\mu}{}_{\nu}\mathrm{d}'_{\mu}} \tag{2.15a}$$

$$A'_{\nu} = \Lambda^{-1\mu}{}_{\nu}\,A_{\mu} \qquad A_{\nu} = \Lambda^{\mu}{}_{\nu}A'_{\mu} \tag{2.15b}$$

If a four-component quantity B is transformed from one reference system to another by the same transformation matrix as the four-dimensional position vector, then B is a *contra-variant vector*, and $B = (B^{\nu})$ is being written with superscript index:

$$\boxed{\mathrm{d}x'^{\nu} = \Lambda^{\nu}{}_{\mu}\mathrm{d}x^{\mu}} \qquad \boxed{\mathrm{d}x^{\nu} = \Lambda^{-1\,\nu}{}_{\mu}\mathrm{d}x'^{\mu}} \tag{2.16a}$$

$$B'^{\nu} = \Lambda^{\nu}{}_{\mu}B^{\mu} \qquad B^{\nu} = \Lambda^{-1\,\nu}{}_{\mu}B'^{\mu} \tag{2.16b}$$

A *tensor of rank* n is a quantity with n indices (for example the tensor $(D_{\kappa}{}^{\mu}{}_{\nu})$ has 3 indices), which is transformed component-by-component like a vector:

$$D'_{\kappa}{}^{\mu}{}_{\nu} = \Lambda^{-1\rho}{}_{\kappa}\Lambda^{\mu}{}_{\sigma}\Lambda^{-1\tau}{}_{\nu}D_{\rho}{}^{\sigma}{}_{\tau} \tag{2.17}$$

Thus a tensor of rank n has 4^n components. Vectors are tensors of rank one. Tensors of rank zero are called *scalars*. Due to *contraction*, a tensor or rank n becomes a tensor of rank $n-2$:

$$F_{\sigma\rho}{}^{\sigma}{}_{\tau} = F_{0\rho}{}^{0}{}_{\tau} + F_{1\rho}{}^{1}{}_{\tau} + F_{2\rho}{}^{2}{}_{\tau} + F_{3\rho}{}^{3}{}_{\tau} = F_{\rho\tau} \tag{2.18}$$

$$E_{\nu}{}^{\nu} = E_{0}{}^{0} + E_{1}{}^{1} + E_{2}{}^{2} + E_{3}{}^{3} = E$$

$$C_{\nu}{}^{\nu} \equiv A_{\nu}B^{\nu} = A_0B^0 + A_1B^1 + A_2B^2 + A_3B^3 = AB = C$$

Note that not every tensor of rank $n+m$ can be factorized into the direct product of a tensor of rank n and a tensor of rank m. In example (2.18) the tensor $(E_{\sigma}{}^{\tau})$ has 16 linearly independent components, while the tensor $(C_{\sigma}{}^{\tau}) = (A_{\sigma}B^{\tau})$ has only 8 .

Any product of a covariant vector and a contravariant vector is a scalar, which is invariant under a transformation of the space-time coordinates:

$$C' = A'_{\nu}B'^{\nu} = \underbrace{\Lambda^{-1\,\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma}}_{\delta_{\mu\sigma}} A_{\mu}B^{\sigma} = A_{\mu}B^{\mu} = C \tag{2.19}$$

Next we investigate the transformation properties of the product of the metric tensor $(g_{\mu\nu})$ and a contravariant vector (B^{ν}). The components

$$g'_{\mu\nu}B'^{\nu} = \Lambda^{-1\sigma}{}_{\mu}\underbrace{\Lambda^{-1\tau}{}_{\nu}\Lambda^{\nu}{}_{\rho}}_{\delta_{\tau\rho}} g_{\sigma\tau}B^{\rho} = \Lambda^{-1\sigma}{}_{\mu}g_{\sigma\tau}B^{\tau} \tag{2.20}$$

are transformed like the components of a covariant vector. For the components of this covariant vector we define the shortcut notation

$$B_{\sigma} \equiv g_{\sigma\tau}B^{\tau} \quad . \tag{2.21}$$

According to (2.5), the metric tensor is symmetric. Therefore also

$$g_{\tau\sigma}B^{\tau} = g_{\sigma\tau}B^{\tau} \overset{(2.21)}{=} B_{\sigma} \ . \tag{2.22}$$

Definition (2.21) can be read as well in inverse direction: Whenever a covariant vector (A_{μ}) with subscript index shows up, we implicitly define by

$$g_{\mu\nu}A^{\nu} \equiv A_{\mu} \tag{2.23}$$

its contravariant complement (A^{ν}). This definition is extended to tensors of arbitrary rank:

$$D_{\kappa\mu\nu} = g_{\kappa\rho}D^{\rho}{}_{\mu\nu} = g_{\kappa\rho}g_{\nu\tau}D^{\rho}{}_{\mu}{}^{\tau} = g_{\kappa\rho}g_{\mu\sigma}g_{\nu\tau}D^{\rho\sigma\tau} \tag{2.24}$$

The definition holds as well for the metric tensor:

$$g_{\mu\nu} = g_{\mu\sigma}g^{\sigma}{}_{\nu} = \tag{2.25a}$$

$$= g_{\nu\sigma}g_{\mu}{}^{\sigma} \overset{(2.5)}{=} g_{\sigma\nu}g_{\mu}{}^{\sigma} = \tag{2.25b}$$

$$= g_{\mu\sigma}g_{\nu\tau}g^{\sigma\tau} \overset{(2.5)}{=} g_{\mu\sigma}g_{\nu\tau}g^{\tau\sigma} \tag{2.25c}$$

From these equations, the relations

$$g^{\sigma}{}_{\nu} \overset{(2.25a)}{=} \delta_{\sigma\nu} \quad , \quad g_{\mu}{}^{\sigma} \overset{(2.25b)}{=} \delta_{\mu\sigma} \quad , \quad g_{\nu\tau}g^{\tau\sigma} \overset{(2.25c)}{=} \delta_{\nu\sigma} \tag{2.26}$$

follow. Therefore for the metric tensor

$$(g^{\nu}{}_{\sigma}) = (g_{\nu}{}^{\sigma}) = (g_{\nu\tau}g^{\tau\sigma}) = \mathbb{1} \quad \text{and} \quad (g^{\nu\sigma}) = (g_{\nu\sigma})^{-1} \ . \tag{2.27}$$

Remarkably these relations hold for arbitrarily curved space-time. *Only* for the metric (2.2) of inertial systems in addition the relation $(g^{\nu\sigma}) = (g_{\nu\sigma})^{-1} = (g_{\nu\sigma})$ is valid.

By multiplication with $g_{\mu\nu}$, superscript indices of vectors and tensors are "pulled down". By multiplication with $g^{\mu\nu}$, subscript indices of vectors and tensors are "pulled up":

$$g_{\nu\tau}A^{\tau} \overset{(2.23)}{=} A_{\nu} \overset{(2.26)}{=} \underbrace{g_{\nu\tau}g^{\tau\sigma}}_{\delta_{\nu\sigma}}A_{\sigma} \quad \Longrightarrow \quad g^{\tau\sigma}A_{\sigma} = A^{\tau} \tag{2.28}$$

Different from mathematically oriented authors, we do *not* define the

$$\text{Kronecker symbol:} \quad \delta_{fg} \equiv \begin{cases} 1 \text{ if } f = g \\ 0 \text{ if } f \neq g \end{cases} \tag{2.29}$$

as a tensor, but place it's indices *always* bottom. We stick to this rule even for the Kronecker symbol of two space-time indices $\delta_{\mu\nu} = g_\mu{}^\nu$. If we would want to treat the Kronecker symbol δ_{kf} with indices different from space-time indices as a tensor, we would need to introduce besides the 4-component Lorentz space further n-component tensor spaces (with n finite or countable infinite), in which a Kronecker tensor $\delta_f{}^g$ then had to be defined as a twofold indexed tensor. In this book, we don't want to consider any tensors different from the tensors of 4-dimensional space-time. To stay consistent, we consider $g_\mu{}^\nu$ as a component of a tensor (which may be called the Kronecker tensor of space-time). But we do not consider $\delta_{\mu\nu}$ as a tensor component, not even if μ and ν are space-time indices.

Due to multiplication by the metric tensor, indices are pulled up or down, i.e. covariant tensor components are transformed to contravariant components, and vice versa. The transformations $\Lambda^\mu{}_\nu$ are no tensors, they are not transformed like the direct product of co- and contravariant vectors. They are *not at all* transformed. Instead they transform vectors and tensors from one reference system to another. Thus the notion of co- or contravariant index position would be pointless for $\Lambda^\mu{}_\nu$. Still the relation

$$\begin{aligned} g'^{\mu\sigma} x'_\sigma &\overset{(2.28)}{=} x'^\mu = \Lambda^\mu{}_\nu x^\nu \overset{(2.28)}{=} \Lambda^\mu{}_\nu g^{\nu\tau} x_\tau \\ g'_{\rho\mu} g'^{\mu\sigma} x'_\sigma &\overset{(2.26)}{=} x'_\rho = g'_{\rho\mu} \Lambda^\mu{}_\nu g^{\nu\tau} x_\tau \overset{(2.15)}{=} \Lambda^{-1\,\tau}{}_\rho x_\tau \end{aligned} \tag{2.30}$$

is suggesting the following definitions:

$$\Lambda_{\mu\nu} \equiv g'_{\mu\sigma} \Lambda^\sigma{}_\nu \tag{2.31a}$$

$$\Lambda^{\mu\nu} \equiv \Lambda^\mu{}_\tau g^{\tau\nu} \tag{2.31b}$$

$$\Lambda_\mu{}^\nu \equiv g'_{\mu\sigma} \Lambda^\sigma{}_\tau g^{\tau\nu} \overset{(2.30)}{=} \Lambda^{-1\,\nu}{}_\mu \tag{2.31c}$$

If Λ transforms vectors from an unprimed to a primed coordinate system, then the row index of Λ must be pulled with g', while the column index must be pulled with the unprimed g. That makes the application of these

definitions complicated and prone to errors. In the case $g' = g$, i. e. in particular for transformations inbetween inertial systems (see the next section), this complication doesn't exist. For transformations inbetween inertial systems therefore (2.31) is often applied.

Above we proved that scalars (i. e. quantities with no un-contracted space-time indices) are invariant under arbitrary coordinate transformations. Actually there also exist two tensors (i. e. not scalars), which are invariant under coordinate transformations:

The invariance of the unit tensor $\mathbb{1} = (g_\alpha{}^\beta)$ can be checked easily:

$$g'_\alpha{}^\beta = \Lambda^{-1}{}^\sigma{}_\alpha \Lambda^\beta{}_\tau g_\sigma{}^\tau = \Lambda^\beta{}_\tau \Lambda^{-1}{}^\tau{}_\alpha = g_\alpha{}^\beta \tag{2.32a}$$

While the unit tensor is invariant under *all* transformations of coordinates, which are admitted in GRT, the tensor

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.32b}$$

is invariant only under *global* transformations of the space-time coordinates, i.e. under transformations, whose elements are not defined differently at different points of space-time, but which are globally identical at all points. These global transformations are called Lorentz transformations. In section 5.5.2 we will derive the Lorentz transformations explicitly, and will thoroughly discuss their characteristics.

2.4 Inertial Systems

By definition, a reference system is an inertial system, if no inertial forces are acting on massive bodies which are at rest or in uniform rectilinear motion in this system. This is the case if and only if the metric tensor has the simple form $(g_{\mu\nu}) = (2.32b)$. Obviously this matrix is identical with it's inverse — and therefore also with $(g^{\mu\nu})$:

$$(g_{\mu\nu}) = (g_{\mu\nu})^{-1} \overset{(2.27)}{=} (g^{\mu\nu}) \tag{2.33}$$

GRT considers gravity forces as inertial forces. Therefore no gravitation must exist in inertial systems, i.e. inertial systems are fixed to free falling laboratories. Because in a free falling laboratory of sufficient size tidal forces will be measurable (to compensate gravity exactly, the laboratory's deeper, ahead falling part must fall faster than the upper, back part), the inertial systems of GRT are of finite size only. Strictly speaking they must be infinitesimal small.

SRT on the other hand considers gravity not as an inertial force. Accelerations, which can be interpreted — following Newton's theory of gravity — as the attractive effect of massive bodies, therefore are subtracted when the metric field g is measured. Consequently, all inertial systems of SRT have infinite size, and all of them are moving rectilinear and without acceleration relative to another.

In this book, we will exclusively use reference systems, which are moving rectilinear and without acceleration relative to another, and in which deviations of the metric tensor from (2.32b) are to small to be detected. And as transformations Λ of the coordinates of four-dimensional space-time we will use exclusively the Lorentz transformations, which are defined globally, i.e. which are independent of time and position. Under these preconditions,

$$x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu} \tag{2.34}$$

is a linear equation, because Λ does not depend on x.

3 Lagrangian and Hamiltonian

3.1 Classical Point-Particle Mechanics

3.1.1 Hamilton's Principle

The mechanical state of a systems of n mass points can be computed for any point of time in the future or the past, if its generalized coordinates $q_1, q_2, \ldots, q_n$, and their time derivatives, the generalized velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$, are known at an arbitrary point of time t. To actually perform this computation, one needs to know the system's equation of motion. This equation can be found by means of the Lagrangian, and an extremal principle, which was discovered by Hamilton.

The system's Lagrangian is a function of time t, and of the generalized coordinates and the generalized velocities of the n mass points:

$$\text{Lagrangian:} \quad L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \tag{3.1}$$

For brevity, we often will use the notation $L(q, \dot{q}, t)$.

At time t_1 the system has coordinates $q(t_1)$ and velocities $\dot{q}(t_1)$, at time t_2 it has coordinates $q(t_2)$ and velocities $\dot{q}(t_2)$. In the time interval between t_1 and t_2, according to experience $q(t)$ and $\dot{q}(t)$ evolve such, that the integral

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q, \dot{q}, t) \tag{3.2}$$

will have an extreme value. This is Hamilton's extremal principle, which we mentioned above. S is called the system's action. Let us assume that we have found the coordinates $q(t)$ and consequently the velocities $\dot{q}(t)$, for which the action S is extreme. Then in linear approximation, S will not be changed by small variations $\delta q(t)$ of $q(t)$ and consequently small variations

$\delta\dot{q}(t)$ of $\dot{q}(t)$. Note that $q(t)$ is varied in the time interval between t_1 and t_2, but not at these boundary points themselves. For these variations we demand

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right) = 0 \ . \tag{3.3}$$

Using the identity

$$\delta\dot{q} = \lim_{t'\to t} \delta \left(\frac{q(t') - q(t)}{t' - t} \right) = \lim_{t'\to t} \left(\frac{\delta q(t') - \delta q(t)}{t' - t} \right) = \frac{\mathrm{d}(\delta q)}{\mathrm{d}t} \ , \tag{3.4}$$

the second term can be reshaped:

$$\begin{aligned} \int_{t_1}^{t_2} \mathrm{d}t \, \frac{\partial L}{\partial \dot{q}}\delta\dot{q} &= \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}}\delta q \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right] \\ &= \frac{\partial L}{\partial \dot{q}} \, \delta q \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \end{aligned} \tag{3.5}$$

Because of $\delta q(t_1) = \delta q(t_2) = 0$, only the last term is different from zero, and the extremal condition becomes

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} \right) \delta q = 0 \ . \tag{3.6}$$

For arbitrary δq this condition can only be met, if the term in brackets is zero. Thus one finds the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \ . \tag{3.7}$$

For a system with n coordinates q_j, these must be varied independently, resulting in the n equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad , \quad j = 1 \dots n \ . \tag{3.8}$$

The equation of motion follows uniquely from the Lagrangian, but different Lagrangians may lead to the same equation of motion. An important example are two Lagrangians L and L', which differ only by the total time derivative of an arbitrary function $G(q, \dot{q}, t)$:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t} G(q, \dot{q}, t) \tag{3.9}$$

According to Hamilton's principle,

$$\begin{aligned} \delta S' &= \delta \int_{t_1}^{t_2} \mathrm{d}t \, \left(L(q,\, \dot{q},\, t) + \frac{\mathrm{d}}{\mathrm{d}t} G(q, \dot{q}, t) \right) \\ &= \delta \int_{t_1}^{t_2} \mathrm{d}t \, L(q,\, \dot{q},\, t) + \underbrace{\delta \left(G(q, \dot{q}, t) \Big|_{t_1}^{t_2} \right)}_{=0} \\ &= \delta S = 0 \ . \end{aligned} \tag{3.10}$$

The additional term $G(q, \dot{q}, t)$ contributes nothing to the variation $\delta S'$, because q and $\dot{q}$ are varied only within the integration interval, but not at it's borders. Therefore the condition $\delta S' = 0$ and the condition $\delta S = 0$ result in the same equations of motion.

The Lagrangian of a classical system of mass points in most cases is equal to the difference of the mass points' kinetic energy T, and their potential energy U:

$$\begin{aligned} L = T - U &= \sum_{j=1}^{n} T_j - U(q_1, \dots, q_n) \\ &= \sum_{j,k=1}^{n} \frac{1}{2} \, a_{jk}(q_1, \dots, q_n) \dot{q}_j \dot{q}_k - U(q_1, \dots, q_n) \ , \end{aligned} \tag{3.11}$$

with the coefficients a_{jk} being functions of the generalized coordinates

$q \equiv (q_1, \ldots, q_n)$. If Cartesian coordinates $x \equiv (x_1, \ldots, x_n)$ are used, then the matrix of coefficients is diagonal, with

$$a_{jk} = m_j \delta_{jk} \tag{3.12}$$

being the mass of the j^{th} mass point. The equation of motion of this mass point

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 \tag{3.13}$$

becomes in Cartesian coordinates

$$m_j \ddot{x}_j = -\frac{\partial U}{\partial x_j} \ . \tag{3.14}$$

Historically speaking, the mass points equations of motion are much longer known than the Lagrangian, from which we derived them here. And most of the field equations, with which we will become acquainted in this book, were found directly, but not derived from a Lagrangian density (the field-theoretical counterpart of the point mechanics Lagrangian). Only after the field equations were known, Lagrangian densities were constructed retroactively, from which the field equations can be derived using Hamilton's principle of least action. The occupation with Lagrangian densities therefore might seem to be a superfluous detour.

But that is not at all the case, because the field equations "inherit" the symmetry properties of the Lagrangian densities, from which they are derived. We will see, that the symmetries usually can be investigated much simpler on the basis of Lagrangian densities, than on the basis of the field equations themselves. The small detour via the Lagrangian densities therefore will soon pay off.

3.1.2 Canonical Equations

The equations of motion, which have been derived from the Lagrangian by means of Hamilton's principle, are second order differential equations. Alternatively and equivalently, two first order differential equations can be

stated. To this purpose, consider the total differential

$$\mathrm{d}L = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\mathrm{d}q_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\mathrm{d}\dot{q}_j + \frac{\partial L}{\partial t}\mathrm{d}t \ . \tag{3.15}$$

Using the definition of the canonically conjugate momentum

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} \tag{3.16}$$

and the equation of motion

$$\frac{\partial L}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} = \dot{p}_j \ , \tag{3.17}$$

the total differential of L becomes

$$\begin{aligned} \mathrm{d}L &= \sum_{j=1}^{n} \dot{p}_j \mathrm{d}q_j + \sum_{j=1}^{n} p_j \mathrm{d}\dot{q}_j + \frac{\partial L}{\partial t}\mathrm{d}t \\ &= \sum_{j=1}^{n} \dot{p}_j \mathrm{d}q_j + \mathrm{d}\left(\sum_{j=1}^{n} p_j \dot{q}_j \right) - \sum_{j=1}^{n} (\mathrm{d}p_j)\dot{q}_j + \frac{\partial L}{\partial t}\mathrm{d}t \ . \end{aligned} \tag{3.18}$$

We change the sequence of terms

$$\underbrace{\mathrm{d}\left(\sum_{j=1}^{n} p_j \dot{q}_j - L \right)}_{\equiv \mathrm{d}H} = -\sum_{j=1}^{n} \dot{p}_j \mathrm{d}q_j + \sum_{j=1}^{n} (\mathrm{d}p_j)\dot{q}_j - \frac{\partial L}{\partial t}\mathrm{d}t \ , \tag{3.19}$$

and define the Hamiltonian

$$H \equiv \sum_{j=1}^{n} p_j \dot{q}_j - L \ , \tag{3.20}$$

whose total differential is

$$\mathrm{d}H(q,p,t) = \sum_{j=1}^{n} \frac{\partial H}{\partial q_j}\mathrm{d}q_j + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j}\mathrm{d}p_j + \frac{\partial H}{\partial t}\mathrm{d}t \ . \tag{3.21}$$

Comparing (3.19) with (3.21), we find the two first order canonical differential equations

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \tag{3.22a}$$

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \ . \tag{3.22b}$$

They often are called Hamilton equations. These two equations together are equivalent to the Lagrange equation (3.8). Formally the third equation

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{3.22c}$$

results from the comparison of (3.19) and (3.21), but it does not add useful informations to the equations of motion.

To evaluate the Hamiltonian's time dependence, we divide (3.19) by $\mathrm{d}t$:

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \sum_{j=1}^{n} \frac{\partial H}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j}\dot{p}_j + \frac{\partial H}{\partial t} \\ &= -\sum_{j=1}^{n} \dot{p}_j\dot{q}_j + \sum_{j=1}^{n} \dot{q}_j\dot{p}_j + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t} \end{aligned} \tag{3.23}$$

If L, and consequently H, does not explicitly depend on time, then the system does not change if it is translated in time, i. e. $\mathrm{d}H/\mathrm{d}t$ is zero. The translation in time then is called a symmetry of the system. In section 4.1 we will establish, that any continuous symmetry of a system (in this case the invariance under a translation in time) correlates to a conservation law. The quantity, which is conserved due to invariance of a system under translation in time, is *by definition* called energy. Thus the Hamiltonian H

is representing the system's energy.

If Cartesian coordinates are used, the Lagrangian $L =$ (3.11) becomes

$$L = T - U = \sum_{j=1}^{n} \frac{1}{2} m_j \dot{x}_j^2 - U(x_1, \ldots, x_n) \ . \tag{3.24}$$

With Cartesian coordinates, the canonical conjugate momentum of particle j is

$$p_j = \frac{\partial L}{\partial \dot{x}_j} = m_j \dot{x}_j \ , \tag{3.25}$$

and the Hamiltonian becomes

$$\begin{aligned} H &= \sum_{j=1}^{n} p_j \dot{x}_j - L \\ &= \sum_{j=1}^{n} \frac{1}{2} m_j \dot{x}_j^2 + U = T + U \\ &= \sum_{j=1}^{n} \frac{1}{2m_j} p_j^2 + U(x_1, \ldots, x_n) \ . \end{aligned} \tag{3.26}$$

From this, the canonical equations in Cartesian coordinates

$$\frac{\partial H}{\partial p_j} = \frac{1}{m_j} p_j = \dot{x}_j \quad \Longrightarrow \quad p_j = m_j \dot{x}_j \tag{3.27a}$$

$$\frac{\partial H}{\partial x_j} = \frac{\partial U}{\partial x_j} = -\dot{p}_j \quad \Longrightarrow \quad \dot{p}_j = -\frac{\partial U}{\partial x_j} \tag{3.27b}$$

follow, which together are equivalent to the equation of motion (3.14).

3.2 Classical Field Theory

3.2.1 Hamilton's Generalized Principle

The variables in the action integral

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q,\, \dot{q},\, t) \tag{3.28}$$

of classical point mechanics are (besides time t) the generalized coordinates q and the generalized velocities $\dot{q}$ of discrete mass points. To make Hamilton's principle usable for field theory, several modifications are required: In field theory we are dealing with field amplitudes ϕ, which are defined for a space-time continuum:

$$\phi(ct, x, y, z) \equiv \phi(x^0, x^1, x^2, x^3) \equiv \phi(x^0, \boldsymbol{x}) \equiv \phi(x) \tag{3.29}$$

We construct a Lagrangian density $\mathcal{L}$, which is a function of the field amplitude $\phi(x)$, of the amplitude's derivatives to all coordinates of space and time, and of the space-time-coordinates x:

$$\begin{aligned} \mathcal{L} &\equiv \mathcal{L}\left(\phi(x), \frac{\mathrm{d}\phi(x)}{\mathrm{d}x^0}, \frac{\mathrm{d}\phi(x)}{\mathrm{d}x^1}, \frac{\mathrm{d}\phi(x)}{\mathrm{d}x^2}, \frac{\mathrm{d}\phi(x)}{\mathrm{d}x^3}, x\right) \\ &\equiv \mathcal{L}(\phi, \mathrm{d}_0\phi, \mathrm{d}_1\phi, \mathrm{d}_2\phi, \mathrm{d}_3\phi, x) \end{aligned} \tag{3.30}$$

We often will call the Lagrangian density simply "Lagrangian" for brevity.

In three-dimensional position space, we fix a normalization volume Ω of sufficient size, such that at any time only within Ω the amplitude $\phi(x)$ differs significantly from zero. (Many authors prefer an infinitely large normalization volume. In section 7 we will state reasons, why an arbitrarily large but finite normalization volume is physically more sensible.) The field's Lagrangian is the space integral over the Lagrangian density:

$$L(t) = \int_{\Omega} \mathrm{d}^3x\, \mathcal{L} \tag{3.31}$$

In practice, we will almost always use the Lagrangian density, but not the Lagrangian.

The action integral is defined by

$$S \equiv \int_{x_a^0}^{x_b^0} \frac{\mathrm{d}x^0}{c} \int_{\Omega} \mathrm{d}^3x\, \mathcal{L} \equiv \int_{x_a^0}^{x_b^0} \frac{\mathrm{d}^4x}{c}\, \mathcal{L}\ . \tag{3.32}$$

On the right side, a shortcut notation for the combination of a time integral over the interval x_a to x_b and the space integral over Ω was defined, which will be used in the following.

According to Hamilton's principle (adapted to field theory), the field will evolve in the time interval from x_a^0 to x_b^0 such, that the action integral becomes extreme. Thus the variation of S with respect to ϕ and to the derivatives of ϕ must be zero in linear approximation. The variation thereby is – same as in mass point mechanics – to be performed only within the integration range, but not on its borders. The borders of the integration range in this case are fixed by the surface of Ω and the points of time x_a^0 and x_b^0.

$$\delta S = \int_{x_a^0}^{x_b^0} \frac{\mathrm{d}^4x}{c} \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \delta(\mathrm{d}_\mu \phi) \right) = 0 \tag{3.33}$$

Here Einstein's summation convention was used (see K.4), according to which the sum over all four space-time coordinates is to be computed, if in a product an index is showing up twice, i.e. in this case

$$\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \delta(\mathrm{d}_\mu \phi) \equiv \sum_{\mu=0}^{3} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \delta(\mathrm{d}_\mu \phi)\ . \tag{3.34}$$

Using the identity $\delta(\mathrm{d}_\mu \phi) = \mathrm{d}_\mu(\delta\phi)$, the last term in (3.33) can be modified:

$$\int_{x_a^0}^{x_b^0} \frac{\mathrm{d}^4x}{c} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \delta(\mathrm{d}_\mu \phi) = \int_{x_a^0}^{x_b^0} \frac{\mathrm{d}^4x}{c} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \mathrm{d}_\mu(\delta\phi) =$$

$$= \int_{x_a^0}^{x_b^0} \frac{\mathrm{d}^4x}{c}\, \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \delta\phi - \int_{x_a^0}^{x_b^0} \frac{\mathrm{d}^4x}{c} \left(\mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \right) \delta\phi \tag{3.35}$$

The second to last term is zero, because according to Gauß' theorem the integral of a vector fields divergence over the volume V equals the same vector field's integral over the surface of V. Everywhere on the surface of the four-dimensional integration volume $\delta\phi = 0$, because the field is varied only within that volume.

The remaining term of (3.35) is inserted into (3.33):

$$\delta S = \int_{x_a}^{x_b} \frac{\mathrm{d}^4x}{c} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \right) \delta\phi = 0 \tag{3.36}$$

This condition can only be met for arbitrary $\delta\phi$ with the field equation

$$\boxed{\mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0} \,. \tag{3.37a}$$

If $\mathcal{L}$ depends on n fields $\phi_r, \phi_s, \phi_t, \ldots$ and their derivatives, then all these fields must be varied independently from another, and one finds n field equations of the form

$$\mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi_r)} - \frac{\partial \mathcal{L}}{\partial \phi_r} = 0 \quad , \quad r = 1 \ldots n \,. \tag{3.37b}$$

The components of vector fields, which are defined in four-dimensional space-time, must be varied separately. For each of the components one gets a particular field equation. In contrast a field with several spinor components is varied on the whole, i. e. there is only one field equation for a spinor field. The different treatment of fields with several components is caused by the fact, that the variation of (3.32) is done in four-dimensional space-time, but not in spinor space.

If the Lagrangian depends on the field $\phi(x)$ and its complex-conjugate $\phi^*(x)$, then $\mathcal{L}$ must be varied with respect to $\phi(x)$ and — separately — with

respect to $\phi^*(x)$, as if $\phi(x)$ and $\phi^*(x)$ were two independent fields. Thereby one gets the two field equations

$$\mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{and} \quad \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0 \,. \tag{3.37c}$$

This is surprising at first moment, as $\phi^*(x)$ clearly is not independent, but uniquely determined by $\phi(x)$. But $\phi(x)$ can be split in it's real and imaginary parts by

$$\begin{aligned} a(x) &\equiv \tfrac{1}{2}\Big(\phi(x) + \phi^*(x)\Big) & b(x) &\equiv -\tfrac{i}{2}\Big(\phi(x) - \phi^*(x)\Big) \\ \phi(x) &\equiv a(x) + i b(x) & &\text{with } a(x) \in \mathbb{R}\,,\, b(x) \in \mathbb{R}\,. \end{aligned} \tag{3.38}$$

$a(x)$ and $b(x)$ actually are two independent fields, for which — using the chain rule — one finds the field equations

$$\begin{aligned} 0 &= \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu a)} - \frac{\partial \mathcal{L}}{\partial a} && \text{(3.39a)} \\ &= \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \underbrace{\frac{\partial(\mathrm{d}_\mu \phi)}{\partial(\mathrm{d}_\mu a)}}_{1} + \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi^*)} \underbrace{\frac{\partial(\mathrm{d}_\mu \phi^*)}{\partial(\mathrm{d}_\mu a)}}_{1} - \frac{\partial \mathcal{L}}{\partial \phi} \underbrace{\frac{\partial \phi}{\partial a}}_{1} - \frac{\partial \mathcal{L}}{\partial \phi^*} \underbrace{\frac{\partial \phi^*}{\partial a}}_{1} \\ 0 &= \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu b)} - \frac{\partial \mathcal{L}}{\partial b} && \text{(3.39b)} \\ &= \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} \underbrace{\frac{\partial(\mathrm{d}_\mu \phi)}{\partial(\mathrm{d}_\mu b)}}_{+i} + \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi^*)} \underbrace{\frac{\partial(\mathrm{d}_\mu \phi^*)}{\partial(\mathrm{d}_\mu b)}}_{-i} - \frac{\partial \mathcal{L}}{\partial \phi} \underbrace{\frac{\partial \phi}{\partial b}}_{+i} - \frac{\partial \mathcal{L}}{\partial \phi^*} \underbrace{\frac{\partial \phi^*}{\partial b}}_{-i} \,. \end{aligned}$$

From these expressions, the two equations

$$0 = \tfrac{1}{2}\Big((3.39a) - i(3.39b)\Big) = \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} \tag{3.40a}$$

$$0 = \tfrac{1}{2}\Big((3.39a) + i(3.39b)\Big) = \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} \tag{3.40b}$$

result, which are identical to (3.37c). The independent variations of $\mathcal{L}$ with

respect to $\phi(x)$ and to $\phi^*(x)$ are nothing but a shortcut, which give the same result as the formally correct method (3.39).

Equations (3.37) follow uniquely from the Lagrangian, but the inverse is not true. Different Lagrangians may lead to the same field equation. In particular, one finds the identical field equation for two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$, which differ only in the four-divergence of an arbitrary vector function $\mathcal{G}$:

$$\mathcal{L}' = \mathcal{L} + \mathrm{d}_\mu \mathcal{G}^\mu \tag{3.41}$$

We compute the action's variation:

$$\delta S' = \delta \int_{x_a}^{x_b} \frac{\mathrm{d}^4 x}{c} \left(\mathcal{L} + \mathrm{d}_\mu \mathcal{G}^\mu \right) = \underbrace{\delta \int_{x_a}^{x_b} \frac{\mathrm{d}^4 x}{c} \mathcal{L}}_{\delta S} + \underbrace{\delta \int_{x_a}^{x_b} \frac{\mathrm{d}^4 x}{c} \mathrm{d}_\mu \mathcal{G}^\mu}_{=0} \tag{3.42}$$

According to Gauß' theorem, the four-dimensional volume integral over the four-dimensional divergence of $\mathcal{G}$ equals the integral over $\mathcal{G}$ on the volume's surface. On this surface, $\mathcal{G}$ is not varied, so that the integral's variation is zero.

Hamilton's principle (adapted to field theory) can not be "derived". It is a law of nature, i.e. a postulate found by guessing, which can be legitimated solely by the fact, that the consequences, which are derived from that law, coincide with all experimental experiences.

3.2.2 Canonical Equations

It's not immediately obvious, how the method of point mechanics, to derive canonical equations from the Lagrangian, can be transferred to field theory. The field amplitude $\phi(t, \boldsymbol{x})$ and its canonically conjugate momentum $p(t, \boldsymbol{x})$ – which we now want to construct – have instead of the discrete index j of $q_j(t)$ and $p_j(t)$ the continuous, innumerable coordinates $\boldsymbol{x}$ of three-dimensional position space. We will define the conjugate momenta of continuous fields by two different methods: First by discretization of the field, and then once again by a more elegant, but less transparent method on page 56, equation

(4.40), by means of Noether's theorem. For discretization of the field, we construct numerable field amplitudes $\phi_j(t)$ by splitting the position space into countably infinitely many small cells with volume V. Then we take the average of $\phi(t, \boldsymbol{x})$ in the cell $V(\boldsymbol{x}_j)$, which is enclosing the point $\boldsymbol{x}_j$:

$$\phi_j(t) \equiv \frac{1}{V} \int\limits_{V(\boldsymbol{x}_j)} \mathrm{d}^3x \, \phi(t, \boldsymbol{x}) \tag{3.43}$$

Thus $\phi(t)$ at $\boldsymbol{x}_j$ is

$$\phi(t, \boldsymbol{x}_j) = \lim_{V \to 0} \frac{1}{V} \int\limits_{V(\boldsymbol{x}_j)} \mathrm{d}^3x \, \phi(t, \boldsymbol{x}) = \lim_{V \to 0} \phi_j(t) \ . \tag{3.44}$$

The countable conjugate momenta $p_j(t)$ are defined as averages in the same volumina:

$$p_j(t) \equiv \frac{1}{V} \int\limits_{V(\boldsymbol{x}_j)} \mathrm{d}^3x \, p(t, \boldsymbol{x}) \tag{3.45}$$

Thus $p(t)$ at $\boldsymbol{x}_j$ is

$$p(t, \boldsymbol{x}_j) = \lim_{V \to 0} \frac{1}{V} \int\limits_{V(\boldsymbol{x}_j)} \mathrm{d}^3x \, p(t, \boldsymbol{x}) = \lim_{V \to 0} p_j(t) \ . \tag{3.46}$$

The Lagrangian of field theory

$$\mathcal{L} = \mathcal{L}\Big(\phi(t, \boldsymbol{x}), \mathrm{d}_\mu \phi(t, \boldsymbol{x}), t, \boldsymbol{x}\Big) \quad \text{with } \mu = 0, 1, 2, 3 \tag{3.47}$$

depends on the field amplitude's derivatives with respect to all four space-time coordinates. If $V(\boldsymbol{x}_{jk})$ is the neighbor cell of $V(\boldsymbol{x}_j)$ in direction x^k, then for the derivative with respect to space coordinate x^k

$$\left.\frac{\mathrm{d}\,\phi(t,\boldsymbol{x})}{\mathrm{d}\,x^k}\right|_{\boldsymbol{x}=\boldsymbol{x}_j} = \frac{\mathrm{d}}{\mathrm{d}\,x^k}\lim_{V\to 0}\phi_j(t)$$
$$= \lim_{V\to 0}\frac{\phi_{jk}(t)-\phi_j(t)}{V^{1/3}} \equiv \lim_{V\to 0}\Delta_k\phi_j(t)\ . \tag{3.48}$$

The field is discretized in position space only, but not in time.

$$\frac{\mathrm{d}\phi(t,\boldsymbol{x}_j)}{\mathrm{d}t} = \dot\phi(t,\boldsymbol{x}_j) = \lim_{V\to 0}\frac{\mathrm{d}\phi_j(t)}{\mathrm{d}t} = \lim_{V\to 0}\dot\phi_j(t) \tag{3.49}$$

Thus the discretized field's Lagrangian density $\overline{\mathcal{L}_j}$ in cell $V(\boldsymbol{x}_j)$ becomes

$$\overline{\mathcal{L}_j} = \overline{\mathcal{L}_j}\Big(\phi_j(t),\dot\phi_j(t),\Delta_1\phi_j(t),\Delta_2\phi_j(t),\Delta_3\phi_j(t),t,\boldsymbol{x}_j\Big)\ . \tag{3.50}$$

The Lagrangian is

$$\overline{L} = \sum_j \overline{\mathcal{L}_j}\cdot V(\boldsymbol{x}_j)\ , \tag{3.51}$$

and thus

$$\mathcal{L} = \lim_{V\to 0}\overline{\mathcal{L}}\quad,\quad L = \lim_{V\to 0}\overline{L}\quad,\quad \int \mathrm{d}^3x = \lim_{V\to 0}\sum_j V(\boldsymbol{x}_j)\ . \tag{3.52}$$

Now we can compute the conjugate momenta explicitly:

$$p_j(t) = \frac{\partial\overline{L}}{\partial\dot\phi_j(t)} = \frac{\partial}{\partial\dot\phi_j(t)}\sum_n \overline{\mathcal{L}_n}\cdot V(\boldsymbol{x}_n) = \frac{\partial\overline{\mathcal{L}_j}}{\partial\dot\phi_j(t)}\cdot V(\boldsymbol{x}_j) \tag{3.53}$$

Taking the limit $\lim_{V\to 0}$ on both sides of the equation, we find

$$p(t,\boldsymbol{x}_j) = \frac{\partial\mathcal{L}}{\partial\dot\phi(t,\boldsymbol{x}_j)}\cdot\mathrm{d}^3x_j\ . \tag{3.54}$$

This holds for arbitrary j. Thus we get

$$p(t,\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial\dot\phi(t,\boldsymbol{x})}\cdot\mathrm{d}^3x\ . \tag{3.55}$$

The conjugate momentum density $\pi(t,\boldsymbol{x})$ is defined by

$$\pi(t,\boldsymbol{x})\mathrm{d}^3x \equiv p(t,\boldsymbol{x}) \ . \tag{3.56}$$

Therefore the conjugate momentum density π of the field ϕ is

$$\boxed{\pi(t,\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(t,\boldsymbol{x})}} \ . \tag{3.57}$$

If the Lagrangian depends on several fields ϕ_r, then in the same way for each field the conjugate momentum density π_r is computed. If ϕ is a vector field with three space components, or a vector field with four space-time-components respectively, then for each component the correspondent momentum density component is computed according to (3.57). Note, that the momentum density, which is conjugate to a covariant field component, is contravariant, and vice versa:

$$\pi_\mu(t,\boldsymbol{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^\mu(t,\boldsymbol{x})} \tag{3.58}$$

The Hamiltonian H of point mechanics was defined in (3.20) by

$$H\Big(q(t),p(t),t\Big) \equiv \sum_{j=1}^{n} p_j(t)\dot{q}_j(t) - L\Big(q(t),\dot{q}(t),t\Big) \ . \tag{3.59}$$

In complete analogy we now define the Hamiltonian H and the Hamiltonian density $\mathcal{H}$ (which usually we will simply call Hamiltonian) of field theory by

$$\begin{aligned} H\Big(\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}),t\Big) &\equiv \int \mathrm{d}^3x\, \mathcal{H}\Big(\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}),t,\boldsymbol{x}\Big) \\ \mathcal{H} &\equiv \sum_r \pi_r(t,\boldsymbol{x})\dot{\phi}_r(t,\boldsymbol{x}) - \mathcal{L}\Big(\phi,\mathrm{d}_\mu\phi,t,\boldsymbol{x}\Big) \ . \end{aligned} \tag{3.60a}$$

In addition we define the physical momentum and the physical momentum density with the three components

$$P^j\Big(\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}),t\Big) \equiv \int \mathrm{d}^3x\, \mathcal{P}^j\Big(\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}),t,\boldsymbol{x}\Big)$$

$$\mathcal{P}^j \equiv \sum_r \pi_r(t,\boldsymbol{x})\mathrm{d}^j\phi_r(t,\boldsymbol{x}) \quad \text{with } j=1,2,3\ . \tag{3.60b}$$

We postpone the motivation and explication for this definition to section 4.2, equation (4.35).

At this point, a warning may be appropriate: In (3.42) we found, that two Lagrangians $\mathcal{L}$ and $\mathcal{L}' = \mathcal{L}+\mathrm{d}_\mu\mathcal{G}^\mu$, which differ only by the four-divergence of an arbitrary vector function, result into identical field equations. This does not at all mean, that different Lagrangians are equivalent in the canonical formalism, as long as their field equations are identical. If the conjugate momentum density, or the energy density, or the physical momentum density of fields shall be computed, this adds further constrains to the definition of the Lagrangian. This becomes immediately obvious by the fact, that a Lagrangian, if multiplied by a constant factor, leads to an unchanged field equation, but to a changed conjugate momentum density.

4 Continuous Symmetries and Conservation Laws

4.1 Noether's Theorem

There exists a systematic correlation between the continuous symmetries of physical systems and their conserved quantities, which was discovered by E. Noether[1], and is called Noether's theorem.

To derive this correlation, we consider a continuous transformation Γ, which acts on a scalar field $\phi(x)$, and causes some modification of the field. The modified field is called $\phi'(x)$:

$$\phi(x) \xrightarrow{\Gamma} \phi'(x) \tag{4.1}$$

Besides such active transformations, we also allow for passive transformations, which change the space-time-coordinates under the field:

$$\phi(x) \xrightarrow{\Gamma} \phi(x') \tag{4.2}$$

The transformations of fields and/or space-time-coordinates in general will also change the Lagrangian:

$$\mathcal{L} \xrightarrow{\Gamma} \mathcal{L}' \tag{4.3}$$

If a transformation Γ leaves the field equation

$$\mathrm{d}_\rho \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{4.4}$$

invariant, then Γ is called a symmetry of the Lagrangian $\mathcal{L}$. Obviously

[1] Emmy (Amalie) Noether, 1882 – 1935

Γ is a symmetry of $\mathcal{L}$ if and only if the variation δS of the action, from which this field equation was extracted, is not changed by Γ. And δS will stay unchanged, if and only if Γ modifies the Lagrangian only by the four-divergence (which may be identical to zero) of an arbitrary vector function $\mathcal{G}$, because that four-divergence disappears – as shown in (3.42) – at the action's variation.

A real parameter W (more precisely, there usually are a finite number of real parameters) specifies the transformation quantitatively, e.g. the angle's value in case of a rotation, or the shift distance in case of a translation, or the phase angle's value in case of a phase transformation, etc. In case of $W = 0$, the transformation turns into the identity I. This is the transformation, which leaves unchanged any object, on which it is acting.

All continuous transformations Γ, which we consider in this book, can be expanded in a Taylor series around the point $W = 0$:

$$\Gamma = \sum_{n=0}^{\infty} \frac{W^n}{n!} \frac{\mathrm{d}^n \Gamma}{\mathrm{d}W^n}\bigg|_{W=0} = I + \sum_{n=1}^{\infty} \frac{W^n}{n!} \frac{\mathrm{d}^n \Gamma}{\mathrm{d}W^n}\bigg|_{W=0} \tag{4.5}$$

Choosing W infinitesimal small, one gets the infinitesimal small transformation Γ_{INF}:

$$w \equiv \lim_{n\to\infty} \frac{W}{n} \quad \text{with } n \in \mathbb{N} \tag{4.6}$$

With w infinitesimal small, the Taylor series may be ended after the linear term, and the infinitesimal transformation becomes

$$\Gamma_{\text{INF}} = I + w \frac{\mathrm{d}\Gamma}{\mathrm{d}W}\bigg|_{W=0} = I + w \frac{i}{\hbar}(-i\hbar) \frac{\mathrm{d}\Gamma}{\mathrm{d}W}\bigg|_{W=0}$$

$$\boxed{\Gamma_{\text{INF}} = I + \frac{i}{\hbar} w \gamma} \quad . \tag{4.7}$$

The operator

$$\boxed{\gamma \equiv -i\hbar \frac{\mathrm{d}\Gamma}{\mathrm{d}W}\bigg|_{W=0}} \tag{4.8}$$

is called the *generator* of the transformation Γ. The real parameter W specifies the transformation's quantity, the generator γ specifies the transformation's type, whether e.g. it is a phase transformation, or a rotation, or whatever. The finite transformation Γ is generated by the concatenation of infinitely many infinitesimal small transformations Γ_{INF}:

$$\Gamma = \lim_{n\to\infty}\left(I+\frac{i}{\hbar}w\,\gamma\right)^n = \lim_{n\to\infty}\left(I+\frac{i}{\hbar}\frac{W}{n}\gamma\right)^n \quad \text{with } n\in\mathbb{N} \tag{4.9}$$

$$\boxed{\Gamma = \exp\left\{\frac{i}{\hbar}W\gamma\right\}} \tag{4.10}$$

Γ will leave $\mathcal{L}$ unchanged (possibly except for the four-divergence of an arbitrary vector function) if and only if the same holds true for the infinitesimal transformation $(I+\frac{i}{\hbar}w\,\gamma)$:

$$\boxed{\begin{aligned}&\Gamma = \lim_{n\to\infty}\left(I+\frac{i}{\hbar}w\,\gamma\right)^n \quad \text{is a symmetry of } \phi \iff\\ &\iff \exists\,\mathcal{G}:\ \mathcal{L}\xrightarrow{I+\frac{i}{\hbar}w\,\gamma}\mathcal{L}\,' = \mathcal{L}+\frac{i}{\hbar}w\,\gamma\mathcal{L} = \mathcal{L}+\mathrm{d}_\rho\mathcal{G}^\rho\end{aligned}} \tag{4.11}$$

Generally, $\mathcal{L}$ is a function of several fields $\phi_r, \phi_s, \phi_t, \ldots$, of the fields derivatives to the space-time-coordinates, and possibly explicitly of the space-time-coordinates x. The effect of the generator γ on $\mathcal{L}$ thus is

$$\gamma\mathcal{L} = \sum_r \frac{\partial\mathcal{L}}{\partial\phi_r}\gamma\phi_r + \sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\gamma(\mathrm{d}_\rho\phi_r) + (\partial_\rho\mathcal{L})\gamma x^\rho \quad . \tag{4.12}$$

If $\mathcal{L}$ is a function of vector fields with four space-time components, each component adds a summand to the sums over r, see the lines after (3.37b).

Using the identity $\gamma(\mathrm{d}_\rho\phi) = \mathrm{d}_\rho(\gamma\phi)$, this equation is modified:

$$\gamma\mathcal{L} = \sum_r \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_r} - \mathrm{d}_\rho \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\right)}_{=0 \text{ see (3.37b)}} \gamma\phi_r + \\ + \mathrm{d}_\rho\left(\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\gamma\phi_r\right) + (\partial_\rho\mathcal{L})\gamma x^\rho \qquad (4.13)$$

Insertion into the symmetry condition (4.11) yields

$$\frac{i}{\hbar} w\,\gamma\mathcal{L} = \mathrm{d}_\rho\left(\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\frac{i}{\hbar} w\,\gamma\phi_r\right) + (\partial_\rho\mathcal{L})\frac{i}{\hbar} w\,\gamma x^\rho = \mathrm{d}_\rho\mathcal{G}^\rho$$

$$\mathrm{d}_\rho\left(\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)} w\gamma\phi_r + i\hbar\mathcal{G}^\rho\right) + (\partial_\rho\mathcal{L}) w\gamma x^\rho = 0 \quad . \qquad (4.14)$$

For arbitrary x this condition can only be fulfilled, if the second term is zero. Therefore a necessary symmetry condition can be formulated :

$$\boxed{\Gamma = \lim_{n\to\infty}(I + \frac{i}{\hbar} w\,\gamma)^n \quad \text{is a symmetry of } \phi \implies \\ \implies (\partial_\rho\mathcal{L}) w\gamma x^\rho = 0} \qquad (4.15)$$

This condition is necessary, but not sufficient. (4.11) is the necessary and sufficient condition for Γ respectively $(I + \frac{i}{\hbar} w\gamma)$ to be a symmetry of $\mathcal{L}$. The necessary condition (4.15) is implicitly included in (4.11). In most cases it's simpler at first only to check the necessary condition (4.15) instead of the sufficient condition. If that criterion already is missed, then one can save the effort to search for a function $\mathcal{G}$ which fulfills the sufficient condition (4.11).

The necessary condition (4.15) is fulfilled, if either the transformation does not act on position vectors but only on fields (then $w\gamma x^\rho = 0$) and/or if the system under evaluation is closed, and the Lagrangian therefore does not explicitly depend on the space-time-coordinates (then $\partial_\rho\mathcal{L} = 0$). The case $w\gamma x^\rho = 0$ is called an "inner symmetry", because this symmetry is not related to the frame of space and time. Inversely, symmetries with $w\gamma x^\rho \neq 0$ (and thus $\partial_\rho\mathcal{L} = 0$) ar called "outer symmetries".

If the sufficient symmetry condition (4.11) is fulfilled, then according to (4.14) the current density j with the components

$$j^\rho \equiv C\left(\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)} w\gamma\phi_r + i\hbar\mathcal{G}^\rho\right) \tag{4.16}$$

is a conserved quantity, for which the equation of continuity

$$\mathrm{d}_\rho j^\rho = 0 \tag{4.17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{j^0}{c}\right) = -\boldsymbol{\nabla}\cdot\boldsymbol{j} \tag{4.17b}$$

is valid. The constant C may be chosen arbitrarily. j^0/c is called charge density, j is called current density. The units of these quantities are

$$[j] = \frac{\text{charge}}{\text{area}\cdot\text{time}} \tag{4.18a}$$

$$[j^0/c] = \frac{\text{charge}}{\text{volume}} \quad . \tag{4.18b}$$

The notion "charge" is to be interpreted quite general, and not constrained to an electrical charge. We soon will get to know examples for pretty different types of charges.

Any equation of continuity is representing a conservation law. This becomes visible by integration over a finite volume V, and conversion of the volume integral over the three-dimensional divergence by Gauß' theorem to a surface integral over the surface of V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int\limits_V \mathrm{d}^3x\,\frac{j^0}{c}}_{Q} = -\int\limits_V \mathrm{d}^3x\,\boldsymbol{\nabla}\cdot\boldsymbol{j} = -\int\limits_{O(V)} \mathrm{d}\boldsymbol{f}\cdot\boldsymbol{j} \tag{4.19}$$

The increase of charge Q inside the volume equals the current, which streams into V through it's surface O. No charge can emerge from nothing or vanish into nothing. Generally holds: The three-dimensional volume integral over a conserved current density's null-component is a conserved quantity.

Note, that this quantity is conserved — as evident from (4.16) — for the sum of all fields contained in the Lagrangian, but not for each single field. E.g. momentum may be exchanged inbetween two fields (and thus the momentum of a single field may change), while the system's overall momentum is conserved.

4.2 Translations

If the necessary symmetry condition $(\partial_\rho \mathcal{L}) w\gamma x^\rho \overset{(4.15)}{=} 0$ is fulfilled by $\partial_\rho \mathcal{L} = 0$, but not by $w\gamma x^\rho = 0$, this is an "outer" symmetry.

An example is the transformation V, which shifts the coordinates of four-dimensional space-time relative to the system under evaluation by a distance A. This is called a passive translation, in contrast to an active translation, which shifts the contents of the world relatively to the fixed coordinates. A maximum of a scalar field ϕ, which has in the shifted coordinate system$'$ (marked by a prime$'$) the coordinates x', has in the unshifted system the coordinates $x + A$:

$$\phi(x) \xrightarrow{V} \phi(x') = \phi(x + A) \tag{4.20}$$

To evaluate the symmetry properties, we consider V as the product of infinitely many concatenated infinitely small transformations. Each infinitesimal transformation step shifts the coordinates by a distance

$$a = \lim_{n\to\infty} \frac{A}{n} \quad \text{with } n \in \mathbb{N} \quad . \tag{4.21}$$

The effect of an infinitesimal transformation step can be expanded in a Taylor series, which may be ended after the linear term because of the infinitesimal small a:

$$\begin{aligned} \phi(x) \xrightarrow{V_{\text{INF}}} \phi(x') &= \phi(x + a) \\ &= \phi(x) + \frac{i}{\hbar}(-i\hbar) a^\nu g_{\nu\mu} \mathrm{d}^\mu \phi(x) + \mathcal{O}(a^2) \end{aligned} \tag{4.22}$$

Comparison with the general form of an infinitesimal transformation

$$\Gamma_{\text{INF}} \overset{(4.7)}{=} I + \frac{i}{\hbar} w\gamma \tag{4.23}$$

with the generator γ and the real parameter w shows, that $-i\hbar(\mathrm{d}^\mu)$ is the generator of the coordinate translation with the real parameter $a \equiv (a^\nu)$. Thus the generator turns out to be the well-known energy-momentum operator (multiplied by -1)

$$\begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} H/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = i\hbar \begin{pmatrix} \mathrm{d}^0 \\ \mathrm{d}^1 \\ \mathrm{d}^2 \\ \mathrm{d}^3 \end{pmatrix} = i\hbar \begin{pmatrix} \mathrm{d}_0 \\ -\mathrm{d}_1 \\ -\mathrm{d}_2 \\ -\mathrm{d}_3 \end{pmatrix} \tag{4.24}$$

with the components

$$\begin{aligned} p_0 = p^0 = H/c &= i\hbar \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}t} \ , \\ p_1 = \frac{\hbar}{i} \frac{\mathrm{d}}{\mathrm{d}x^1} \ , \ p_2 = \frac{\hbar}{i} \frac{\mathrm{d}}{\mathrm{d}x^2} \ , \ p_3 &= \frac{\hbar}{i} \frac{\mathrm{d}}{\mathrm{d}x^3} \ . \end{aligned} \tag{4.25}$$

The energy operator $i\hbar \frac{\mathrm{d}}{\mathrm{d}t}$ thus is (up to the constant factor $-1/c$) the generator of a passive translation in time-direction. And the momentum operator's k-component $\frac{\hbar}{i} \frac{\mathrm{d}}{\mathrm{d}x^k}$ is (up to the constant factor -1) the generator of a passive translation in space-direction x^k.

The necessary symmetry condition (4.15)

$$(\partial_\rho \mathcal{L}) w\gamma x^\rho = -(\partial_\rho \mathcal{L}) a^\nu i\hbar \mathrm{d}_\nu x^\rho = -(\partial_\rho \mathcal{L}) a^\rho i\hbar = 0 \tag{4.26}$$

is fulfilled if and only if the Lagrangian at least does not explicitly depend of those space-time-directions, into which the coordinates system is being shifted (i.e. those directions, for which $a^\rho \neq 0$), because then for these directions $\partial_\rho \mathcal{L} = 0$.

A vector function $\mathcal{G}$, with which the sufficient symmetry condition (4.11) is fulfilled, can be found by applying the infinitesimal transformation to the Lagrangian:

$$\mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}' = \mathcal{L} + \frac{i}{\hbar}a^\rho g_{\rho\sigma}(-i\hbar)\mathrm{d}^\sigma\mathcal{L} = \mathcal{L} + a^\rho \mathrm{d}_\rho\mathcal{L} = \mathcal{L} + \mathrm{d}_\rho\mathcal{G}^\rho \tag{4.27}$$

As a is constant, this condition can be met for arbitrary Lagrangians $\mathcal{L}$ by choosing

$$\mathcal{G}^\rho \equiv a^\rho\mathcal{L} \ . \tag{4.28}$$

The conserved current density's components follow from

$$j^\rho \overset{(4.16)}{\equiv} C\Big(\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)} w\gamma\phi_r + i\hbar\mathcal{G}^\rho\Big) \ . \tag{4.29}$$

Inserting $w = (a^\nu)$, $\gamma = -i\hbar(\mathrm{d}^\mu)$, and $\mathcal{G}^\rho = a^\rho\mathcal{L}$, one finds

$$j^\rho = Ca^\nu\Big(-\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)} i\hbar g_{\nu\mu}\mathrm{d}^\mu\phi_r + i\hbar g^\rho{}_\nu\mathcal{L}\Big) \ . \tag{4.30}$$

As the four components of the coordinate translation a are independent from another, there are four conserved current densities. Thus we may choose the constant

$$Ca^\nu \equiv -\frac{g^{\sigma\nu}}{i\hbar} \quad \text{with } \sigma = 0,1,2,3 \ , \tag{4.31}$$

and the four conserved current densities become

$$j^\rho = \sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\mathrm{d}^\sigma\phi_r - g^{\rho\sigma}\mathcal{L} \equiv \mathcal{T}^{\rho\sigma} \quad \text{with } \sigma = 0,1,2,3 \ . \tag{4.32}$$

Each of the four current densities fulfills the continuity equation

$$\mathrm{d}_\rho j^\rho = \mathrm{d}_\rho\mathcal{T}^{\rho\sigma} = 0 \quad \text{with } \sigma = 0,1,2,3 \ . \tag{4.33}$$

For reasons, which will become clear immediately, the tensor $\mathcal{T}$ is called energydensity-stress-tensor, or simply energy-stress-tensor for brevity, or even shorter ES-tensor. Special names are used for this tensor's elements, which also will be explained immediately.

$$(\mathcal{T}^{\rho\sigma}) = \begin{pmatrix} \mathcal{H} & c\mathcal{P}^1 & c\mathcal{P}^2 & c\mathcal{P}^3 \\ \mathcal{S}^{E1}/c & \mathcal{S}^{P11} & \mathcal{S}^{P21} & \mathcal{S}^{P31} \\ \mathcal{S}^{E2}/c & \mathcal{S}^{P12} & \mathcal{S}^{P22} & \mathcal{S}^{P32} \\ \mathcal{S}^{E3}/c & \mathcal{S}^{P13} & \mathcal{S}^{P23} & \mathcal{S}^{P33} \end{pmatrix} \tag{4.34}$$

The four continuity equations are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_0\phi_r)}\,\mathrm{d}^0\phi_r - \mathcal{L}}_{\mathcal{H}}\Bigg) = -\frac{\mathrm{d}}{\mathrm{d}\,x^j}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_j\phi_r)}\,c\,\mathrm{d}^0\phi_r}_{\mathcal{S}^{Ej}}\Bigg) \tag{4.35a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(c\,\mathrm{d}_0\phi_r)}\,\mathrm{d}^1\phi_r}_{\mathcal{P}^1}\Bigg) = -\frac{\mathrm{d}}{\mathrm{d}\,x^j}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_j\phi_r)}\,\mathrm{d}^1\phi_r - g^{j1}\mathcal{L}}_{\mathcal{S}^{P1j}}\Bigg) \tag{4.35b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(c\,\mathrm{d}_0\phi_r)}\,\mathrm{d}^2\phi_r}_{\mathcal{P}^2}\Bigg) = -\frac{\mathrm{d}}{\mathrm{d}\,x^j}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_j\phi_r)}\,\mathrm{d}^2\phi_r - g^{j2}\mathcal{L}}_{\mathcal{S}^{P2j}}\Bigg) \tag{4.35c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(c\,\mathrm{d}_0\phi_r)}\,\mathrm{d}^3\phi_r}_{\mathcal{P}^3}\Bigg) = -\frac{\mathrm{d}}{\mathrm{d}\,x^j}\Bigg(\underbrace{\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_j\phi_r)}\,\mathrm{d}^3\phi_r - g^{j3}\mathcal{L}}_{\mathcal{S}^{P3j}}\Bigg) \tag{4.35d}$$

In the continuity equation (4.35a), which follows from the field's symmetry under translation along the time axis, we recognize the energy density (= Hamiltonian) $\mathcal{H}$ of equation (3.60a):

$$\sum_r \frac{\partial\mathcal{L}}{\partial\dot{\phi}_r}\,\dot{\phi}_r - \mathcal{L} = \sum_r \pi_r\dot{\phi}_r - \mathcal{L} = \mathcal{H} \tag{4.36}$$

The decrease of energy density $\mathcal{H}$ of the fields ϕ_r at any point of space-time equals the the three-dimensional divergence of the energy current density at the same point. Thus energy is the "charge" of the fields, which is conserved due to their symmetry under time-like translations.

Alternatively we could have proceeded in reverse direction: Instead of tediously deriving in section 3.2.2 the canonically conjugate momentum density and the Hamiltonian by discretization of the fields, one can immedi-

ately write down equation (4.35) and postulate that the the null-component of the current density, which is conserved due to invariance of the fields under translation along the time-like coordinates axis, must be – up to a possible constant factor f – the fields energy density.

$$f \cdot \int \mathrm{d}^3x \left(\frac{1}{c} \sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_0 \phi_r)} \mathrm{d}^0 \phi_r - \frac{\mathcal{L}}{c} \right) = \int \mathrm{d}^3x \, \mathcal{H} \tag{4.37}$$

The space integral over the energy density must correspond to the Hamilton function (3.20) of point mechanics:

$$H = \sum_{j=1}^{n} p_j \dot{q}_j - L \tag{4.38}$$

Thus for the fields

$$\int \mathrm{d}^3x \, \mathcal{H} = \int \mathrm{d}^3x \left(\sum_r \pi_r \dot{\phi}_r - \mathcal{L} \right) \tag{4.39}$$

must hold. From comparison of (4.37) and (4.39) follows

$$f = c \quad \text{and} \quad \pi_r = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \, . \tag{4.40}$$

The "charge", wich is conserved due to translation symmetry of the fields in space-like x^k-direction, is *by definition* their momentum in k-direction. The decrease of momentum density $\mathcal{P}^k$ equals the divergence of momentum current density $\mathcal{S}^{Pk}$. The meaning of these definitions becomes more clear, when (4.35) is integrated over the space volume V, and the volume integral over the divergence is according to Gauß' theorem converted into an integral over the surface of V:

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_V \mathrm{d}^3x \, \mathcal{H} = \int\limits_{O(V)} \mathrm{d}f_j \, \mathcal{S}^{Ej} \tag{4.41a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V}\mathrm{d}^3x\,\mathcal{P}^k=\int\limits_{O(V)}\mathrm{d}f_j\,\mathcal{S}^{Pkj} \tag{4.41b}$$

The decrease of energy within the volume V equals the amount of energy, which streams out through the surface $O(V)$. Thus $\mathcal{S}^{Ej}$ is the density of the energy stream, which flows in space direction x^j. Respectively $\mathcal{S}^{Pkj}$ is the density of the k-component of the momentum flowing in x^j-direction.

Different from energy density

$$\mathcal{H}\overset{(4.35)}{=}\sum_r\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_0\phi_r)}\mathrm{d}^0\phi_r-\mathcal{L}\;, \tag{4.42}$$

the total momentum density

$$\mathcal{P}^j\overset{(4.35)}{=}\sum_r\frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_0\phi_r)}\mathrm{d}^j\phi_r\equiv\sum_r\mathcal{P}^j_r \tag{4.43}$$

can be written as the sum of the momentum densities $\mathcal{P}^j_r$ of the single fields. But only the sum of the momentum densities is a conserved quantity, not the momentum density of a single field.

The ES-tensor's dimension is

$$\left[(\mathcal{T}^{\rho\sigma})\right]=\frac{\text{energy}}{\text{volume}}\;. \tag{4.44}$$

Thus the dimension of the momentum flow densities is

$$\left[\mathcal{S}^{Pkj}\right]=\frac{\text{energy}}{\text{volume}}=\frac{\text{force}}{\text{area}}\;.$$

In the surface integral on the right-hand side of (4.41b), in case $k=i$ the integrand refers to the pressure, which the field is exerting onto the volume's surface. In case $k\neq i$ it refers to the shear forces, which the field is exerting tangentially to the surface of V. For this reason, $(\mathcal{T}^{\rho\sigma})$ is called energydensity-stress-tensor. We want to introduce a notation, which clearly discriminates between shear forces and pressure. We define the mean pressure

$$\mathrm{P} \equiv \frac{1}{3}\left(\mathcal{S}^{P11} + \mathcal{S}^{P22} + \mathcal{S}^{P33}\right) \tag{4.45a}$$

(not to be confused with momentum density $\mathcal{P}$!), and the stress tensor with the 9 components

$$\sigma^{kj} \equiv \mathcal{S}^{Pkj} + g^{kj}\mathrm{P} \ . \tag{4.45b}$$

With these definitions, the ES-tensor can be written in the form

$$(\mathcal{T}^{\rho\sigma}) = (4.34) = \begin{pmatrix} \mathcal{H} & c\mathcal{P}^j \\ \mathcal{S}^{Ek}/c & (\sigma^{kj} - g^{kj}\mathrm{P}) \end{pmatrix} \ . \tag{4.46}$$

In case of isotropic field we get:

$$\begin{aligned} \text{Isotropy} \implies \mathcal{P}^j &= \mathcal{S}^{Ej} = \sigma^{kj} = 0 \\ \mathcal{S}^{P11} &= \mathcal{S}^{P22} = \mathcal{S}^{P33} = P \end{aligned} \tag{4.47}$$

Assuming isotropy when averaging over very large space scales, thus the ES-tensor of arbitrary fields becomes a diagonal matrix:

$$\text{Isotropy} \implies (\mathcal{T}^{\rho\sigma}) = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & -g^{kj}\mathrm{P} \end{pmatrix}$$

$(-\mathrm{P}g^{\rho\sigma})$ with constant P obviously is a Lorentz-tensor, as we know for sure that $(g^{\rho\sigma})$ is a Lorentz-tensor. $(\mathcal{T}^{\rho\sigma})$ could not be a Lorentz-tensor if it was identical with $(-\mathrm{P}g^{\rho\sigma})$ in all components except for $(\mathcal{T}^{00}) \neq (-\mathrm{P}g^{00})$. Thus Lorentz-invariance implies

$$\text{Isotropy} \implies (\mathcal{T}^{\rho\sigma}) = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & -g^{kj}\mathrm{P} \end{pmatrix} = (-g^{\rho\sigma}\mathrm{P}) \ . \tag{4.48}$$

Final remark: The the momentum density $\mathcal{P}_r^j$ of a field ϕ_r must not be confused with its canonically conjugate momentum density

$$\pi_r \overset{(3.57)}{\equiv} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_0\phi_r)} \ . \tag{4.49}$$

For the components of the physical momentum,

$$\mathcal{P}_r^k \overset{(4.35)}{=} \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_0\phi_r)}\mathrm{d}^k\phi_r \overset{(3.57)}{=} \pi_r \mathrm{d}^k\phi_r \ . \tag{4.50}$$

The physical momentum density $\mathcal{P}$ is defined by its correlation with the space-like translation symmetry of the field, same as energy density $\mathcal{H}$ is defined by its correlation with the time-like translation symmetry of the field. The canonically conjugate momentum density π_r is a construct of Hamilton-formalism, and is defined by its correlation to the field's amplitude. It even may have a dimension different from momentum/volume. The product $\pi_r\phi_r$ of canonically conjugate momentum density and field amplitude always has the dimension action/volume. Thus the conjugate momentum density has the dimension momentum/volume if and only if the field's amplitude has the dimension length. We often will investigate fields, which have the dimension $1/\sqrt{\text{volume}}$. Their canonically conjugate momentum density has the dimension action$/\sqrt{\text{volume}}$. The physical momentum is canonically conjugate to the position vector $\boldsymbol{x}$. Thus both have three components. $\pi_r(x)$ has, same as the scalar field $\phi_r(x)$, only one component. In the case of vector fields or spinor fields, for each field component one component of its canonically conjugate momentum density is defined according to (4.49).

4.3 Rotations

After we got to know the four-dimensional momentum operator (multiplied by -1) as generator of passive translations of scalar fields, it is plausible to guess, that the four-dimensional angular momentum operator (multiplied by -1) might be the generator of passive rotations of scalar fields. But it will turn out immediately, that the *positive* angular momentum operator is that generator. This holds true at least if the coordinate axes are defined as usual as a right-handed system, and rotation angles are counted counter-clockwise.

There are six mutually orthogonal planes in four-dimensional space-time: The space-time planes $10 = xct$, $20 = yct$, $30 = zct$, and the space-space planes $23 = yz$, $31 = zx$, $12 = xy$. Suppose that a primed coordinates

system K' is rotated relatively to the unprimed coordinates system K in 12-plane by an infinitesimal small positive angle ω^{12} as depicted in figure 4.1. At space-time point A, a maximum of a scalar field $\phi(x)$ is located:

$$\phi_{\max} = \phi \begin{pmatrix} 0' \\ \mathsf{A}' \\ 0' \\ 0' \end{pmatrix} = \phi \begin{pmatrix} 0 \\ \mathsf{A} \\ \omega^{12}\,\mathsf{A} \\ 0 \end{pmatrix} \quad . \tag{4.51}$$

A minimum of the field is located at space-time point B:

$$\phi_{\min} = \phi \begin{pmatrix} 0' \\ 0' \\ \mathsf{B}' \\ 0' \end{pmatrix} = \phi \begin{pmatrix} 0 \\ -\omega^{12}\,\mathsf{B} \\ \mathsf{B} \\ 0 \end{pmatrix} \quad . \tag{4.52}$$

Apparently the general relation is

$$\phi \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \phi \begin{pmatrix} x^0 \\ x^1 - \omega^{12}\,x^2 \\ x^2 + \omega^{12}\,x^1 \\ x^3 \end{pmatrix} = \phi(x) + \Big(\omega^{12} x^1 \mathrm{d}_2 - \omega^{12} x^2 \mathrm{d}_1\Big)\phi(x) \; . \tag{4.53}$$

The Taylor series may be ended after the linear term because of the infinitesimal small angle ω^{12}. We insert the factor $1 = -g_{11} = -g_{22}$ two times, and

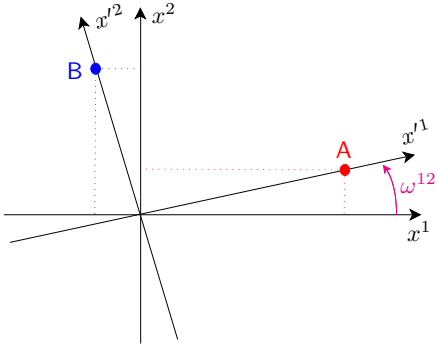


Fig. 4.1 : coordinate rotation

pull up the indices of d_2 and of d_1:

$$\begin{aligned}\phi(x') &= \phi(x) - \omega^{12}\Big(g_{11}x^1\mathrm{d}_2 - g_{22}x^2\mathrm{d}_1\Big)\phi(x)\\ &= \phi(x) - \omega^{\alpha\beta}\Big(g_{\alpha 1}x^1 g_{\beta 2}\mathrm{d}^2 - g_{\beta 2}x^2 g_{\alpha 1}\mathrm{d}^1\Big)\phi(x)\end{aligned}$$

$$\begin{aligned}\phi(x') &= \phi(x) - \omega_{\sigma\tau}g^{\sigma\alpha}g^{\tau\beta}\Big(g_{\alpha 1}x^1 g_{\beta 2}\mathrm{d}^2 - g_{\beta 2}x^2 g_{\alpha 1}\mathrm{d}^1\Big)\phi(x)\\ &= \phi(x) - \omega_{\sigma\tau}\Big(g^{\sigma}{}_1 x^1 g^{\tau}{}_2\mathrm{d}^2 - g^{\tau}{}_2 x^2 g^{\sigma}{}_1\mathrm{d}^1\Big)\phi(x)\end{aligned} \tag{4.54}$$

This can be generalized to an arbitrary coordinates plane: Any infinitesimal rotation of coordinates can be achieved by linear combination of the rotation angles Ω^{10},Ω^{20},Ω^{30},Ω^{23},Ω^{31},Ω^{12} in the six space-time planes. The angles are anti-symmetric under index permutations: $\Omega^{\sigma\tau} = -\Omega^{\tau\sigma}$. We arrange them in a skew-symmetric matrix:

$$\Omega = (\Omega^{\sigma\tau}) \equiv \begin{pmatrix} 0 & -\Omega^{10} & -\Omega^{20} & -\Omega^{30} \\ \Omega^{10} & 0 & \Omega^{12} & -\Omega^{31} \\ \Omega^{20} & -\Omega^{12} & 0 & \Omega^{23} \\ \Omega^{30} & \Omega^{31} & -\Omega^{23} & 0 \end{pmatrix} \tag{4.55}$$

Because of $g^{00} = -g^{11} = -g^{22} = -g^{33} = +1$, space-time-angles change sign, if the indices are pulled to the covariant (bottom) position, while the sign of space-time-angles remains unchanged. Normally we don't need to take care of that, as our formulas "automatically" account for it.

We replace the indices 1 and 2 in (4.54) by the general indices κ and μ. Furthermore a factor $\frac{1}{2}$ is inserted, which will be justified immediately. The coordinates rotations are named R:

$$\begin{aligned}\phi(x') = R_{\text{INF}}\phi(x) &= \Big(I - \frac{1}{2}\omega_{\sigma\tau}(g^{\sigma}{}_{\kappa}x^{\kappa}g^{\tau}{}_{\mu}\mathrm{d}^{\mu} - g^{\tau}{}_{\nu}x^{\nu}g^{\sigma}{}_{\rho}\mathrm{d}^{\rho})\Big)\phi(x)\\ &= \Big(I + \frac{i}{2\hbar}\omega_{\sigma\tau}i\hbar(x^{\sigma}\mathrm{d}^{\tau} - x^{\tau}\mathrm{d}^{\sigma})\Big)\phi(x)\\ \phi(x') = R_{\text{INF}}\phi(x) &= \Big(I + \frac{i}{2\hbar}\omega_{\sigma\tau}J^{\sigma\tau}\Big)\phi(x)\end{aligned} \tag{4.56}$$

Thus

$$\boxed{J^{\sigma\tau} \equiv i\hbar(x^\sigma \mathrm{d}^\tau - x^\tau \mathrm{d}^\sigma)} \tag{4.57}$$

$$= i\hbar(x^\sigma g^{\tau\mu} - x^\tau g^{\sigma\mu})\mathrm{d}_\mu \tag{4.58}$$

is the generator of an infinitesimal passive rotation

$$\boxed{R_{\text{INF}} = I + \frac{i}{2\hbar}\omega_{\sigma\tau}J^{\sigma\tau}} \tag{4.59}$$

of the scalar field $\phi(x)$.

$$J^{\tau\sigma} = -J^{\sigma\tau} \tag{4.60}$$

is evident from (4.57). As the diagonal elements of Ω are zero, the summation over $\sigma = 0, 1, 2, 3$ and $\tau = 0, 1, 2, 3$ results in the product

$$\begin{aligned}\frac{1}{2}\Omega_{\sigma\tau}J^{\sigma\tau} = \frac{1}{2}(&\Omega_{10}J^{10} + \Omega_{20}J^{20} + \Omega_{30}J^{30} + \Omega_{10}J^{10} + \\ &+ \Omega_{12}J^{12} + \Omega_{31}J^{31} + \Omega_{20}J^{20} + \Omega_{12}J^{12} + \\ &+ \Omega_{23}J^{23} + \Omega_{30}J^{30} + \Omega_{31}J^{31} + \Omega_{23}J^{23}) = \\ = &\Omega_{10}J^{10} + \Omega_{20}J^{20} + \Omega_{30}J^{30} + \\ &+ \Omega_{23}J^{23} + \Omega_{31}J^{31} + \Omega_{12}J^{12}\ .\end{aligned} \tag{4.61}$$

This is just the correct sum over the products of the six parameters and the six generators, and elucidates, why in (4.56) the factor $\frac{1}{2}$ had to be inserted.

We now will demonstrate that the generator $J = (4.57)$ is nothing other than the four-dimensional angular momentum operator. The angular momentum in three-dimensional position space can be written by means of the cross product

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} = \begin{pmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_z \end{pmatrix} = \begin{pmatrix} x^2p^3 - x^3p^2 \\ x^3p^1 - x^1p^3 \\ x^1p^2 - x^2p^1 \end{pmatrix} \equiv \begin{pmatrix} J^{23} \\ J^{31} \\ J^{12} \end{pmatrix}\ .$$

In the syntax J^{jk}, the indices jk specify a rotation in the $x^j x^k$-plane. The generalization to angular momenta in four-dimensional space-time is evident:

$$J^{\sigma\tau} \equiv x^\sigma p^\tau - x^\tau p^\sigma \quad \text{with} \begin{cases} \sigma = 0,1,2,3 \\ \tau = 0,1,2,3 \end{cases} \tag{4.62}$$

The angular momentum operator $J^{\sigma\tau}$ can be found by replacing the components p^ν of four-dimensional momentum by the components $i\hbar \mathrm{d}^\nu$ of the four-dimensional momentum operator. This directly leads to equation (4.57). As usual, we use identical letters for operators and for their Eigenvalues, i.e. we (usually) don't mark operators by hats, but leave the discrimination to the reader's attention.

The angles $\Omega^{\sigma\tau}$ of finite coordinates rotations are related to infinitesimal rotation angles by

$$\omega^{\sigma\tau} \equiv \lim_{n\to\infty} \frac{1}{n} \Omega^{\sigma\tau} \quad \text{with } n \in \mathbb{N} \,. \tag{4.63}$$

Finite passive rotations of scalar fields are achieved by concatenating infinitely many infinitesimal transformations:

$$R = \lim_{n\to\infty} \left(I + \frac{1}{n} \frac{i}{2\hbar} \Omega_{\sigma\tau} J^{\sigma\tau} \right)^n \quad \text{with } n \in \mathbb{N}$$

$$\boxed{R = \exp\left\{ \frac{i}{2\hbar} \Omega_{\sigma\tau} J^{\sigma\tau} \right\}} \tag{4.64}$$

Under which conditions are rotations of the four-dimensional space-time-coordinates symmetries of scalar fields $\phi_r(x)$, and which conservation laws correlate with these symmetries? The equations of the fields ϕ_r are derived from the Lagrangian $\mathcal{L}$. According to (4.15), the necessary (but not sufficient) condition for a transformation Γ with generator γ to be a symmetry of ϕ is given by

$$(\partial_\rho \mathcal{L}) w\gamma x^\rho = 0 \,. \tag{4.65}$$

Insertion of $\frac{1}{2}\omega_{\sigma\tau} J^{\sigma\tau}$ for $w\gamma$ with $J^{\sigma\tau} \overset{(4.58)}{=} i\hbar(x^\sigma g^{\tau\mu} - x^\tau g^{\sigma\mu})\mathrm{d}_\mu$ results in

$$0 = (\partial_\rho \mathcal{L}) \frac{1}{2} \omega_{\sigma\tau} i\hbar (x^\sigma g^{\tau\mu} - x^\tau g^{\sigma\mu}) \mathrm{d}_\mu x^\rho$$
$$= (\partial_\rho \mathcal{L}) \frac{1}{2} \omega_{\sigma\tau} i\hbar (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \ . \tag{4.66}$$

$(x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho})$ can only be different from zero, if $\rho = \sigma$ or $\rho = \tau$. Thus the condition is met, if the Lagrangian at least does not explicitly depend on the space-time directions σ and τ, in which the rotation takes place, because then $\partial_\sigma \mathcal{L} = \partial_\tau \mathcal{L} = 0$.

The sufficient symmetry condition (4.11)

$$\exists\, \mathcal{G} : \quad \mathcal{L} \xrightarrow{I + \frac{i}{\hbar} w\gamma} \mathcal{L}' = \mathcal{L} + \frac{i}{\hbar} w\gamma \mathcal{L} = \mathcal{L} + \mathrm{d}_\rho \mathcal{G}^\rho$$

results by insertion of $\frac{1}{2}\omega_{\sigma\tau} J^{\sigma\tau}$ for $w\gamma$ into

$$\frac{i}{\hbar} w\gamma \mathcal{L} = \mathrm{d}_\rho \mathcal{G}^\rho = \frac{i}{\hbar} \frac{1}{2} \omega_{\sigma\tau} J^{\sigma\tau} \mathcal{L}$$
$$\overset{(4.58)}{=} \frac{i}{\hbar} \frac{1}{2} \omega_{\sigma\tau} i\hbar (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathrm{d}_\rho \mathcal{L}$$
$$= -\mathrm{d}_\rho \frac{1}{2} \omega_{\sigma\tau} (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathcal{L} \ . \tag{4.67}$$

We check the last step:

$$\mathrm{d}_\rho (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathcal{L} =$$
$$= (\underbrace{g^\sigma{}_\rho g^{\tau\rho} - g^\tau{}_\rho g^{\sigma\rho}}_{0}) \mathcal{L} + (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathrm{d}_\rho \mathcal{L}$$

The first summand in fact is zero, therefore in (4.67) the differential operator may be pulled left, and one finds the vector function

$$\mathcal{G}^\rho \equiv -\frac{1}{2} \omega_{\sigma\tau} (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathcal{L} \ , \tag{4.68}$$

which fulfills the sufficient symmetry condition. Into the conserved current density formula

$$j^\rho \overset{(4.16)}{=} C\left(\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)} w\gamma \phi_r + i\hbar \mathcal{G}^\rho\right)$$

we insert $\frac{1}{2}\omega_{\sigma\tau} J^{\sigma\tau}$ for $w\gamma$ and (4.68) for $\mathcal{G}^\rho$:

$$\begin{aligned} j^\rho &= C\left(\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)} \frac{1}{2}\omega_{\sigma\tau} J^{\sigma\tau}\phi_r - i\hbar \frac{1}{2}\omega_{\sigma\tau}(x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho})\mathcal{L}\right) \\ &= \sum_{\sigma\tau=10,20,30,23,31,12} C i\hbar \omega_{\sigma\tau} \cdot \\ &\quad \cdot \left(\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)}(x^\sigma \mathrm{d}^\tau - x^\tau \mathrm{d}^\sigma)\phi_r - (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho})\mathcal{L}\right) \end{aligned}$$

Here the summation is restricted to the six linearly independent components of the skew-symmetric tensor $\omega_{\sigma\tau}$. Therefore the factor 1/2 was dropped. As the six $\omega_{\sigma\tau}$ are independent, a separate equation of continuity holds for each of them. Choosing the constant

$$C \equiv \frac{1}{ci\hbar\omega_{\sigma\tau}} \quad \text{with } \sigma\tau = 10, 20, 30, 23, 31, 12 \ , \tag{4.69}$$

we find the components of the six conserved current densities

$$\begin{aligned} j^\rho &= x^\sigma\left(\frac{1}{c}\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)}\mathrm{d}^\tau \phi_r - g^{\tau\rho}\frac{\mathcal{L}}{c}\right) - x^\tau\left(\frac{1}{c}\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)}\mathrm{d}^\sigma \phi_r - g^{\sigma\rho}\frac{\mathcal{L}}{c}\right) \\ &\overset{(4.32)}{=} x^\sigma \frac{\mathcal{T}^{\rho\tau}}{c} - x^\tau \frac{\mathcal{T}^{\rho\sigma}}{c} \quad \text{with } \sigma\tau = 10, 20, 30, 23, 31, 12 \ . \end{aligned} \tag{4.70}$$

Here the energydensity-stress tensor $\mathcal{T}$ reappears, which was defined in (4.32). The conserved current densities can be combined to the angular-momentum tensor (strictly speaking: the angular-momentum-density tensor). It's components are

$$\boxed{\mathcal{M}^{\rho\sigma\tau} \equiv x^\sigma \frac{\mathcal{T}^{\rho\tau}}{c} - x^\tau \frac{\mathcal{T}^{\rho\sigma}}{c}} \ . \tag{4.71}$$

The angular-momentum tensor's dimension is

$$[\,\mathcal{M}^{\rho\sigma\tau}\,] = \frac{\text{length}\cdot\text{energy}\cdot\text{time}}{\text{volume}\cdot\text{length}} = \frac{\text{action}}{\text{volume}}\ . \tag{4.72}$$

$\mathcal{M}$ fulfills the six linearly independent equations of continuity

$$\mathrm{d}_\rho \mathcal{M}^{\rho\sigma\tau} = 0 \quad \text{for } \sigma\tau = 10, 20, 30, 23, 31, 12 \quad . \tag{4.73}$$

The six conserved angular momenta are the volume integrals over the null-components of the conserved current densities:

$$M^{\sigma\tau} \equiv \int \mathrm{d}^3x\, \mathcal{M}^{0\sigma\tau} \quad \text{for } \sigma\tau = 10, 20, 30, 23, 31, 12 \tag{4.74}$$

The purely space-like angular momenta M^{23}, M^{31}, M^{12} are well known. They are of the form

$$M^{jk} = \int \mathrm{d}^3x \left(x^j \mathcal{P}^k(x) - x^k \mathcal{P}^j(x)\right) , \tag{4.75}$$

with $(\mathcal{P}^j) \overset{(4.34)}{=} (\mathcal{T}^{0j}/c)$ being the fields momentum densities. In contrast, a pictorial interpretation of the space-time-like angular momenta M^{10}, M^{20}, M^{30} is difficult:

$$M^{j0} = \int \mathrm{d}^3x \left(x^j \frac{\mathcal{H}(\boldsymbol{x})}{c} - c\,t\,\mathcal{P}^j(\boldsymbol{x})\right) \tag{4.76}$$

$\mathcal{H}$ is the fields energy density. M^{j0} can have the unhandy feature to be the small difference of two large numbers, respectively it certainly will get this feature in course of time. M^{j0} is the curiosity of a conserved quantity, which explicitly depends on time. Formally, the space-time-like angular momenta are acceptable conserved quantities, but – being the difference of diverging numbers – they are hardly seizable by measurements, and therefore irrelevant to the experimentalist.

There is a further reason to ignore the M^{j0}: In the second part of this book, we will quantize energy density, momentum density, and the position vector. But a hermitian time operator is unknown to quantum theory. Therefore (4.76) anyway is useless for quantum field theory, and we will

only consider the purely space-like angular momenta M^{23}, M^{31}, M^{12} in the following chapters.

In this section, we restricted our considerations to the angular momentum of scalar fields. In section 5.7, we will extend the theory to the angular momentum of vector fields, and in section 6.4 to the angular momentum of spinor fields.

4.4 Global Phase Transformations

In case of an inner symmetry, the necessary (but not sufficient) symmetry condition

$$(\partial_\rho \mathcal{L}) w\gamma x^\rho \overset{(4.15)}{=} 0$$

is fulfilled by $w\gamma x^\rho = 0$.

We will investigate a global phase transformation U, which rotates the phase of a field $\phi(x)$ in all points x of space-time by the angle $Kq/\hbar$:

$$\phi(x) \xrightarrow{U} \phi'(x) = e^{\frac{i}{\hbar}Kq}\phi(x) \quad \text{with } K, q \in \mathbb{R} \tag{4.77}$$

This should be compared to the general form (4.9) and (4.10)! The generator q in this case is a constant, real quantity, which is called the *charge* of the field ϕ. The charge could be an electrical charge, or any other additive quantity which is characteristic for this field. For a real field, this transformation would be meaningless. Only complex fields can have a conserved charge, which is based on the invariance of the field equation under phase transformations. The phase transformation (4.77) is called "global", because the real parameter K is identical at all points of space-time. As the exponent must be dimension-less, the dimension of K is action/charge.

The space-time coordinates are not affected by the transformation U. Therefore the necessary symmetry condition (4.15)

$$(\partial_\rho \mathcal{L}) \underbrace{w\gamma x^\rho}_{=0} = 0 \tag{4.78}$$

is fulfilled for any field. To find out whether there exists a $\mathcal{G}$, with which

also the sufficient symmetry condition (4.11)

$$\exists \mathcal{G}: \quad \mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}' = \mathcal{L} + \frac{i}{\hbar} w\gamma \mathcal{L} = \mathcal{L} + \mathrm{d}_\rho \mathcal{G}^\rho \tag{4.79}$$

can be fulfilled, we consider the infinitesimal transformation. Because of

$$e^{\frac{i}{\hbar}Kq} = \lim_{n\to\infty} \left(1 + \frac{i}{\hbar} \underbrace{\frac{K}{n}}_{\equiv k} q\right)^n \quad \text{with } n \in \mathbb{N}\ , \tag{4.80}$$

the condition in this case is

$$\phi(x) \xrightarrow{I+\frac{i}{\hbar}w\gamma} \phi'(x) = \phi(x) + \frac{i}{\hbar} kq\phi(x)\ . \tag{4.81}$$

Without detailed informations on $\mathcal{L}$, it's impossible to check the symmetry condition (4.79). We must restrict the evaluation to a certain Lagrangian, and choose the Dirac-field for this purpose. It's field equation can be derived from the Lagrangian

$$\mathcal{L} = \phi^\dagger \gamma^0 (i\hbar c \gamma^\rho \mathrm{d}_\rho - mc^2)\phi\ . \tag{4.82}$$

We will discuss the Dirac-field thoroughly in chapter 8. Here we merely mention, that ϕ is a four-spinor, i.e. a field with four components in an abstract four-dimensional spinor-space, which is a space completely different from four-dimensional position-time space. $\phi^\dagger$ is the spinor, which is adjunct to ϕ, i.e. it is the transposed complex-conjugate form of ϕ. The factors γ^μ are 4×4 matrices in spinor space with constant, complex elements.

The transformed adjunct spinor is

$$\phi^{\dagger\prime}(x) = \phi^\dagger(x) - \frac{i}{\hbar} kq\phi^\dagger(x)\ . \tag{4.83}$$

Note the minus sign. Now we can compute the effect of the infinitesimal transformation on $\mathcal{L}$ (terms $\mathcal{O}(k^2)$ can be neglected):

$$\begin{aligned}\mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}' &= (\phi^\dagger - \frac{i}{\hbar}kq\phi^\dagger)\gamma^0(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)(\phi + \frac{i}{\hbar}kq\phi) \\ &= \mathcal{L} + \phi^\dagger\gamma^0(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\frac{i}{\hbar}kq\phi + \\ &\quad - \frac{i}{\hbar}kq\phi^\dagger\gamma^0(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\phi + \mathcal{O}(k^2) \\ &= \mathcal{L} \end{aligned} \tag{4.84}$$

The sufficient symmetry condition (4.79) is fulfilled by

$$\mathrm{d}_\rho \mathcal{G}^\rho = 0 \qquad \Longleftrightarrow \qquad \mathcal{G} = \text{constant} \ . \tag{4.85}$$

We can avoid unnecessary paperwork by choosing $\mathcal{G} \equiv 0$. Then the components of the conserved current density are

$$\begin{aligned} j^\rho &\overset{(4.16)}{=} C\sum_r \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi_r)} w\gamma\phi_r \\ &= C\left(\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho\phi)}kq\phi - \phi^\dagger \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho\phi^\dagger)}kq\right) \\ &= C\left(\phi^\dagger\gamma^0 i\hbar c\gamma^\rho kq\phi - 0\right) \ , \end{aligned} \tag{4.86}$$

from which with $C \equiv 1/(i\hbar k)$ follows

$$j^\rho = q\phi^\dagger\gamma^0 c\gamma^\rho\phi \ . \tag{4.87}$$

In (4.86) we changed the sequence of factors in the second term, to make sure that both terms are numbers, but not spinor matrices. For j holds the continuity equation

$$\mathrm{d}_\rho j^\rho = 0 \tag{4.88a}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{j^0}{c} &= -\mathrm{d}_k j^k \\ \frac{\mathrm{d}}{\mathrm{d}t}q\phi^\dagger\gamma^0\gamma^0\phi &= -\mathrm{d}_k q\phi^\dagger\gamma^0 c\gamma^k\phi \quad . \end{aligned} \tag{4.88b}$$

Integration over the volume V and application of Gauß' theorem results

into

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \mathrm{d}^3x\, q\phi^\dagger\gamma^0\gamma^0\phi}_{Q(V)} = -\int_{O(V)} \mathrm{d}f_k\, \underbrace{q\phi^\dagger\gamma^0 c\gamma^k\phi}_{j^k} \,. \tag{4.89}$$

The increase of the charge $Q(V)$ contained within V equals the current, which is flowing into the volume through it's surface. No charge can emerge out of nothing or vanish into nothing.

4.5 Gauge Transformations

In section 4.4 we explored a phase transformation, whose real parameter K was a constant. The field's phase shift therefore was identical in all points of space. That is called a "global" phase transformation.

Now we will investigate a transformation $U(x)$, which rotates the phase of the field $\phi(x)$ at space-time point x by the angle $\frac{1}{\hbar}K(x)q$:

$$\phi(x) \xrightarrow{U(x)} \phi'(x) = e^{\frac{i}{\hbar}K(x)q}\phi(x) \quad \text{with } K(x), q \in \mathbb{R} \tag{4.90}$$

The generator q is – same as for the global transformation – a charge of the field ϕ, i. e. a constant, real quantity, which is characteristic for the field. In contrast the real parameter K (whose dimension is action/charge) now is a function of x, and can have different values at different points of space and time. We only fix the restriction, that the function $K(x)$ must be analytic, i. e. it must be differentiable to all four space-time coordinates, and the equation

$$\mathrm{d}_\rho \mathrm{d}_\nu K = \mathrm{d}_\nu \mathrm{d}_\rho K \quad \text{for } \nu, \rho = 0, 1, 2, 3 \tag{4.91}$$

must hold. As the space-time coordinates are not affected by the phase transformation, the necessary symmetry condition (4.15)

$$(\partial_\rho \mathcal{L})\underbrace{w\gamma x^\rho}_{=0} = 0 \tag{4.92}$$

is fulfilled for any field. Because of $w\gamma x^\rho = 0$, the invariance under the phase transformation is an "inner" symmetry. To find out whether a $\mathcal{G}$ exists, with which also the sufficient symmetry condition (4.11)

$$\exists\,\mathcal{G}: \quad \mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}\,' = \mathcal{L} + \frac{i}{\hbar} w\gamma\mathcal{L} = \mathcal{L} + \mathrm{d}_\rho \mathcal{G}^\rho$$

can be fulfilled, we consider the infinitesimal transformation. Because of

$$e^{\frac{i}{\hbar}K(x)q} = \lim_{n\to\infty}\left(1 + \frac{i}{\hbar}\underbrace{\frac{K(x)}{n}}_{\equiv k(x)} q\right)^n \quad \text{with } n \in \mathbb{N} \tag{4.93}$$

it has the form

$$\phi(x) \xrightarrow{I+\frac{i}{\hbar}w\gamma} \phi'(x) = \phi(x) + \frac{i}{\hbar}k(x)q\phi(x)\ . \tag{4.94}$$

For definiteness, we again choose the Dirac field with the Lagrangian

$$\mathcal{L} = \phi^\dagger\gamma^0(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\phi\ . \tag{4.95}$$

As $k(x)$ is a function of the space-time coordinates, now an additional term shows up in the transformation of $\mathcal{L}$:

$$\begin{aligned}
\mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}' &= (\phi^\dagger - \frac{i}{\hbar}kq\phi^+)\gamma^0(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)(\phi + \frac{i}{\hbar}kq\phi) = \\
&= \mathcal{L} + \phi^\dagger\gamma^0 i\hbar c\gamma^\rho \frac{i}{\hbar}(\mathrm{d}_\rho k)q\phi + \phi^\dagger\gamma^0 k(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\frac{i}{\hbar}q\phi - \\
&\quad - \frac{i}{\hbar}kq\phi^\dagger\gamma^0(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\phi + \mathcal{O}(k^2) = \mathcal{L} - \phi^\dagger\gamma^0 c\gamma^\rho(\mathrm{d}_\rho k)q\phi
\end{aligned} \tag{4.96}$$

The local phase transformation $U(x)$ is a symmetry of the Dirac field if and only if there is a vector function $\mathcal{G}$, which fulfills the equation

$$\mathrm{d}_\rho \mathcal{G}^\rho(x) = -\phi^\dagger(x)\gamma^0 c\gamma^\rho\Big(\mathrm{d}_\rho k(x)\Big)q\phi(x)\ . \tag{4.97}$$

Such function does not exist. Therefore the local phase transformation $U(x)$

is *not* a symmetry of the Dirac field.

Instead of being satisfied by this result, we now will extend the local phase transformation to a gauge transformation, and we will postulate the invariance under gauge transformations as a law of nature, with which all charged fields must comply.

The local phase transformation is missing the symmetry criterion due to the term with $\mathrm{d}_\rho k$ in equation (4.96). As we demand k to be a function of the space-time coordinates, but not a constant, such kind of term can not be avoided as long as there are differential operators in the Lagrangian. On the other hand, we cannot do without differential operators, because a Lagrangian without differential operators can only lead to static, but not to dynamic field equations.

Actually it is possible, to make the Dirac field invariant under local phase transformations without having to sacrifice the differential operators in the Lagrangian. The solution is to replace the differential operators d_ρ by "phase-transformation-adapted" differential operators D_ρ, which have the transformation properties

$$\begin{aligned} \mathrm{D}_\rho\,\phi \xrightarrow{U(x)} \mathrm{D}'_\rho\,\phi' &= \mathrm{D}'_\rho\, e^{\frac{i}{\hbar}K(x)q}\phi = \mathrm{D}'_\rho\, U(x)\,\phi \\ &= e^{\frac{i}{\hbar}K(x)q}\,\mathrm{D}_\rho\,\phi = U(x)\;\mathrm{D}_\rho\,\phi \end{aligned} \tag{4.98}$$

instead of

$$\mathrm{d}_\rho\phi(x) \xrightarrow{U(x)} \mathrm{d}_\rho\phi'(x) = \mathrm{d}_\rho e^{\frac{i}{\hbar}K(x)q}\phi = e^{\frac{i}{\hbar}K(x)q}\Big(\mathrm{d}_\rho + \frac{i}{\hbar}(\mathrm{d}_\rho K)q\Big)\phi \tag{4.99}$$

with the annoying second term. Different from the always and everywhere identical normal differential operators d_ρ, the operators $D_\rho(x)$ are modified locally by the transformation, with the result that

$$(4.98) \quad\Longrightarrow\quad \mathrm{D}'_\rho(x) = U(x)\;\mathrm{D}_\rho(x)\,U^\dagger(x)\;. \tag{4.100}$$

holds. Therefore they are named covariant differential operators, and we wrote D'_ρ in (4.98). We multiply the time- and space-depending difference between the two types of differential operators by $\frac{\hbar}{iq}$, and give it the name

$$A_\rho(x) \equiv \frac{\hbar}{iq}\Big(\mathrm{D}_\rho(x) - \mathrm{d}_\rho\Big) \ . \tag{4.101}$$

Thus the covariant differential operator is

$$\mathrm{D}_\rho(x) \equiv \mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho(x) \ . \tag{4.102}$$

The covariant differential quotient $\mathrm{D}_\rho\,\phi$ is transformed as follows:

$$\begin{aligned}(\mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho)\phi \xrightarrow{U(x)} (\mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho{}')\phi' = (\mathrm{d}_\rho + \frac{i}{\hbar} q A'_\rho)\phi e^{\frac{i}{\hbar}Kq} = \\ = e^{\frac{i}{\hbar}Kq}(\frac{i}{\hbar}(\mathrm{d}_\rho K) q\phi + \mathrm{d}_\rho\phi + \frac{i}{\hbar} q A_\rho{}'\phi) \ .\end{aligned} \tag{4.103}$$

According to (4.98), this must equal

$$(\mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho{}')\phi' = e^{\frac{i}{\hbar}Kq}(\mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho)\phi \ , \tag{4.104}$$

from which

$$A_\rho(x) \xrightarrow{U(x)} A_\rho{}'(x) = A_\rho(x) - \mathrm{d}_\rho K(x) \tag{4.105}$$

follows. As the dimension of $K(x)$ is action/charge, the dimension of $A(x)$ is

$$\Big[A(x)\Big] = \text{momentum/charge} \ . \tag{4.106}$$

The effect of $U(x)$ upon A is not linear, and the transformed field $A'(x)$ is not proportional to$A(x)$. Thereby this transformation goes beyond the scope of Noether's theorem, and it is *not* possible to derive a conserved current density by following method (4.16).

Both differential operators

$$\mathrm{d}_\rho\phi(x) = \lim_{\Delta x^\rho\to 0}\frac{\phi(x+\Delta x^\rho)-\phi(x)}{\Delta x^\rho} \tag{4.107a}$$

$$\text{and}\quad \mathrm{D}_\rho\,\phi(x) = \lim_{\Delta x^\rho\to 0}\frac{\phi(x+\Delta x^\rho)-\phi(x)}{\Delta x^\rho} + \frac{i}{\hbar}q\,A_\rho(x)$$

$$\equiv \lim_{\Delta X^\rho\to 0}\frac{\phi(x+\Delta X^\rho)-\phi(x)}{\Delta X^\rho} \tag{4.107b}$$

compare the fields value at two points with same infinitesimal distance Δx^ρ. But the presence of A has the same effect, as if D_ρ would apply instead of measure Δx^ρ a differently gauged measure ΔX^ρ. Therefore $A(x)$ is called a gauge field.

"Presence of A" is to be understood literally. We postulate, that the field $A(x)$ is as real as electrons, stars, or black holes, and that it's interaction with the Dirac field is described correctly, if in the Lagrangian the normal differential operator d_ρ is replaced by the covariant differential operator $\mathrm{D}_\rho(x) = \mathrm{d}_\rho + \frac{i}{\hbar}qA_\rho(x)$:

$$\mathcal{L} = \phi^\dagger\gamma^0\Big(i\hbar c\gamma^\rho(\mathrm{d}_\rho + \frac{i}{\hbar}qA_\rho) - mc^2\Big)\phi \tag{4.108}$$

We combine the phase transformation of the field ϕ and the corresponding transformation (4.105) of the gauge field A to the

gauge transformation:

$$\phi(x) \xrightarrow{U(x)} \phi'(x) = \phi(x)e^{\frac{i}{\hbar}K(x)q}$$
$$A_\rho(x) \xrightarrow{U(x)} A'_\rho(x) = A_\rho(x) - \mathrm{d}_\rho K(x)$$
with $K(x) \in \mathbb{R}$, $q \in \mathbb{R}$ constant

, (4.109)

and check the effect of an infinitesimal gauge transformation onto the Lagrangian (4.108):

$$\mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}' = \Big(\phi^\dagger - \frac{i}{\hbar}kq\phi^\dagger\Big)\gamma^0 \cdot \Big(i\hbar c\gamma^\rho\Big(\mathrm{d}_\rho + \frac{i}{\hbar}q(A_\rho - \mathrm{d}_\rho k)\Big) - mc^2\Big) \cdot$$
$$\cdot \Big(\phi + \frac{i}{\hbar}kq\phi\Big)$$
$$= (4.96) + \phi^\dagger\gamma^0 c\gamma^\rho(\mathrm{d}_\rho k)q\phi + \mathcal{O}(k^2) = \mathcal{L} \qquad (4.110)$$

$\mathcal{L} = (4.108)$ is in fact gauge invariant, i.e. the gauge transformation (4.109) is a symmetry of the combination of Dirac-field $\phi(x)$ and gauge-field $A(x)$.

The invariance of charged fields under gauge transformations – briefly called gauge invariance – can not be derived from the theory. It is an experimental fact, found by guessing, i.e. a law of nature. As any law of nature, it can be justified only by the fact, that the conclusions, which can be drawn from it, coincide with all experimatal observations. In search for the field theories of the electro-weak and strong interactions, the natural law of gauge invariance was an important guideline.

To evaluate the properties of the gauge field in more detail, we derive from the Lagrangian (4.108) the field equations

$$\mathrm{d}_\nu \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\nu\phi_r)} - \frac{\partial\mathcal{L}}{\partial\phi_r} \overset{(3.37b)}{=} 0\ . \qquad (4.111)$$

According to the statements in the lines after (3.37b), ϕ, $\phi^\dagger$, and the four components of A must be considered independent variables of $\mathcal{L}$. Therefore we find six field equations. $\phi_r = \phi^\dagger$ results in the Dirac equation

$$\Big(i\hbar c\gamma^\rho(\mathrm{d}_\rho + \frac{i}{\hbar}qA_\rho) - mc^2\Big)\phi = 0\ , \qquad (4.112)$$

which will be investigated in detail in section 8.1. With $\phi_r = \phi$ one finds the equation adjunct to (4.112). We will demonstrate the derivation of this equation only when we will be familiar with the commutator properties of the γ-matrices.

The equation for the gauge field's μ-component follows with $\phi_r = A_\mu$ as

$$-\phi^\dagger\gamma^0 i\hbar c\gamma^\rho \frac{i}{\hbar} q \frac{\partial A_\rho}{\partial A_\mu}\phi = \phi^\dagger\gamma^0 c\gamma^\mu q\phi = 0 \overset{(4.87)}{=} j^\mu \qquad (4.113)$$

$$\text{with } \mu = 0, 1, 2, 3 \ .$$

j^μ is the μ-component of the conserved current density. It's interesting to note, that the current density shows up in the field equation of $A(x)$, but why is it zero? And why is the field $A(x)$ itself not visible at all in this equation? Apparently, some important terms still are missing in the Lagrangian. Particularly there should be derivatives of $A(x)$ in the Lagrangian, in order that we will get a dynamic field equation.

When amending $\mathcal{L}$ we must be careful not to destroy the just established gauge invariance. This can be achieved if the additional terms have the form of a four-dimensional rotation of $A(x)$. In the gauge transformation (4.109), the four-dimensional gradient $\mathrm{d}_\rho K(x)$ shows up as a summand of $A(x)$, and rotation terms of the form

$$F_{\rho\nu} \equiv \mathrm{d}_\rho A_\nu - \mathrm{d}_\nu A_\rho \qquad (4.114)$$

in the Lagrangian are because of

$$\begin{aligned} &\mathrm{d}_\rho(A_\nu - \mathrm{d}_\nu K) - \mathrm{d}_\nu(A_\rho - \mathrm{d}_\rho K) = \\ &\quad = \mathrm{d}_\rho A_\nu - \mathrm{d}_\nu A_\rho \underbrace{-\mathrm{d}_\rho\mathrm{d}_\nu K + \mathrm{d}_\nu\mathrm{d}_\rho K}_{\overset{(4.91)}{=} 0} \end{aligned} \qquad (4.115)$$

harmless with respect to gauge invariance. Furthermore, the rotation terms should occur minimum quadratically in the Lagrangian, to make sure that by computing $\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\nu A)}$, the result will not be a field equation like (4.113), in which A itself doesn't exist any more. Therefore the most simple and self-evident strategy is, to insert into the Lagrangian a summand $vF_{\rho\nu}F^{\rho\nu}$ with rotation terms $F_{\rho\nu}$ according to (4.114) and some constant v, which will be specified immediately.

Lets consider the dimension of the constant v. The dimension of A is momentum/charge. Thus the dimension of $F_{\rho\nu}$ is momentum/(charge·length).

$$[vF_{\rho\nu}F^{\rho\nu}] = [v] \cdot \frac{\text{momentum}^2}{\text{charge}^2 \cdot \text{length}^2} = [\mathcal{L}] = \frac{\text{energy}}{\text{volume}}$$
$$[v] = \frac{\text{energy}}{\text{volume}} \cdot \frac{\text{charge}^2 \cdot \text{length}^2}{\text{momentum}^2} = \frac{\text{charge}^2}{\text{force} \cdot \text{time}^2} \tag{4.116}$$

At this point, we cannot proceed any further with the general notion "charge". We must restrict our discussion to some certain type of charge. We define q as an electrical charge, because this charge is better known to us than any other thype of charge. Then a physical constant with appropriate dimension is already known, i.e. the

$$\text{magnetic field constant} = \mu_0 = 4\pi \cdot 10^{-7} \frac{\text{N} \cdot \text{s}^2}{\text{C}^2} \ . \tag{4.117}$$

Thus we set

$$v = \frac{u}{\mu_0} \quad \text{with } u \in \mathbb{R} \text{ dimension-less} \ , \tag{4.118}$$

and find the Lagrangian

$$\mathcal{L} = \phi^\dagger \gamma^0 \Big(i\hbar c \gamma^\rho (\mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho) - mc^2 \Big) \phi + \frac{u}{\mu_0} F_{\sigma\tau} F^{\sigma\tau} \ . \tag{4.119}$$

As real number u, we insert $u = -1/4$, because only with this value – as will be checked at the end of this section – Maxwell's well-known equations of classical electrodynamics can be derived from the Lagrangian (4.120). Choosing a different value for u would result in a re-definition of the magnetic field constant μ_0. This would not be "wrong", but we have no reason to re-define μ_0.

$u = -1/4$ results into the Lagrangian:

$$\boxed{\mathcal{L} = \phi^\dagger \gamma^0 \Big(i\hbar c \gamma^\rho (\mathrm{d}_\rho + \frac{i}{\hbar} q A_\rho) - mc^2 \Big) \phi - \frac{1}{4\mu_0} F_{\sigma\tau} F^{\sigma\tau}} \ . \tag{4.120}$$

The field equation for the μ-component of the gauge field A becomes

$$\mathrm{d}_\nu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu A_\mu)} \underbrace{- \frac{\partial \mathcal{L}}{\partial A_\mu}}_{(4.113)} =$$

$$= -\frac{1}{4\mu_0} \mathrm{d}_\nu \frac{\partial F_{\sigma\tau} F^{\sigma\tau}}{\partial(\mathrm{d}_\nu A_\mu)} + \underbrace{\phi^\dagger \gamma^0 c \gamma^\mu q \phi}_{j^\mu} = 0 \ . \tag{4.121}$$

Using

$$F_{\sigma\tau} F^{\sigma\tau} = F^{\nu\rho} g_{\sigma\nu} g_{\tau\rho} F^{\sigma\tau} = F^{\nu\rho} F_{\nu\rho} = F^{\sigma\tau} F_{\sigma\tau} \tag{4.122}$$

we find

$$\frac{\partial F_{\sigma\tau} F^{\sigma\tau}}{\partial(\mathrm{d}_\nu A_\mu)} = \frac{\partial F_{\sigma\tau}}{\partial(\mathrm{d}_\nu A_\mu)} F^{\sigma\tau} + F^{\sigma\tau} \frac{\partial F_{\sigma\tau}}{\partial(\mathrm{d}_\nu A_\mu)} = 2 \frac{\partial F_{\sigma\tau}}{\partial(\mathrm{d}_\nu A_\mu)} F^{\sigma\tau} =$$
$$= 2(g^\nu{}_\sigma g^\mu{}_\tau - g^\nu{}_\tau g^\mu{}_\sigma) F^{\sigma\tau} = 2(F^{\nu\mu} - F^{\mu\nu}) = 4F^{\nu\mu} \ . \tag{4.123}$$

This is inserted into (4.121):

$$-\frac{1}{\mu_0} \mathrm{d}_\nu F^{\nu\mu} + \underbrace{\phi^\dagger \gamma^0 c \gamma^\mu q \phi}_{j^\mu} = 0 \tag{4.124}$$

These equations must hold for $\mu = 0, 1, 2, 3$:

$$\boxed{\mathrm{d}_\nu F^{\nu\mu} = \mu_0 \phi^\dagger \gamma^0 c \gamma^\mu q \phi = \mu_0 j^\mu \quad \text{with } \mu = 0, 1, 2, 3} \tag{4.125}$$

The j^μ in this equation is a component of the invariant current density of the *global* phase transformation, which we found in equation (4.87). In the lines after (4.105) we already pointed out, that no conserved current corresponds to gauge invariance. The benefit of gauge symmetry rather is the fact, that it

1. predicts for any charged field the existence of a gauge field,
2. supplies a clear and unique rule, how to connect a field and its gauge field, i.e. the replacement of the normal derivative by the covariant derivative,
3. to a very large extend fixes the gauge field's properties and it's field equations.

In conclusion of this section we will delve a little bit more into the third point. We have inserted the gauge field into the Lagrangian (4.120) as two terms: Firstly as covariant derivative, due to which gauge invariance was achieved. Secondly as the term $-\frac{1}{4\mu_0}F_{\sigma\tau}F^{\sigma\tau}$, which is – apart from the factor $-\frac{1}{4}$, which we tuned with regard to the Maxwell equations – the simplest method to get a dynamical equation for the gauge field without disturbing gauge invariance. Now we want to verify, that this Lagrangian really determines all four Maxwell equations, and thus the complete classical theory of electrodynamics.

To this purpose we write the gauge field in the usual form

$$A = (A^\rho) = \begin{pmatrix} \Phi/c \\ \boldsymbol{A} \end{pmatrix} \tag{4.126}$$

with the scalar electric potential Φ and the magnetic vector potential $\boldsymbol{A}$, and use for the components of the field strength tensor (4.114) the usual nomenclature

$$E^k/c \equiv F^{k0} = -F^{0k} = \mathrm{d}^k A^0 - \mathrm{d}^0 A^k = -\mathrm{d}_k A^0 - \mathrm{d}_0 A^k \tag{4.127a}$$
$$B^1 \equiv -F^{23} = F^{32} = \mathrm{d}^3 A^2 - \mathrm{d}^2 A^3 = -\mathrm{d}_3 A^2 + \mathrm{d}_2 A^3 \tag{4.127b}$$
$$B^2 \equiv -F^{31} = F^{13} = \mathrm{d}^1 A^3 - \mathrm{d}^3 A^1 = -\mathrm{d}_1 A^3 + \mathrm{d}_3 A^1 \tag{4.127c}$$
$$B^3 \equiv -F^{12} = F^{21} = \mathrm{d}^2 A^1 - \mathrm{d}^1 A^2 = -\mathrm{d}_2 A^1 + \mathrm{d}_1 A^2 \tag{4.127d}$$
$$0 = F^{\rho\rho} = \mathrm{d}^\rho A^\rho - \mathrm{d}^\rho A^\rho \ . \tag{4.127e}$$

This is equivalent to the vector equations

$$\text{electrical field strength:} \quad (E^k) = \boldsymbol{E} \equiv -\boldsymbol{\nabla}\Phi - \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \tag{4.128a}$$
$$\text{magnetic field strength:} \quad (B^k) = \boldsymbol{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{A} \ . \tag{4.128b}$$

With these definitions, the four-dimensional field equation

$$(\mathrm{d}_\nu F^{\nu\rho}) = \mu_0 \phi^\dagger \gamma^0 c \begin{pmatrix} \gamma^0 \\ \boldsymbol{\gamma} \end{pmatrix} q\phi = \mu_0 \begin{pmatrix} j^0 \\ \boldsymbol{j} \end{pmatrix} \tag{4.125}$$

becomes

$$\begin{pmatrix} \mathrm{d}_\nu F^{\nu 0} \\ \mathrm{d}_\nu F^{\nu 1} \\ \mathrm{d}_\nu F^{\nu 2} \\ \mathrm{d}_\nu F^{\nu 3} \end{pmatrix} = \begin{pmatrix} \mathrm{d}_k E^k/c \\ -\mathrm{d}_0 E^1/c + \mathrm{d}_2 B^3 - \mathrm{d}_3 B^2 \\ -\mathrm{d}_0 E^2/c - \mathrm{d}_1 B^3 + \mathrm{d}_3 B^1 \\ -\mathrm{d}_0 E^3/c + \mathrm{d}_1 B^2 - \mathrm{d}_2 B^1 \end{pmatrix} = \\ = \begin{pmatrix} \boldsymbol{\nabla} \cdot \boldsymbol{E}/c \\ -\dfrac{1}{c^2}\dfrac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} + \boldsymbol{\nabla} \times \boldsymbol{B} \end{pmatrix} = \mu_0 \begin{pmatrix} j^0 \\ \boldsymbol{j} \end{pmatrix} \ . \tag{4.129}$$

It's useful to define the

$$\text{electrical field constant:} \quad \epsilon_0 \equiv \frac{1}{c^2 \mu_0} \ , \tag{4.130}$$

with which eventually the two Maxwell equations for the divergence of $\boldsymbol{E}$ and for the rotation of $\boldsymbol{B}$ follow:

$$\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{j^0}{c} \tag{4.131a}$$

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j} + \epsilon_0 \frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} \tag{4.131b}$$

These two equations relate the fields strength to their sources. The two remaining Maxwell equations for the divergence of $\boldsymbol{B}$ and the rotation of $\boldsymbol{E}$ don't depend on j, and therefore are called inner field equations. They are implicitly already fixed by the fact, that the field strength tensor

$$F_{\sigma\tau} \overset{(4.114)}{\equiv} \mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma$$

was defined due to rotation terms, and thus

$$\mathrm{d}_\rho F_{\sigma\tau} + \mathrm{d}_\tau F_{\rho\sigma} + \mathrm{d}_\sigma F_{\tau\rho} = \\ = \mathrm{d}_\rho \mathrm{d}_\sigma A_\tau - \mathrm{d}_\rho \mathrm{d}_\tau A_\sigma + \mathrm{d}_\tau \mathrm{d}_\rho A_\sigma - \mathrm{d}_\tau \mathrm{d}_\sigma A_\rho + \\ + \mathrm{d}_\sigma \mathrm{d}_\tau A_\rho - \mathrm{d}_\sigma \mathrm{d}_\rho A_\tau = 0 \ . \tag{4.132}$$

As each index can have one out of four values, these are $4^3 = 64$ equations.

But all equations with two – or even three – identical indices are because of

$$\mathrm{d}_\sigma F_{\sigma\tau} + \mathrm{d}_\tau \underbrace{F_{\sigma\sigma}}_{=0} + \mathrm{d}_\sigma \underbrace{F_{\tau\sigma}}_{=-F_{\sigma\tau}} = 0 \tag{4.133}$$

trivial (namely $0 = 0$). Thus only $4! = 24$ non-trivial equations with $\rho \neq \sigma \neq \tau \neq \rho$ remain. If the first term in (4.132) instead of $\mathrm{d}_\rho F_{\sigma\tau}$ is chosen as $\mathrm{d}_\tau F_{\rho\sigma}$ or as $\mathrm{d}_\sigma F_{\tau\rho}$, then only the sequence of the three terms changes, but the equations are identical. For this reason there are only $24/3 = 8$ different non-trivial equations. Because of $F_{\sigma\tau} = -F_{\tau\sigma}$, half of these equations differ only by a factor of (-1) from the rest, and there are only $8/2 = 4$ linear independent non-trivial equations. With the definitions from (4.127), these equations are

$$\begin{aligned}
&\mathrm{d}_1 F_{23} + \mathrm{d}_3 F_{12} + \mathrm{d}_2 F_{31} = \\
&\quad = -\mathrm{d}_1 B^1 - \mathrm{d}_3 B^3 - \mathrm{d}_2 B^2 = 0 && \text{(4.134a)}\\
&\mathrm{d}_2 F_{30} + \mathrm{d}_0 F_{23} + \mathrm{d}_3 F_{02} = \\
&\quad = -\mathrm{d}_2 E^3/c - \mathrm{d}_0 B^1 + \mathrm{d}_3 E^2/c = 0 && \text{(4.134b)}\\
&\mathrm{d}_3 F_{01} + \mathrm{d}_1 F_{30} + \mathrm{d}_0 F_{13} = \\
&\quad = \mathrm{d}_3 E^1/c - \mathrm{d}_1 E^3/c + \mathrm{d}_0 B^2 = 0 && \text{(4.134c)}\\
&\mathrm{d}_0 F_{12} + \mathrm{d}_2 F_{01} + \mathrm{d}_1 F_{20} = \\
&\quad = -\mathrm{d}_0 B^3 + \mathrm{d}_2 E^1/c - \mathrm{d}_1 E^2/c = 0\ , && \text{(4.134d)}
\end{aligned}$$

from which by a little re-arrangement

$$\begin{aligned}
\mathrm{d}_1 B^1 + \mathrm{d}_2 B^2 + \mathrm{d}_3 B^3 &= 0 && \text{(4.135a)}\\
\mathrm{d}_2 E^3 - \mathrm{d}_3 E^2 &= -c\,\mathrm{d}_0 B^1 && \text{(4.135b)}\\
\mathrm{d}_3 E^1 - \mathrm{d}_1 E^3 &= -c\,\mathrm{d}_0 B^2 && \text{(4.135c)}\\
\mathrm{d}_1 E^2 - \mathrm{d}_2 E^1 &= -c\,\mathrm{d}_0 B^3 && \text{(4.135d)}
\end{aligned}$$

respectively in vector notation

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{4.136a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \tag{4.136b}$$

follows. Thus all four Maxwell's equations (4.131) and (4.136) result from the postulate, that the Dirac field shall be gauge-invariant. A very remarkable fact!

5 The Lorentz Group

In chapter 4, we found out by clever guessing how scalar fields are transformed under translations (section 4.2) or rotations (section 4.3) of the space-time coordinates. In the following chapters we also will need to know the transformations of vectors respectively vector fields, and of spinors respectively spinor fields under modifications of the space-time coordinates. We will proceed more easily in deriving those transformations, if we adopt more systematic methods than before.

The transformations of scalar fields, of vectors resp. vector fields, and of spinors resp. spinor fields under coordinate rotations and translations are representations of certain groups. And the scalar fields, vectors, and spinors are the bases of those representations. In this chapter, we explicate this group-theoretical point of view, acquire a lot of most useful tools, and quote explicitly two transformations, which are representations of the Lorentz group. The transformations of spinors will be treated in chapter 6 .

5.1 The Homogeneous Proper Lorentz Group $\{\ell\}$

The inhomogeneous Lorentz group (sometimes named Poincaré group) contains translations (these are shifts of the coordinate system's origin, which leave unchanged the coordinate axes directions), proper and improper rotations (these are changes of the direction of one or several coordinate axes, while the system's origin remains unchanged), and combinations of translations and rotations. Any combination can be considered as the concatenation of pure rotations and pure translations. Therefore there are no benefits but only unnecessary complications, if translations and rotations are treated in combination. The analysis of translations can be completely separated from the analysis of rotations.

The transformation of scalar fields under translations of the coordinate

system has already been delineated in section 4.2. Vectors (with the exception of the position vector) and spinors don't change at all under translations. Position vectors, vector fields, and spinor fields are transformed under coordinate translations exactly like scalar fields, i.e. by merely adding the translation to the position vector resp. to the field's argument. Therefore no further investigation of translations is needed, and we will focus in this chapter exclusively on rotations.

Proper rotations can be considered as concatenations of infinitely many rotations with infinitely small rotation angles.

Improper rotations are combinations of proper rotations with the inversion of one or several coordinate axes. In practice, one uses the time inversion T, which inverts (mirrors) the x^0 axis, and the parity transformation P, which mirrors the three x^1, x^2, x^3 axes. The inversion of any single axis or of any combination of axes can be achieved by appropriate combinations of T, P, and proper rotations.

In this book, we will use the simple notion "Lorentz group" for the homogeneous proper Lorentz group, i.e. the group of coordinate rotations of four-dimensional spacetime, which can be composed of infinitesimal small rotations, and leave unchanged the coordinate system's origin. Attention: The literature is not uniform regarding this naming convention.

5.2 Groups and their Representations

We refered to the set of all proper coordinate rotations as a "group", without precisely stating until now, what is meant by this notion. We close this gap due to the following

Definition: A set $G = \{g, h, j, \ldots\}$ with a group operation $\circ$, which is uniquely defined for any pair of elements, is a *group*, if the following 4 conditions are met:

(1) Closure: $\forall g, h \in G : \; g \circ h \in G$ (5.1)

(2) Associative law: $\forall g, h, j \in G : \; (g \circ h) \circ j = g \circ (h \circ j)$

(3) Unit element: $\exists\, e \in G : \; g \circ e = e \circ g = g \; \forall g \in G$

(4) Inverse element:
$\forall g \in G \, \exists g^{-1} \in G : \; g \circ g^{-1} = g^{-1} \circ g = e$

To explain the symbols $\exists$, $\in$, :, $\forall$, we repeat the four conditions verbally:

(1) Closure: For all g, h in G holds: The product $g \circ h$ is again an element in G.

(2) Associative law: For all g, h, j in G holds: The group operation of $g \circ h$ with j equals the group operation of g with $h \circ j$.

(3) Unit element: There is an element e in G with the property: For all g in G holds $g \circ e = e \circ g = g$. The element e is called identity of the group G.

(4) Inverse element: For each element g in the group G there is in G an element g^{-1} with the property: $g \circ g^{-1} = g^{-1} \circ g = e$. The element g^{-1} is called inverse element of g.

The group operation mostly is called *product*, but it does not necessarily need to be the algebraic product of two numbers. In case of the Lorentz group, the group operation resp. the product is the concatenation of two coordinate rotations. The sign $\circ$ frequently is not written explicitly. Instead we define

$$hg \equiv h \circ g \ .$$

The product does not need to be commutative. For example the product of rotations generally depends on whether we first rotate around axis a and then around axis b, or first rotate around axis b and then around axis a.

Definition: A *map* A from a set G into a set D is an instruction with the following two properties:
(1) To each element in G uniquely one element in D is assigned:

$$A : G \longrightarrow D$$
$$\forall g \in G : \ g \xrightarrow{A} d(g) \in D \tag{5.2}$$

(2) Minimum one element in $g \in G$ is mapped onto each element $d \in D$.

Maps with the property (2) are called "surjective". As in this book we will exclusively be concerned with surjective maps, we integrated this property into the map's definition. The map must be unique, but it does not need to

be reversibly unique (= bijective). Several different elements in G may be mapped onto the same element in D.

Definition: Suppose that the elements in a group D are transformations acting on a vector space spanned by the vectors $v_1, v_2, v_3, \ldots$. D is called a *representation* of a group G, if a map $A : G \longrightarrow D$ exists, which is conserving the group's structure:

$$\forall g, h \in G : \; d(h \circ g) = d(h)d(g) \tag{5.3}$$

The vector space V is called *representation space*. The vectors $v_1, v_2, v_3, \ldots \in V$ spanning the representation space are called the *basis* of the representation D.

If the map A is reversibly unique (another word for this is bijective), then D and G are called *isomorph*. If the map A is merely unique, but not reversibly unique, that is if there are elements $d \in D$, onto which several elements $g \in G$ are mapped, then the group D is *homomorph* to the group G. The definition of representation merely requires D to be homomorph to G. If in additon the groups are isomorph, then D is called a *true representation* of G.

Attention: Some authors define the notion "map" differently. They don't require the map to be unique, but admit also ambivalent maps. Consequently also their representations are not unique. Instead they define "multi-valued representations". For example they refer to the matrix group SU(2) as a double-valued representation of the rotation group $\{\overline{\ell}\}$ of three-dimensional position space. We will come back to that at the end of section 6.1.3. In this book, we stick consistently to the definition, that maps and representations must be unique.

Attention: Some authors apply for the notion "representation" a definition, which is more restrictive than (5.3): Their representations exclusively are groups of $n \times n$-dimensional matrices, and the bases of their representations are n-component column vectors. Even if $n = \infty$ is admitted, we consider that restriction counterproductive. According to our definition, representations may be groups of matrices or not. We also need representations, which are transformation groups, but no matrix groups. Therefore we stick to the more general definition (5.3). In section 5.4 we will get back to this

point.

Representations are groups, whose elements are transformations. The transformations transform the representation's basis vectors. Abstract groups are defined solely by their group operation $\circ$, their elements are not acting onto anything outside the group. There are groups, with which we usually become acquainted directly as transformation groups, not as abstract groups. Four important examples: The group $\{\Lambda\}$ of Lorentz transformations consists of 4×4-dimensional matrices, which transform vectors with four space-time components. The matrix group SO(3) consists of 3×3-dimensional matrices, which transform vectors with three position-space components. The elements of the matrix groups SU(2) and SL(2,$\mathbb{C}$) are 2×2-dimensional matrices, which transform two-dimensional spinors. We consider these transformation groups as representations of abstract groups (abstract groups = groups, which are solely defined by their group structure):

Abstract Group	Representation
Lorentz group $\{\ell\}$	$\{\Lambda\}$
3-D space group $\{\overline{\ell}\}$	SO(3)
$\{\overline{\mathcal{B}}\}$	SU(2)
$\{\mathcal{B}\}$	SL(2,$\mathbb{C}$)

The four mentioned representations are but examples for infinitely many, because for any abstract group, infinitely many different representations can be constructed.

We write the Lorentz group as $\{\ell\}$. For it's elements – the rotations of space-time coordinates – we write $\ell^B, \ell^C, \ell^D, \ldots$ (The letter A is spared for a different purpose). To construct a representation of the Lorentz group, we consider a vector space V with vectors $v_j, v_k, v_l, \ldots$. The rotation ℓ^B of space-time coordinates will transform the vector $v_j \in V$ into an element $v_p \in V$ of the same vector space:

$$v_j \xrightarrow{\ell^B} d(\ell^B) v_j = v_p \quad \text{with } v_j, v_p \in V \tag{5.4}$$

The transformations d transform the elements $v \in V$ into other (or potentially in some cases into the same) elements $v \in V$. These transformations

are reversible, as $v_j = d(\ell^{B\,-1})v_p = d^{-1}(\ell^B)v_p$. With the concatenation of transformations as group operation, and with the identical transformation I as unit element (= identity), the set D of the transformations d constitutes a group, as it meets all conditions of definition (5.1).

From (5.4) a map

$$A : \{\ell\} \longrightarrow D \tag{5.5}$$

can be extracted, which maps the Lorentz group onto the set D of invertible transformations d, which are defined on the vector space V:

$$\boxed{\begin{array}{l} \ell^B \xrightarrow{A} d(\ell^B) \\ \text{is defined by} \\ v_j \xrightarrow{\ell^B} d(\ell^B)v_j \quad \forall v_j \in V \end{array}} \tag{5.6}$$

With the just defined map A, the group D becomes a representation of the Lorentz group $\{\ell\}$, as D is conserving – as requested by definition (5.3) – the group structure of the Lorentz group:

$$\forall \ell^B, \ell^C \in \{\ell\} : \; d(\ell^C \ell^B) = d(\ell^C) d(\ell^B)$$

The representation D may be a true representation, or not. For example one gets a representation of the Lorentz group, which is no true representation, if V is a one-dimensional vector space, which is spanned by the basis vector (1). The representation D then is consisting of the one element $d = (1)$, and A is mapping all elements $\ell^B, \ell^C, \ell^D, \ldots \in \{\ell\}$ onto the element $(1) \in D$. This is a one-dimensional representation according to the following

> **Definition:** The *dimension* of a representation D equals the dimension of the vector space V, on which the transformations $d \in D$ are defined. The dimension of the vector space V again equals the number of linearly independent vectors $v \in V$, which are spanning the vector space. (5.7)

Summary of this section: If we want to find out, how certain mathematical objects are transformed under a rotation of the coordinate system, then we

look for a group of transformations, which are defined on the vector space of these mathematical objects, and which form a representation of the Lorentz group. By re-phrasing the question like that, we have not yet found the wanted transformations. But this new perspective will turn out to be quite helpful for the explicit construction of the wanted transformations.

5.3 Lie Groups

Marius Sophus Lie (1842 - 1899) did pioneer work in research of continuous groups. A class of continuous groups with analytic parameter-manifold is named in his honor. Among the Lie groups is also the Lorentz group. A Lie group is uniquely defined by it's parameter manifold and it's Lie algebra. In this section, we will explicate these two notions, and in particular we will specify the Lorentz group's parameter manifold and Lie algebra.

5.3.1 Parameter Manifold

Any element of a Lie group is parameterized by k real numbers $\Omega = (\Omega_1, \ldots, \Omega_k)$. In the specific case of the Lorentz group there are six real parameters: The rotation angles Ω^{10},Ω^{20},Ω^{30} in the three space-time planes 10, 20, 30, and the rotation angles Ω^{23},Ω^{31},Ω^{12} in the purely space-like planes 23, 31, 12 of four-dimensional spacetime. The k real parameters of arbitrary Lie groups may be considered as points in a k-dimensional parameter space. Each point in the parameter space corresponds to a parameter multiplet, and thus to a certain element of the Lie group. There is a topological relation in between the parameter points, as the distance between the points A and B, which are representing the parameter multiplets Ω^A and Ω^B, can be defined by

$$\text{distance}(A, B) \equiv + \sqrt{(\Omega_1^A - \Omega_1^B)^2 + \ldots + (\Omega_k^A - \Omega_k^B)^2} \; . \qquad (5.8)$$

It may happen that several different parameter multiplets are representing the same element. In case of the Lorentz group, a rotation of the three purely space-like coordinate axes by an angle $\mathbf{\Omega}$ with unchanged time axes is defined by the infinitely many different parameters $\mathbf{\Omega} + z \cdot 2\pi\mathbf{\Omega}/|\Omega|$ with arbitrary

integer $z \in \mathbb{Z}$. No such redundancy does exist for space-time-rotations. In the Lorentz group's parameter multiplet (Ω^{10},Ω^{20},Ω^{30},Ω^{23},Ω^{31},Ω^{12}), the first three components uniquely define one group element, but the last three components are periodical with 2π, and thus infinitely redundant.

We would like to get rid of the ballast of infinitely many redundant parameters in the parameter manifold. Therefore we define, that out of all points in parameter space, which parameterize the same element of the group, only that one is belonging to the parameter manifold, which has the shortest distance to the parameter multiplet of the group's identity (= unit element). By that definition, in case of the Lorentz group the manifold of parameters Ω^{23},Ω^{31},Ω^{12} is constrained to the three-dimensional sphere with radius π and center point $\Omega = (0, 0, 0, 0, 0, 0)$. Any purely space-like rotation now is represented by exactly one point in the parameter manifold, except for rotations with the angles $\pm\pi\,\boldsymbol{\Omega}/|\Omega|$. Because of $\ell(+\pi\,\boldsymbol{\Omega}/|\Omega|) = \ell(-\pi\,\boldsymbol{\Omega}/|\Omega|)$, each π-rotation is defined by two parameter points with diametral opposite positions on the parameter sphere's surface. It is common practice to count both of these points to the parameter manifold, but to consider them as just one single topological point of the manifold.

By the way (as this detail is of no relevance for our further considerations) we note, that this definition of the parameter manifold calls for a more precise definition of the distance between two points in the manifold. For example: What is the distance between the topological point $+0.5\pi\,\boldsymbol{\Omega}/|\Omega|$ and the topological point $\pm\pi\,\boldsymbol{\Omega}/|\Omega|$ in the three purely space-like components of the Lorentz group's parameter manifold? Is the distance 0.5π or 1.5π? We define the distance between topological points to be always the smallest of all possible values.

After these preparations, we can formulate the parameter manifold's definition:

Definition: Each multiplet of the k real parameters of a Lie group is represented by a point in a k-dimensional space. The set of these points is called the Lie group's topological *parameter manifold*.
If different parameter multiplets are specifying one and the same element of the Lie group, then only that point is belonging to the parameter manifold, which has the smallest distance to the parameter multiplet of the group's unit element. If several euclidean points meet this criterion of minimal distance, then all of them are belonging to the parameter manifold, but they are considered as being one single topological point. (5.9)

The parameter manifold $\{\Omega^{10},\Omega^{20},\Omega^{30},\Omega^{23},\Omega^{31},\Omega^{12}\}$ of the Lorentz group is infinite in its first three components, i.e. $-\infty \leq \Omega_{j0} \leq +\infty$. In the other three components it is a sphere with radius π around the origin. Any two diametral euclidean points on the sphere's surface constitute one single topological point, as they are parameterizing the identical element of the Lorentz group.

5.3.2 Lie Algebra

It is one of the defining properties of Lie groups, that their parameter manifold is analytic. This implies, that any group element $\ell(\Omega)$, which is specified by the parameter multiplet Ω, can be expanded in a Taylor series around the unit element $\ell(0)$. If we constrain to infinitesimal small parameters

$$\omega_k \equiv \lim_{n\to\infty} \frac{\Omega_k}{n} \quad \text{with } n \in \mathbb{N} \ , \tag{5.10}$$

then the series may be ended after the linear term:

$$\ell_{\text{INF}}(\omega) = \ell(0) + \omega_k \frac{\mathrm{d}\,\ell(\Omega)}{\mathrm{d}\,\Omega_k}\Big|_{\Omega=0} = \ell(0) + \frac{i}{\hbar}\omega_k l^k \tag{5.11}$$

$$\boxed{l^k \equiv -i\hbar \frac{\mathrm{d}\,\ell(\Omega)}{\mathrm{d}\,\Omega_k}\Big|_{\Omega=0}} \tag{5.12}$$

The k operators l^k are called *generators* of the Lie group. We already encountered this definition in (4.8). A finite transformation is achieved by concatenating infinitely many infinitesimal small transformations:

$$\begin{aligned}\ell(\Omega) &= \lim_{n\to\infty}\left(\ell(0) + \frac{i}{\hbar}\frac{\Omega_k}{n}l^k\right)^n \\ &= \exp\left\{\frac{i}{\hbar}\Omega_k l^k\right\}\end{aligned} \tag{5.13}$$

In case of the Lorentz group it's convenient to arrange the six real parameters in a skew-symmetric matrix:

$$(\Omega^{\sigma\tau}) = \begin{pmatrix} 0 & -\Omega^{10} & -\Omega^{20} & -\Omega^{30} \\ \Omega^{10} & 0 & \Omega^{12} & -\Omega^{31} \\ \Omega^{20} & -\Omega^{12} & 0 & \Omega^{23} \\ \Omega^{30} & \Omega^{31} & -\Omega^{23} & 0 \end{pmatrix} \tag{5.14}$$

We attach the same double indices to the six generators l, and arrange them as well in a skew-symmetric matrix:

$$(l^{\sigma\tau}) = \begin{pmatrix} 0 & -l^{10} & -l^{20} & -l^{30} \\ l^{10} & 0 & l^{12} & -l^{31} \\ l^{20} & -l^{12} & 0 & l^{23} \\ l^{30} & l^{31} & -l^{23} & 0 \end{pmatrix} \tag{5.15}$$

Thus in the sum $\Omega_{\sigma\tau}l^{\sigma\tau}$ each parameter and each generator show's up twice. Therefore a factor $1/2$ must be inserted, when the Lorentz group's elements are written in the exponential form (5.13):

$$\boxed{\ell(\Omega) = \exp\left\{\frac{i}{2\hbar}\Omega_{\sigma\tau}l^{\sigma\tau}\right\}} \tag{5.16}$$

Note the factor -1 in the product $\Omega_{\sigma\tau}l^{\sigma\tau} = \Omega^{\alpha\beta}g_{\alpha\sigma}g_{\beta\tau}l^{\sigma\tau}$ with the space-time index combinations $j0$ and $0j$, which does not exist with the space-spaces indices jj.

Some care is needed if finite elements of Lie groups are to be concatenated. The obvious assumption

$$\ell(\Omega^C) \equiv \ell(\Omega^B)\,\ell(\Omega^A) = \exp\Big\{\frac{i}{2\hbar}\,\Omega^B{}_{\gamma\delta}\,l^{\gamma\delta}\Big\}\exp\Big\{\frac{i}{2\hbar}\,\Omega^A{}_{\alpha\beta}\,l^{\alpha\beta}\Big\}$$
$$\stackrel{??}{=} \exp\Big\{\frac{i}{2\hbar}(\Omega^B{}_{\sigma\tau} + \Omega^A{}_{\sigma\tau})\,l^{\sigma\tau}\Big\} \quad \text{is wrong!} \tag{5.17}$$

Here the contracted indices in the products $\Omega^B{}_{\gamma\delta}\,l^{\gamma\delta}$ and $\Omega^A{}_{\alpha\beta}\,l^{\alpha\beta}$ have been re-named to $\sigma\tau$. In the following derivations, we will often exercise the option of re-naming contracted indices. The failure in (5.17) can best be understood and corrected by considering the infinitesimal element:

$$\ell_{\text{INF}}(\omega^C) \stackrel{(5.11)}{=} (\ell(0) + \frac{i}{2\hbar}\,\omega^B{}_{\gamma\delta}\,l^{\gamma\delta})(\ell(0) + \frac{i}{2\hbar}\,\omega^A{}_{\alpha\beta}\,l^{\alpha\beta}) \tag{5.18}$$
$$= \ell(0) + \frac{i}{2\hbar}(\omega^A{}_{\sigma\tau} + \omega^B{}_{\sigma\tau})l^{\sigma\tau} - \frac{1}{4\hbar^2}\,\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\gamma\delta}l^{\alpha\beta}$$

The product $\omega^B\omega^A$ in the last term does not show up in the exponential form (5.17), and exactly this is the failure. $\omega^B\omega^A$ is not necessarily of magnitude $\mathcal{O}(\omega^2)$, but very well might be $\mathcal{O}(\omega)$. If for example ω^A and ω^B are the rotation angles around to mutually orthogonal axes, then the resulting "diagonal" total rotation is of same order as the two rotation components. Terms with $\omega^B\omega^A$ therefore must not be generally ignored – in contrast to the terms $\omega^A\omega^A$ or $\omega^B\omega^B$, which are $\mathcal{O}(\omega^2)$ in any case.

For this reason, in the exponent of (5.17) there must show up a term with $\omega^B\omega^A$. For this purpose we make the ansatz

$$\omega^C{}_{\sigma\tau} = \omega^B{}_{\sigma\tau} + \omega^A{}_{\sigma\tau} + \frac{1}{2}\,\omega^B{}_{\gamma\delta}\frac{1}{2}\,\omega^A{}_{\alpha\beta}\,\frac{i}{\hbar}f^{\gamma\delta\alpha\beta}{}_{\sigma\tau} \tag{5.19}$$

with for the time being unknown structure constants $f^{\gamma\delta\alpha\beta}{}_{\sigma\tau}$. The two factors $\frac{1}{2}$ are inserted again, because in the summation over $\gamma, \delta, \alpha, \beta$ each of the six parameters ω^B and each of the six parameters ω^A does show up twice. Remark: Mathematically oriented authors usually define – in contrast to us – the generator (5.12) without the factor $\hbar$. Consequentially they then – again in contrast to us – don't factor out $1/\hbar$ from the structure constants in the ansatz (5.19).

An infinitesimal small element $\ell_{\text{INF}}(\omega)$ of the Lorentz group is created, if in the Taylor series expansion around the element $\ell(0)$ only those terms are

kept, which are linear in the real parameter ω. In the case of ℓ^C_{INF}, the series must be continued including the quadratic term, because in our ansatz there are terms of the type $\omega^B\,\omega^A$ within $\omega^{C\,2}$, which may be $\mathcal{O}(\omega)$:

$$\begin{aligned}\ell_{\text{INF}}(\omega^C) &= \ell(0) + \frac{i}{2\hbar}\,\omega^C{}_{\sigma\tau}\,l^{\sigma\tau} - \frac{1}{8\hbar^2}\,\omega^C{}_{\sigma\tau}\,\omega^C{}_{\mu\nu}\,l^{\sigma\tau}l^{\mu\nu} = \\ &= \ell(0) + \frac{i}{2\hbar}(\,\omega^B{}_{\sigma\tau} + \omega^A{}_{\sigma\tau} + \omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,\frac{i}{4\hbar}f^{\gamma\delta\alpha\beta}{}_{\sigma\tau})l^{\sigma\tau} \\ &\quad - \frac{1}{8\hbar^2}(\,\omega^B{}_{\sigma\tau}\,\omega^A{}_{\mu\nu} + \omega^A{}_{\sigma\tau}\,\omega^B{}_{\mu\nu})l^{\sigma\tau}l^{\mu\nu} \qquad (5.20)\end{aligned}$$

Terms containing $\omega^{B\,2}$ or $\omega^{A\,2}$ clearly always are $\mathcal{O}(\omega^2)$ and could be neglected. For the same reason, the series expansion up to the linear term is sufficient for ℓ^A_{INF} and for ℓ^B_{INF} – but not for ℓ^C_{INF}.

The factor $\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\gamma\delta}l^{\alpha\beta}$ in the last term of (5.18) can be modified as follows:

$$\begin{aligned}\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\gamma\delta}l^{\alpha\beta} &= \frac{1}{2}\Big(\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\gamma\delta}l^{\alpha\beta} + \omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\gamma\delta}l^{\alpha\beta} \\ &\quad - \omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\alpha\beta}l^{\gamma\delta} + \omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,l^{\alpha\beta}l^{\gamma\delta}\Big) \\ &= \frac{1}{2}\Big(\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,[l^{\gamma\delta}l^{\alpha\beta}, l^{\alpha\beta}l^{\gamma\delta}] \\ &\quad + (\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta} + \omega^B{}_{\alpha\beta}\,\omega^A{}_{\gamma\delta})l^{\gamma\delta}l^{\alpha\beta}\Big) \qquad (5.21)\end{aligned}$$

This is inserted into the last term of (5.18):

$$\begin{aligned}\ell_{\text{INF}}(\omega^C) &= \ell(0) + \frac{i}{2\hbar}(\,\omega^A{}_{\sigma\tau} + \omega^B{}_{\sigma\tau})l^{\sigma\tau} \\ &\quad - \frac{1}{8\hbar^2}\Big(\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta}\,[l^{\gamma\delta}l^{\alpha\beta}, l^{\alpha\beta}l^{\gamma\delta}] \\ &\quad + (\omega^B{}_{\gamma\delta}\,\omega^A{}_{\alpha\beta} + \omega^B{}_{\alpha\beta}\,\omega^A{}_{\gamma\delta})l^{\gamma\delta}l^{\alpha\beta}\Big) \qquad (5.22)\end{aligned}$$

This result is compatible with (5.20) if and only if

$$\boxed{[l^{\gamma\delta}l^{\alpha\beta}, l^{\alpha\beta}l^{\gamma\delta}] = f^{\gamma\delta\alpha\beta}{}_{\sigma\tau}l^{\sigma\tau}}\ . \tag{5.23a}$$

Now we are in a position to write (5.17) correctly by inserting $\Omega^C =$ (5.19):

$$\boxed{\begin{aligned}&\ell(\Omega^B)\ell(\Omega^A) = \exp\Big\{\frac{i}{2\hbar}\,\Omega^B{}_{\beta\gamma}\,l^{\beta\gamma}\Big\}\exp\Big\{\frac{i}{2\hbar}\,\Omega^A{}_{\alpha\beta}\,l^{\alpha\beta}\Big\}\\ &= \exp\Big\{\frac{i}{2\hbar}(\Omega^B{}_{\sigma\tau} + \Omega^A{}_{\sigma\tau} + \Omega^B{}_{\gamma\delta}\Omega^A{}_{\alpha\beta}\frac{i}{4\hbar}f^{\gamma\delta\alpha\beta}{}_{\sigma\tau})l^{\sigma\tau}\Big\}\\ &= \ell(\Omega^B + \Omega^A + \Omega^B{}_{\gamma\delta}\Omega^A{}_{\alpha\beta}\,\frac{i}{4\hbar}f^{\gamma\delta\alpha\beta})\\ &\quad\text{with } f^{\gamma\delta\alpha\beta}{}_{\sigma\tau} \text{ resp. } f^{\gamma\delta\alpha\beta} \equiv (f^{\gamma\delta\alpha\beta}{}_{\sigma\tau}) \text{ acc. (5.23a)}\end{aligned}} \tag{5.23b}$$

In this equation, $\Omega^B\Omega^A$ appears with the weighing factor — i.e. it is considered a term linear in Ω to the extend — that the generator of the respective generators differs from zero. Therefore in the commutators, resp. in the structure constants f built from the commutators according to (5.23a), the information on the structural properties of a Lie group is enclosed.

(5.23) is the *Lie-Algebra's* fundamental equation system. In its derivation we nowhere made use of any special property of the Lorentz group. Thus (5.23) is generally valid for arbitrary Lie groups.

For the Lorentz group $\{\ell\}$ we fix the structure constants *by definition* as follows:

$$\boxed{\begin{aligned}&\text{structure constants of the Lorentz group:}\\ &f^{\alpha\beta\gamma\delta}{}_{\sigma\tau} \equiv i\hbar(g^{\beta\gamma}g^{\alpha}{}_{\sigma}g^{\delta}{}_{\tau} - g^{\beta\delta}g^{\alpha}{}_{\sigma}g^{\gamma}{}_{\tau}\\ &\qquad\qquad - g^{\alpha\gamma}g^{\beta}{}_{\sigma}g^{\delta}{}_{\tau} + g^{\alpha\delta}g^{\beta}{}_{\sigma}g^{\gamma}{}_{\tau})\end{aligned}} \tag{5.24a}$$

From this follows the

$$\boxed{\begin{aligned}&\text{Lie algebra of the Lorentz group:}\\ &[l^{\alpha\beta}, l^{\gamma\delta}] = f^{\alpha\beta\gamma\delta}{}_{\sigma\tau}l^{\sigma\tau}\\ &\qquad = i\hbar(g^{\beta\gamma}l^{\alpha\delta} - g^{\beta\delta}l^{\alpha\gamma} - g^{\alpha\gamma}l^{\beta\delta} + g^{\alpha\delta}l^{\beta\gamma})\end{aligned}} \tag{5.24b}$$

Clearly this quite complex definition does not come out of the blue. Actually (5.24) is the Lie algebra of the Lorentz transformations $\{\Lambda\}$. Therefore it

will be no surprise, when in section 5.5 we will find out, that the Lorentz transformations have the same Lie algebra as the Lorentz group. Still we want to stick to our systematic point of view, which consistently discriminates abstract groups from their representations. Therefore the Lie algebra (5.24) of the abstract Lorentz group was *defined* without reference to any representation.

Any representation D of the Lorentz group with element $d \in D$ according to (5.3) must conserve the group structure, i.e. must be homomorph to the Lorentz group $\{\ell\}$:

$$\begin{aligned} &\forall \ell(\Omega^A), \ell(\Omega^B) \in \{\ell\} \ : \\ &\quad d\Big(\ell(\Omega^B)\Big) d\Big(\ell(\Omega^A)\Big) = d\Big(\ell(\Omega^B)\ell(\Omega^A)\Big) \\ &\quad \overset{(5.23b)}{=} d\Big(\ell(\Omega^B + \Omega^A + \Omega^B{}_{\gamma\delta}\Omega^A{}_{\alpha\beta}\,\frac{i}{4\hbar} f^{\gamma\delta\alpha\beta})\Big) \\ &\qquad \text{with } f^{\gamma\delta\alpha\beta} \equiv (f^{\gamma\delta\alpha\beta}{}_{\sigma\tau}) \text{ according to (5.24a)} \end{aligned} \tag{5.25}$$

As the map $\{\ell\} \to D$ is defined by the real parameters Ω in the form $\ell(\Omega^B) \to d(\Omega^B)$, this can be written more compactly:

$$\begin{aligned} &\forall \Omega^A, \Omega^B \in \text{ the parameter manifold of } \{\ell\} \ : \\ &\quad d(\Omega^B) d(\Omega^A) = d(\Omega^B + \Omega^A + \Omega^B{}_{\gamma\delta}\Omega^A{}_{\alpha\beta}\,\frac{i}{4\hbar} f^{\gamma\delta\alpha\beta}) \\ &\qquad \text{with } f^{\gamma\delta\alpha\beta} \equiv (f^{\gamma\delta\alpha\beta}{}_{\sigma\tau}) \text{ acc. (5.24a)} \end{aligned} \tag{5.26}$$

In plain words:

Theorem: A transformation group D is a representation of the Lorentz group, if and only if there exists an unique map $\{\ell\} \to D$, and D has the same Lie algebra as the Lorentz group. (5.27)

In this theorem, the condition in definition (5.3), that a representation must conserve the group's structure, is replaced by the condition that the representation must have the group's Lie algebra. As for groups with an uncountably infinite number of elements the proof of structure conservation

would be quite difficult, this theorem will turn out to be most helpful to identify representations of the Lorentz group.

5.4 A Finite-Dimensional Representation

The elements $\ell(\Omega^B)$ of the Lorentz group $\{\ell\}$ are coordinate rotations with certain rotation angles Ω^B. All six parameters of the rotation are determined by the skew-symmetric tensor Ω^B. In section 4.3 we described – without using the notion "representation" – a representation $\{R\}$ of the Lorentz group, whose basis is a scalar field $\phi(x)$:

$$R(\Omega^B)\phi(x) \overset{(4.64)}{=} \exp\Big\{-\frac{i}{2\hbar}\,\Omega^B_{\sigma\tau}\,J^{\sigma\tau}\Big\}\phi(x) \tag{5.28}$$

To prove that the group of transformations $\{R\}$ in fact is a representation of $\{\ell\}$, according to (5.27) we must check firstly, whether there exists a unique map $\{\ell\} \to \{R\}$, and secondly, whether $\{R\}$ has the same Lie algebra as the Lorentz group $\{\ell\}$.

The first criterion is fulfilled, as the elements of $\{R\}$ are specified by the identical six parameters Ω as the elements of $\{\ell\}$. Due to this fact, the map $\ell(\Omega) \leftrightarrow R(\Omega)$ in between the elements of both groups even is bijective. Thus $\{R\}$ actually is a true representation of the Lorentz group, provided that the second criterion is met as well.

To check the second criterion, we compute the commutators of the generators of $\{R\}$, i.e. the angular momentum operators:

$$\begin{aligned}
[J^{\alpha\beta}, J^{\gamma\delta}] &\overset{(4.58)}{=} (i\hbar)^2\Big((x^\alpha g^{\beta\mu} - x^\beta g^{\alpha\mu})\mathrm{d}_\mu(x^\gamma g^{\delta\kappa} - x^\delta g^{\gamma\kappa})\mathrm{d}_\kappa \\
&\qquad - (x^\gamma g^{\delta\kappa} - x^\delta g^{\gamma\kappa})\mathrm{d}_\kappa(x^\alpha g^{\beta\mu} - x^\beta g^{\alpha\mu})\mathrm{d}_\mu\Big) \\
&= (i\hbar)^2\Big((x^\alpha g^{\beta\gamma} g^{\delta\kappa} - x^\alpha g^{\beta\delta} g^{\gamma\kappa} - x^\beta g^{\alpha\gamma} g^{\delta\kappa} + x^\beta g^{\alpha\delta} g^{\gamma\kappa})\mathrm{d}_\kappa \\
&\qquad - (x^\gamma g^{\delta\alpha} g^{\beta\mu} - x^\gamma g^{\delta\beta} g^{\alpha\mu} - x^\delta g^{\gamma\alpha} g^{\beta\mu} + x^\delta g^{\gamma\beta} g^{\alpha\mu})\mathrm{d}_\mu\Big)
\end{aligned}$$

$$= (i\hbar)^2\Big(g^{\beta\gamma}(x^\alpha g^{\delta\mu} - x^\delta g^{\alpha\mu})\mathrm{d}_\mu - g^{\beta\delta}(x^\alpha g^{\gamma\mu} - x^\gamma g^{\alpha\mu})\mathrm{d}_\mu$$
$$- g^{\alpha\gamma}(x^\beta g^{\delta\mu} - x^\delta g^{\beta\mu})\mathrm{d}_\mu + g^{\alpha\delta}(x^\beta g^{\gamma\mu} - x^\gamma g^{\beta\mu})\mathrm{d}_\mu\Big)$$
$$= i\hbar(g^{\beta\gamma}J^{\alpha\delta} - g^{\beta\delta}J^{\alpha\gamma} - g^{\alpha\gamma}J^{\beta\delta} + g^{\alpha\delta}J^{\beta\gamma}) \tag{5.29}$$

This complies with (5.24b). Therefore $\{R\}$ is a true representation of the Lorentz group.

The representation $\{R\}$ is uncountably infinite dimensional. It's generator J contains differential operators, which scan $\phi(x)$ not only at the point x, but also in an infinitesimal small neighborhood around this point. Within that infinitesimal neighborhood there are uncountable infinitely many space-time points, and the totality of all $\phi(x)$ at these uncountable infinitely many space-time points is spanning the vector space, on which the transformations $\{R\}$ are defined.

Subsequent to the definition (5.3) of representations we mentioned, that some authors acknowledge only groups of $n \times n$-dimensional matrices (with possibly $n = \infty$) as representations. According to those authors, the group of transformations $\{R\}$ wouldn't be a representation of the Lorentz group. We stick to the definition, according to which representations must be groups of transformations, but not necessarily groups of matrix transformations. The uncountably infinite-dimensional representation $\{R\}$ of the Lorentz group is as "good" and important as the matrix representations, which we will get to know in the following sections.

5.5 A Four-Dimensional Representation

We now want to construct the *Lorentz transformations* $\{\Lambda\}$ as a four-dimensional representation of the Lorentz group $\{\ell\}$.

5.5.1 Generators

As basis for the sought representation we will use the four-vectors (P^μ), and make for an infinitesimal Lorentz transformation Λ_{INF} the ansatz

$$P'^{\mu} = \Lambda_{\text{INF}}{}^{\mu}{}_{\nu} P^{\nu} = (g^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu}) P^{\nu} \tag{5.30}$$

with infinitesimal small angle $\omega^{\mu}{}_{\nu}$. We attached a minus sign to $\omega^{\mu}{}_{\nu}$ from start on, as we want to describe a passive rotation, and a coordinates rotation by a positive angle is correlated to a rotation of the vectors P by a negative angle. It's the central postulate of relativity theory, that the square of four-vectors is invariant under this transformation:

$$\begin{aligned} P_{\nu} P^{\nu} &= P'_{\mu} P'^{\mu} = P'^{\sigma} g_{\sigma\mu} P'^{\mu} \\ &= (g^{\sigma}{}_{\tau} - \omega^{\sigma}{}_{\tau}) P^{\tau} g_{\sigma\mu} (g^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu}) P^{\nu} \\ &= g^{\sigma}{}_{\tau} P^{\tau} g_{\sigma\mu} g^{\mu}{}_{\nu} P^{\nu} - g^{\sigma}{}_{\tau} P^{\tau} g_{\sigma\mu} \omega^{\mu}{}_{\nu} P^{\nu} \\ &\qquad - \omega^{\sigma}{}_{\tau} P^{\tau} g_{\sigma\mu} g^{\mu}{}_{\nu} P^{\nu} + \mathcal{O}(\omega^2) \\ &= P_{\nu} P^{\nu} - P^{\tau} \omega_{\tau\nu} P^{\nu} - \omega_{\nu\tau} P^{\tau} P^{\nu} \\ \Longrightarrow \quad & \omega_{\tau\nu} = -\omega_{\nu\tau} \end{aligned} \tag{5.31}$$

It will be useful to make the skew symmetry of the angle matrix $\omega_{\tau\nu}$ explicitly visible in the ansatz for Λ_{INF}:

$$\begin{aligned} \Lambda_{\text{INF}}{}^{\mu}{}_{\nu} &= g^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} - g^{\mu\kappa} \omega_{\kappa\nu} \\ &= g^{\mu}{}_{\nu} - g^{\mu\kappa} \frac{1}{2} \omega_{\sigma\tau} (g^{\sigma}{}_{\kappa} g^{\tau}{}_{\nu} - g^{\tau}{}_{\kappa} g^{\sigma}{}_{\nu}) \\ &= g^{\mu}{}_{\nu} + \frac{i}{2\hbar} \omega_{\sigma\tau} i\hbar (g^{\sigma\mu} g^{\tau}{}_{\nu} - g^{\tau\mu} g^{\sigma}{}_{\nu}) \\ &= g^{\mu}{}_{\nu} + \frac{i}{2\hbar} \omega_{\sigma\tau} B^{\sigma\tau\mu}{}_{\nu} \end{aligned} \tag{5.32}$$

Here we defined

$$\boxed{B^{\sigma\tau\mu}{}_{\nu} \equiv i\hbar (g^{\sigma\mu} g^{\tau}{}_{\nu} - g^{\tau\mu} g^{\sigma}{}_{\nu})} \tag{5.33}$$

as generator of the Lorentz transformations. (5.32) contains the 4×4-dimensional matrices $(\Lambda_{\text{INF}}{}^{\mu}{}_{\nu})$ and $(g^{\mu}{}_{\nu})$ with row index μ and column index ν. Therefore we consider as well the four-fold indexed tensor $(B^{\sigma\tau\mu}{}_{\nu})$ as a 4×4 matrix with row index μ, column index ν, and "name" $\sigma\tau$. The infinitesimal Lorentz transformation is

$$\boxed{(\Lambda_{\text{INF}}{}^{\mu}{}_{\nu}) \equiv (g^{\mu}{}_{\nu}) + \frac{i}{2\hbar}\,\omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})} \quad . \tag{5.34}$$

Due to concatenation of infinitely many infinitesimal small transformations we get the finite Lorentz transformations

$$\boxed{(\Lambda^{\mu}{}_{\nu}) \equiv \exp\Big\{\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big\}} \quad . \tag{5.35}$$

The group $\{\Lambda\}$ of Lorentz transformations is a representation of the Lorentz group, if and only if there firstly exists a unique map from the Lorentz group $\{\ell\}$ onto the group $\{\Lambda\}$, and if secondly the Lie algebra of $\{\Lambda\}$ is identical to the Lie algebra of the Lorentz group. The first condition is evidently met, as both groups are parameterized by the same angle matrix Ω. To check point two, the commutation relations of the generators $B^{\sigma\tau}$ must be evaluated.

$$\begin{aligned}
&[B^{\alpha\beta}, B^{\gamma\delta}]^{\mu}{}_{\nu} = B^{\alpha\beta\mu}{}_{\kappa}B^{\gamma\delta\kappa}{}_{\nu} - B^{\gamma\delta\mu}{}_{\kappa}B^{\alpha\beta\kappa}{}_{\nu} \\
&\overset{(5.33)}{=} (i\hbar)^2\Big((g^{\alpha\mu}g^{\beta}{}_{\kappa} - g^{\beta\mu}g^{\alpha}{}_{\kappa})(g^{\gamma\kappa}g^{\delta}{}_{\nu} - g^{\delta\kappa}g^{\gamma}{}_{\nu}) \\
&\qquad\qquad - (g^{\gamma\mu}g^{\delta}{}_{\kappa} - g^{\delta\mu}g^{\gamma}{}_{\kappa})(g^{\alpha\kappa}g^{\beta}{}_{\nu} - g^{\beta\kappa}g^{\alpha}{}_{\nu})\Big) \\
&= (i\hbar)^2\Big(g^{\alpha\mu}g^{\beta\gamma}g^{\delta}{}_{\nu} - g^{\alpha\mu}g^{\beta\delta}g^{\gamma}{}_{\nu} - g^{\beta\mu}g^{\alpha\gamma}g^{\delta}{}_{\nu} + g^{\beta\mu}g^{\alpha\delta}g^{\gamma}{}_{\nu} \\
&\qquad\qquad - g^{\gamma\mu}g^{\delta\alpha}g^{\beta}{}_{\nu} + g^{\gamma\mu}g^{\delta\beta}g^{\alpha}{}_{\nu} + g^{\delta\mu}g^{\gamma\alpha}g^{\beta}{}_{\nu} - g^{\delta\mu}g^{\gamma\beta}g^{\alpha}{}_{\nu}\Big) \\
&= (i\hbar)^2\Big(g^{\beta\gamma}(g^{\alpha\mu}g^{\delta}{}_{\nu} - g^{\delta\mu}g^{\alpha}{}_{\nu}) - g^{\beta\delta}(g^{\alpha\mu}g^{\gamma}{}_{\nu} - g^{\gamma\mu}g^{\alpha}{}_{\nu}) \\
&\qquad\qquad - g^{\alpha\gamma}(g^{\beta\mu}g^{\delta}{}_{\nu} - g^{\delta\mu}g^{\beta}{}_{\nu}) + g^{\alpha\delta}(g^{\beta\mu}g^{\gamma}{}_{\nu} - g^{\gamma\mu}g^{\beta}{}_{\nu})\Big) \\
&\overset{(5.33)}{=} i\hbar(g^{\beta\gamma}B^{\alpha\delta} - g^{\beta\delta}B^{\alpha\gamma} - g^{\alpha\gamma}B^{\beta\delta} + g^{\alpha\delta}B^{\beta\gamma})^{\mu}{}_{\nu}
\end{aligned} \tag{5.36}$$

These commutation relations comply with (5.24b).

The group $\{\Lambda\}$ of Lorentz transformations is a four-dimensional (true) representation of the Lorentz group, as the Lorentz transformations are defined on basis of the vectors (P^{μ}), which are spanning a four-dimensional

vector space.

The coordinate rotations, which are generated by B^{k0} in the planes $10, 20, 30$ of four-dimensional spacetime, are called passive "Lorentz boosts" in k-direction. No special name exists for the coordinate rotations in the spacetime planes $23, 31, 12$. By means of

$$B^{\sigma\tau\mu}{}_{\nu} \overset{(5.33)}{=} i\hbar(g^{\sigma\mu}g^{\tau}{}_{\nu} - g^{\tau\mu}g^{\sigma}{}_{\nu}) \ , \tag{5.37}$$

it's not difficult to compute the generators of the six linearly independent transformations.

Boost along the x^1 axis:

$$K^1 \equiv B^{10} = (B^{10\mu}{}_{\nu}) = -B^{01} = -i\hbar \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.38a}$$

Boost along the x^2 axis:

$$K^2 \equiv B^{20} = (B^{20\mu}{}_{\nu}) = -B^{02} = -i\hbar \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.38b}$$

Boost along the x^3 axis:

$$K^3 \equiv B^{30} = (B^{30\mu}{}_{\nu}) = -B^{03} = -i\hbar \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{5.38c}$$

Rotation in the x^2-x^3 plane:

$$L^1 \equiv B^{23} = (B^{23\mu}{}_{\nu}) = -B^{32} = -i\hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{5.38d}$$

Rotation in the x^3-x^1 plane:

$$L^2 \equiv B^{31} = (B^{31\mu}{}_{\nu}) = -B^{13} = -i\hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.38e}$$

Rotation in the x^1-x^2 plane:

$$L^3 \equiv B^{12} = (B^{12\mu}{}_{\nu}) = -B^{21} = -i\hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.38f}$$

The generators B^{jk} of purely space-like rotations are hermitean operators (they are identical to the transposed complex-conjugated operators), while the boost generators B^{k0} are not. In the formalism of quantum theory, only the generators B^{jk} of space-like rotations, but not the generators B^{k0} of boosts are representing observable quantities, as only hermitean operators have real eigenvalues.

5.5.2 Lorentz Transformations

To derive the finite Lorentz transformations explicitly, we make use of the series expansion of the exponential function:

$$(\Lambda^{\mu}{}_{\nu}) \overset{(5.35)}{=} \exp\Big\{\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big\} \tag{5.39}$$

$$= \sum_{n=0,1,2,\ldots}^{\infty} \frac{1}{n!}\Big(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big)^n \tag{5.40}$$

At a coordinate rotation in 10 plane, there is $\Omega_{10} = -\Omega_{01} \neq 0$, $\Omega_{\sigma\tau} = 0$ for $10 \neq \sigma\tau \neq 01$, and $\Omega_{10} = g_{\alpha 1} g_{\beta 0} \Omega^{\alpha\beta} = -\Omega^{10}$.

$$\Big(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big)^n \overset{(5.38a)}{=} (-\Omega^{10})^n \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } n = 1, 3, 5, \ldots$$

$$\Big(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big)^n \overset{(5.38a)}{=} (-\Omega^{10})^n \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } n = 2, 4, 6, \ldots$$

$$\frac{1}{0!}\Big(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big)^0 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{5.41}$$

Thus the Lorentz transformation related to a coordinate rotation by the angle Ω^{10} in the 10 plane is

$$\Lambda \overset{(5.40)}{=} \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \cdot \sum_{n=0,2,4,\ldots}^{\infty} \frac{1}{n!}\Omega^{10n} - \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \cdot \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n!}\Omega^{10n}\,.$$

These are the series expansions of the hyperbola functions cosh and sinh, i.e. we get the Lorentz transformation for the

$$\text{Boost}^{\,10}\!: \qquad \Lambda = \begin{pmatrix}\cosh\Omega^{10} & -\sinh\Omega^{10} & 0 & 0\\ -\sinh\Omega^{10} & \cosh\Omega^{10} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{5.42a}$$

The Lorentz transformations of boosts in other space directions are constructed in complete analogy:

$$\text{Boost}^{\,20}\!: \qquad \Lambda = \begin{pmatrix}\cosh\Omega^{20} & 0 & -\sinh\Omega^{20} & 0\\ 0 & 1 & 0 & 0\\ -\sinh\Omega^{20} & 0 & \cosh\Omega^{20} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{5.42b}$$

$$\text{Boost}^{30}: \quad \Lambda = \begin{pmatrix} \cosh\Omega^{30} & 0 & 0 & -\sinh\Omega^{30} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\Omega^{30} & 0 & 0 & \cosh\Omega^{30} \end{pmatrix} \tag{5.42c}$$

Boosts in arbitrary directions are described by appropriate combinations of (5.42). The rotation angle Ω^{k0} of a boost is called *rapidity*. For unique specification of a boost, three linearly independent real parameters must be fixed, namely the rapidities $\Omega^{10}, \Omega^{20}, \Omega^{30}$.

The boost ^{10} is a transformation of a vector from one reference system to another with parallel axes directions, whose's origin is moving with velocity v along the x^1 axis of the first system. The functional dependence of velocity v and rapidity Ω^{10} is

$$\tanh\Omega^{10} = v/c \; . \tag{5.43}$$

$\tanh\Omega^{10} \to 1$ holds for $\Omega^{10} \to +\infty$, and $\tanh\Omega^{10} \to -1$ for $\Omega^{10} \to -\infty$. Thus the parameter ranges of Ω^{10} and of v/c are $-\infty \leq \Omega^{10} \leq +\infty$ and $-1 \leq v/c \leq +1$.
Using the property

$$\cosh\Omega = \sqrt{\frac{1}{1-\tanh^2\Omega}} \tag{5.44}$$

of hyperbola functions and the definition

$$\gamma \equiv \sqrt{\frac{1}{1-(v/c)^2}} \; , \tag{5.45}$$

the Lorentz boost alternatively may be written as

$$\Lambda \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \overset{(5.42a)}{=} \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \frac{v}{c}x) \\ \gamma(x - vt) \\ y \\ z \end{pmatrix} \; . \tag{5.46}$$

While this form possibly is more familiar from undergraduate studies, it is advisable to parametrize the boost by the rapidity instead of the velocity, because the rapidity, but not the velocity, is additive when several transformations are concatenated.

The inverse transformations Λ^{-1} of the Lorentz boosts are found by replacing the rapidity by the negative rapidity. For the boost along the x^1 axis holds

$$\Lambda^{-1} = \begin{pmatrix} \cosh\Omega^{10} & \sinh\Omega^{10} & 0 & 0 \\ \sinh\Omega^{10} & \cosh\Omega^{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \neq \Lambda^{\sim} \tag{5.47}$$

$^{\sim}$ signifies the transposed matrix. The Lorentz boost is no orthogonal transformation, as the inverse transformation does not equal the transposed! We already noticed that the boost generators are not hermitean. This property is transmitted to the non-orthogonality of the transformations.

Next we consider the transformations of vectors under purely space-like rotations of the coordinate system by an angle Ω^{12} in the 12 plane.

$$\left(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\right)^n \overset{(5.38f)}{=} \Omega_{12}{}^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Big| \begin{array}{l} \text{for } n = \\ 1, 5, 9, 13, \ldots \end{array} \tag{5.48}$$

$$\left(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\right)^n \overset{(5.38f)}{=} \Omega_{12}{}^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Big| \begin{array}{l} \text{for } n = \\ 2, 6, 10, 14, \ldots \end{array} \tag{5.49}$$

$$\left(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\right)^n \overset{(5.38f)}{=} \Omega_{12}{}^n \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Big| \begin{array}{l} \text{for } n = \\ 3, 7, 11, 15, \ldots \end{array} \tag{5.50}$$

$$\Big(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big)^{n} \overset{(5.38f)}{=} \Omega_{12}{}^{n}\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\Big|\begin{matrix}\text{for } n=\\4,8,12,16,\dots\end{matrix} \tag{5.51}$$

$$\frac{1}{0!}\Big(\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big)^{0} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{5.52}$$

From this follows the Lorentz transformation for a coordinate rotation in the 12 plane:

$$\begin{aligned}\Lambda \overset{(5.40)}{=}& \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} + \\ &+\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\cdot\sum_{n=0,4,8,12,\dots}^{\infty}\Big(\frac{1}{n!}\Omega_{12}{}^{n}-\frac{1}{(n+2)!}\Omega_{12}{}^{n+2}\Big)\\ &+\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\cdot\sum_{n=1,5,9,13,\dots}^{\infty}\Big(\frac{1}{n!}\Omega_{12}{}^{n}-\frac{1}{(n+2)!}\Omega_{12}{}^{n+2}\Big)\end{aligned} \tag{5.53}$$

These are the series expansions of the angular functions cos and sin. Taking into account $\Omega_{12} = g_{1\alpha}g_{2\beta}\Omega^{\alpha\beta} = +\Omega^{12}$, the Lorentz transformation for a coordinate rotation in 12 plane becomes

$$\text{Rotation}^{12}\text{:}\qquad \Lambda = \begin{pmatrix}1&0&0&0\\0&\cos\Omega^{12}&\sin\Omega^{12}&0\\0&-\sin\Omega^{12}&\cos\Omega^{12}&0\\0&0&0&1\end{pmatrix}. \tag{5.54a}$$

The Lorentz transformations for coordinate rotations in the planes 23 and

31 are found by the same method:

$$\text{Rotation}^{23}: \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Omega^{23} & \sin\Omega^{23} \\ 0 & 0 & -\sin\Omega^{23} & \cos\Omega^{23} \end{pmatrix} \tag{5.54b}$$

$$\text{Rotation}^{31}: \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega^{31} & 0 & -\sin\Omega^{31} \\ 0 & 0 & 1 & 0 \\ 0 & \sin\Omega^{31} & 0 & \cos\Omega^{31} \end{pmatrix} \tag{5.54c}$$

The inverse transformations Λ^{-1} of the space-like rotations are found by replacing the rotation angles by the negative rotation angles:

$$\Lambda^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega^{12} & -\sin\Omega^{12} & 0 \\ 0 & \sin\Omega^{12} & \cos\Omega^{12} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^{\sim} \quad . \tag{5.55}$$

The Lorentz transformations of purely space-like coordinate rotations are orthogonal: The inverse transformations equal the transposed. An arbitrary space-like rotation is uniquely specified due to three real parameters, namely the angles Ω^{23}, Ω^{31}, Ω^{12}.

In total, six linearly independent real parameters, three for the boosts and three for the space-like rotations, uniquely determine an arbitrary rotation of the four-dimensional space-time coordinates.

Without proof we cite the following theorem from the representation theory of Lie groups:

> **Theorem:** If the parameter-manifold of a Lie group is not compact, then this group has no finite-dimensional unitary representation. (5.56)

Compact means bounded and closed. The boost-transformations' parameters, i.e. the rapidities, are not bounded, but can have any value in the range $-\infty \leq \Omega^{j0} \leq +\infty$. Therefore according to the theorem there is no chance to find a four-dimensional representation of the Lorentz group, whose elements

exclusively are unitary transformations. A transformation T is unitary, if it's adjoint (= transposed complex-conjugate) transformation equals it's inverse transformation:

$$\begin{aligned} T \text{ is unitary} \quad &\Longleftrightarrow \quad TT^\dagger = I \\ \text{with } T^\dagger = T^{*\sim}\,,\ I &= \text{identical transformation} \end{aligned} \tag{5.57}$$

Orthogonal transformations are the real subset of the unitary transformations. The boost-transformations (5.42) neither are orthogonal nor unitary, while the purely space-like rotations (5.54) are orthogonal and therefore a fortiori unitary. The property of transformations, to be unitary or not, is closely related to the property of their generators, to be hermitean or not. To elucidate this relation, we consider the exponential form of an arbitrary transformation:

$$\begin{aligned} T \equiv \exp\Big\{\frac{i}{\hbar}\,W_k\,\gamma^k\Big\} \text{ is unitary} \quad &\Longleftrightarrow \\ \Longleftrightarrow \quad TT^\dagger = \exp\Big\{\frac{i}{\hbar}\,W_k\,\gamma^k\Big\}\exp\Big\{-\frac{i}{\hbar}\,W_k\,\gamma^{k\dagger}\Big\} \\ = \exp\Big\{\frac{i}{\hbar}\,W_k\left(\gamma^k - \gamma^{k\dagger}\right)\Big\} = I \quad &\Longleftrightarrow \\ \Longleftrightarrow \quad \gamma^k = \gamma^{k\dagger} \quad \text{with } I = \text{identical transformation} \end{aligned} \tag{5.58}$$

This equation can be read in both directions: A transformation necessarily is unitary, if it's generator is hermitean (= self-adjoint). If a transformation is unitary, then it's generator necessarily is hermitean. Therefore it is not by accident, that the boost-transformations are not unitary, as their generators are not hermitean. In contrast, the purely space-like rotations are – thanks to their hermitean generators – unitary, in this case even orthogonal.

Actually a subgroup of the Lorentz group – namely the group of coordinate rotations of three-dimensional position space – has orthogonal representations, which we will evaluate in section 6.1.1 .

Attention: The systematic names of matrix groups are compiled on page 124. The group of the 4×4 matrices of the Lorentz transformations has the systematic name SO(3,1). The O in this name means orthogonal.

Didn't we just find out, that at least some of the Lorentz transformations are *not* orthogonal? After all in that table the name is already reduced to "pseudo-orthogonal". The discrepancy can be traced to differing definitions. Our definition of the notion orthogonal characterizes a feature of the transformation. The definition behind the name SO(3,1) characterizes a feature of the scalar product:

In three-dimensional position space of Newton's physics, the scalar product $(\boldsymbol{a},\boldsymbol{b})$ of two vectors $\boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ is defined by

$$(\boldsymbol{a},\boldsymbol{b}) \equiv \boldsymbol{a}^{\sim}\boldsymbol{b} = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z \ . \tag{5.59}$$

This scalar product is invariant under orthogonal transformations O:

$$(O\boldsymbol{a}, O\boldsymbol{b}) = (O\boldsymbol{a})^{\sim} O\boldsymbol{b} = \boldsymbol{a}^{\sim} O^{\sim} O\boldsymbol{b} = \boldsymbol{a}^{\sim} O^{-1} O\boldsymbol{b} = \boldsymbol{a}^{\sim}\boldsymbol{b} \tag{5.60}$$

In the Hilbert space of the state vectors $|u\rangle, |v\rangle$ of quantum theory, a scalar product is defined, which is invariant under unitary transformations U:

$$(|Uu\rangle, |Uv\rangle) \equiv \langle Uu|Uv\rangle = \langle u|U^{\dagger}Uv\rangle = \langle u|U^{-1}Uv\rangle = \langle u|v\rangle \tag{5.61}$$

In the four-dimensional space-time of special relativity theory, the scalar product (a,b) of two vectors a and b is defined by

$$\begin{aligned}(a,b) \equiv a^{\sim} g b &= \begin{pmatrix} a^0 & a^1 & a^2 & a^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{pmatrix} = \\ &= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \end{aligned} \tag{5.62}$$

with the metric tensor g. This scalar product is invariant under Lorentz transformations:

$$(\Lambda a, \Lambda b) = (\Lambda a)^{\sim} g \Lambda b = a^{\sim} \Lambda^{\sim} g \Lambda b = a^{\sim} g b \tag{5.63}$$
$$\text{thanks to} \quad \Lambda^{\sim} g \Lambda = g$$
$$\text{resp.} \quad g^{-1} \Lambda^{\sim} g = \Lambda^{-1}$$

The Lorentz transformations would be "strictly orthogonal" under the condition $\Lambda^{\sim} = \Lambda^{-1}$. They are "pseudo-orthogonal", as they in fact meet the condition $g^{-1}\Lambda^{\sim} g = \Lambda^{-1}$, because multiplication from left side by $g^{-1} = g$ multiplies the three bottom rows of $\Lambda^{\sim}$ by -1, and multiplication from right side by g multiplies the three right columns of $\Lambda^{\sim}$ by -1. In total, the parameters of purely space-like rotations remain invariant, while the boost-parameters are multiplied by -1 . To this extend, all six linearly independent Lorentz transformations are pseudo-orthogonal (or orthogonal), justifying the matrix group's name SO(3,1).

5.5.3 Single Indices

So far we attached double indices to the Lorentz group's 6 real parameters and to the 6 generators, and arranged them in skew-symmetric 4×4 matrices, see (5.14) and (5.15). This notation often is very useful and elegant, but not always. Particularly for the Lorentz transformations, sometimes a different notation is advantageous, which has only one index, but in exchange clearly emphasizes the difference between boost- and rotation-parameters respectively between boost- and rotation-generators:

$$(\Omega^{\sigma\tau}) = \begin{pmatrix} 0 & -\eta^1 & -\eta^2 & -\eta^3 \\ \eta^1 & 0 & \theta^3 & -\theta^2 \\ \eta^2 & -\theta^3 & 0 & \theta^1 \\ \eta^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix} \tag{5.64a}$$

$$(B^{\sigma\tau}) = \begin{pmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & L^3 & -L^2 \\ K^2 & -L^3 & 0 & L^1 \\ K^3 & L^2 & -L^1 & 0 \end{pmatrix} \tag{5.64b}$$

Because of

$$\begin{aligned}\frac{1}{2}\,\Omega_{\sigma\tau}\,B^{\sigma\tau} = \frac{1}{2}\Big(&\Omega_{j0}\,B^{j0} + \Omega_{23}\,B^{23} + \Omega_{31}\,B^{31} + \Omega_{12}\,B^{12} \\ &+ \Omega_{0j}\,B^{0j} + \Omega_{32}\,B^{32} + \Omega_{13}\,B^{13} + \Omega_{21}\,B^{21}\Big) \\ &= \eta_j K^j + \Theta_j L^j \ , \end{aligned} \tag{5.65}$$

the Lorentz transformation in this notation becomes

$$(\Lambda^{\mu}{}_{\nu}) \overset{(5.35)}{=} \exp\Big\{\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,(B^{\sigma\tau\mu}{}_{\nu})\Big\} \;=\; \exp\Big\{\frac{i}{\hbar}(\eta_j K^j + \Theta_j L^j)\Big\} \ . \tag{5.66}$$

We introduce the totally antisymmetric tensor

$$\epsilon^{jkl} \begin{cases} = 1 & \text{for even permutations of } jkl \\ = -1 & \text{for odd permutations of } jkl \\ = 0 & \text{if two indices are identical ,} \end{cases} \tag{5.67}$$

and observe

$$\eta_j = \Omega_{j0} = g_{j\alpha} g_{0\beta} \Omega^{\alpha\beta} = -\Omega^{j0} = -\eta^j \tag{5.68a}$$

$$\Theta_j = \epsilon^{jkl}\Omega_{kl} = \epsilon^{jkl} g_{k\alpha} g_{l\beta} \Omega^{\alpha\beta} = +\epsilon^{jkl}\Omega^{kl} = +\Theta^j \tag{5.68b}$$

$$K_j = \Omega_{j0} = g_{j\alpha} g_{0\beta} \Omega^{\alpha\beta} = -\Omega^{j0} = -K^j \tag{5.68c}$$

$$L_j = \epsilon^{jkl}\Omega_{kl} = \epsilon^{jkl} g_{k\alpha} g_{l\beta} \Omega^{\alpha\beta} = +\epsilon^{jkl}\Omega^{kl} = +L^j \ . \tag{5.68d}$$

Note that $\boldsymbol{\eta}$ and $\boldsymbol{K}$ are vectors, while $\boldsymbol{\Theta}$ and $\boldsymbol{L}$ are pseudo-vectors.

5.5.4 Commutators

We have seen that the Lorentz transformations with the six generators

Generator	What does it generate?
$K^1 = B^{10}$	boost in direction x^1
$K^2 = B^{20}$	boost in direction x^2
$K^3 = B^{30}$	boost in direction x^3
$L^1 = B^{23}$	rotation in plane x^2x^3
$L^2 = B^{31}$	rotation in plane x^3x^1
$L^3 = B^{12}$	rotation in plane x^1x^2

(5.69)

have the same Lie algebra as the Lorentz group:

$$[B^{\alpha\beta}, B^{\gamma\delta}] \overset{(5.36)}{=} i\hbar(g^{\beta\gamma}B^{\alpha\delta} - g^{\beta\delta}B^{\alpha\gamma} - g^{\alpha\gamma}B^{\beta\delta} + g^{\alpha\delta}B^{\beta\gamma})$$

Now we want to list all commutators explicitely. With 6 generators, there are $5+4+3+2+1 = 15$ commutators:

$$\begin{aligned}
[B^{10}, B^{20}] = [K^1, K^2] &= i\hbar(g^{02}B^{10} - g^{00}B^{12} - g^{12}B^{00} + g^{10}B^{02}) \\
&= -i\hbar B^{12} = -i\hbar L^3 && (5.70a)
\end{aligned}$$

$$\begin{aligned}
[B^{20}, B^{30}] = [K^2, K^3] &= i\hbar(g^{03}B^{20} - g^{00}B^{23} - g^{23}B^{00} + g^{20}B^{03}) \\
&= -i\hbar B^{23} = -i\hbar L^1 && (5.70b) \\
[B^{30}, B^{10}] = [K^3, K^1] &= i\hbar(g^{01}B^{30} - g^{00}B^{31} - g^{31}B^{00} + g^{30}B^{01}) \\
&= -i\hbar B^{31} = -i\hbar L^2 && (5.70c) \\
[B^{10}, B^{23}] = [K^1, L^1] &= i\hbar(g^{02}B^{13} - g^{03}B^{12} - g^{12}B^{03} + g^{13}B^{02}) \\
&= 0 && (5.70d) \\
[B^{20}, B^{31}] = [K^2, L^2] &= i\hbar(g^{03}B^{21} - g^{01}B^{23} - g^{23}B^{01} + g^{21}B^{03}) \\
&= 0 && (5.70e) \\
[B^{30}, B^{12}] = [K^3, L^3] &= i\hbar(g^{01}B^{32} - g^{02}B^{31} - g^{31}B^{02} + g^{32}B^{01}) \\
&= 0 && (5.70f) \\
[B^{10}, B^{31}] = [K^1, L^2] &= i\hbar(g^{03}B^{11} - g^{01}B^{13} - g^{13}B^{01} + g^{11}B^{03}) \\
&= +i\hbar B^{30} = +i\hbar K^3 && (5.70g)
\end{aligned}$$

$$
\begin{aligned}
[B^{20}, B^{12}] = [K^2, L^3] &= i\hbar(g^{01}B^{22} - g^{02}B^{21} - g^{21}B^{02} + g^{22}B^{01}) \\
&= +i\hbar B^{10} = +i\hbar K^1 && \text{(5.70h)} \\
[B^{30}, B^{23}] = [K^3, L^1] &= i\hbar(g^{02}B^{33} - g^{03}B^{32} - g^{32}B^{03} + g^{33}B^{02}) \\
&= +i\hbar B^{20} = +i\hbar K^2 && \text{(5.70i)} \\
[B^{23}, B^{20}] = [L^1, K^2] &= i\hbar(g^{32}B^{20} - g^{30}B^{22} - g^{22}B^{30} + g^{20}B^{32}) \\
&= +i\hbar B^{30} = +i\hbar K^3 && \text{(5.70j)} \\
[B^{31}, B^{30}] = [L^2, K^3] &= i\hbar(g^{13}B^{30} - g^{10}B^{33} - g^{33}B^{10} + g^{30}B^{13}) \\
&= +i\hbar B^{10} = +i\hbar K^1 && \text{(5.70k)} \\
[B^{12}, B^{10}] = [L^3, K^1] &= i\hbar(g^{21}B^{10} - g^{20}B^{11} - g^{11}B^{20} + g^{10}B^{21}) \\
&= +i\hbar B^{20} = +i\hbar K^2 && \text{(5.70l)} \\
[B^{23}, B^{31}] = [L^1, L^2] &= i\hbar(g^{33}B^{21} - g^{31}B^{23} - g^{23}B^{31} + g^{21}B^{33}) \\
&= +i\hbar B^{12} = +i\hbar L^3 && \text{(5.70m)} \\
[B^{31}, B^{12}] = [L^2, L^3] &= i\hbar(g^{11}B^{32} - g^{12}B^{31} - g^{31}B^{12} + g^{32}B^{11}) \\
&= +i\hbar B^{23} = +i\hbar L^1 && \text{(5.70n)}
\end{aligned}
$$

$$
\begin{aligned}
[B^{12}, B^{23}] = [L^3, L^1] &= i\hbar(g^{22}B^{13} - g^{23}B^{12} - g^{12}B^{23} + g^{13}B^{22}) \\
&= +i\hbar B^{31} = +i\hbar L^2 && \text{(5.70o)}
\end{aligned}
$$

Using the totally antisymmetric tensor (5.67), the 15 commutators (5.70) can be combined into three equations:

$$
\begin{aligned}
[K^j, K^k] &= -i\hbar\epsilon^{jkl}L^l && \text{(5.71a)} \\
[K^j, L^k] &= +i\hbar\epsilon^{jkl}K^l && \text{(5.71b)} \\
[L^j, L^k] &= +i\hbar\epsilon^{jkl}L^l && \text{(5.71c)}
\end{aligned}
$$

It's evident, that the purely space-like rotations constitute a subgroup of the Lorentz group, because the concatenation of purely space-like angular momentum operators again results into a space-like angular momentum operator, see (5.71c). On the other hand, from (5.71a) and (5.71b) we learn that the boosts are no subgroup of the Lorentz group, because in the

concatenation of boosts always a space-like angular momentum shows up.

We could have extracted the same result already from the matrices (5.42) of boost-transformations and the matrices (5.54) of space-like rotations: In the matrices (5.54) of space-like rotations there are non-zero non-diagonal elements only in a 3×3 block bottom right. These transformations therefore only are mixing the components 1 to 3 of the vectors onto which they are acting, but leave the 0-component unchanged. In contrast, the matrices (5.42) have non-zero non-diagonal elements in all rows and in all columns. Thus they are mixing all four components of the vectors onto which they are acting, and can not be separated as a subgroup from the Lorentz transformations.

5.6 Vector Field Transformations

First we consider again the transformation of scalar fields. In sections 4.3 and 5.4 we introduced on the basis of scalar fields $\phi(x)$ an infinite-dimensional representation

$$R\phi(x) \overset{(4.64)}{=} \exp\Big\{\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,J^{\sigma\tau}\Big\}\phi(x) \tag{5.72}$$

of the Lorentz group, whose generators are the angular momentum operators J.

We now are going to proof, that the same transformation of the scalar field can be achieved by applying the inverse Lorentz transformation Λ^{-1} onto the scalar field's argument:

$$\boxed{R\phi(x) \equiv \phi(\Lambda^{-1}x)} \tag{5.73}$$

For this purpose it's sufficient to evaluate the infinitesimal transformation. Once

$$R_{\text{INF}}\phi(x) \equiv \phi(\Lambda^{-1}_{\text{INF}}\,x) \tag{5.74}$$

is proved, (5.73) is proved as well. The infinitesimal Lorentz transformation

of the position vector has the ρ-component

$$\Lambda_{\text{INF}}{}^{\rho}{}_{\nu} x^{\nu} \overset{(5.34)}{=} (g^{\rho}{}_{\nu} + \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, B^{\sigma\tau\rho}{}_{\nu}) x^{\nu} \ . \tag{5.75}$$

Thus the inverse infinitesimal transformation is

$$\Lambda^{-1}_{\text{INF}}{}^{\rho}{}_{\nu} x^{\nu} = (g^{\rho}{}_{\nu} - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, B^{\sigma\tau\rho}{}_{\nu}) x^{\nu} \ . \tag{5.76}$$

As $B^{\sigma\tau\rho}{}_{\nu} x^{\nu}$ is the ρ-component of the four-vector

$$B^{\sigma\tau}{}_{\nu} x^{\nu} \equiv \begin{pmatrix} B^{\sigma\tau 0}{}_{\nu} x^{\nu} \\ B^{\sigma\tau 1}{}_{\nu} x^{\nu} \\ B^{\sigma\tau 2}{}_{\nu} x^{\nu} \\ B^{\sigma\tau 3}{}_{\nu} x^{\nu} \end{pmatrix} \ , \tag{5.77}$$

(5.76) is the ρth row of the matrix equation

$$\Lambda^{-1}_{\text{INF}}\, x = x - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, B^{\sigma\tau}{}_{\nu} x^{\nu} \ . \tag{5.78}$$

We expand the term

$$\phi(\Lambda^{-1}_{\text{INF}}\, x) = \phi(x - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, B^{\sigma\tau}{}_{\nu} x^{\nu}) \tag{5.79}$$

in a Taylor series around $\phi(x)$. The Taylor series may be ended after the linear term, because ω is infinitesimally small:

$$\phi(x - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, B^{\sigma\tau}{}_{\nu} x^{\nu}) = \phi(x) - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, B^{\sigma\tau\mu}{}_{\nu} x^{\nu} \mathrm{d}_{\mu}\phi(x) \tag{5.80}$$

Because of

$$B^{\sigma\tau\mu}{}_{\nu} \overset{(5.33)}{=} i\hbar(g^{\sigma\mu} g^{\tau}{}_{\nu} - g^{\tau\mu} g^{\sigma}{}_{\nu}) \tag{5.81a}$$

$$\begin{aligned} J^{\sigma\tau} &\overset{(4.58)}{=} i\hbar(x^{\sigma} g^{\tau\mu} - x^{\tau} g^{\sigma\mu})\mathrm{d}_{\mu} \\ &= i\hbar(g^{\sigma}{}_{\nu} g^{\tau\mu} - g^{\tau}{}_{\nu} g^{\sigma\mu}) x^{\nu} \mathrm{d}_{\mu} \ , \end{aligned} \tag{5.81b}$$

J and B are related by

$$J^{\sigma\tau} = -B^{\sigma\tau\mu}{}_{\nu} x^{\nu} \mathrm{d}_{\mu} \ . \tag{5.82}$$

Combination with (5.79) and (5.80) results into

$$\begin{aligned} \phi(\Lambda^{-1}_{\mathrm{INF}}\, x) &= \phi(x) - \frac{i}{2\hbar}\, \omega_{\sigma\tau}\, B^{\sigma\tau\mu}{}_{\nu} x^{\nu} \mathrm{d}_{\mu} \phi(x) = \\ &= \phi(x) + \frac{i}{2\hbar}\, \omega_{\sigma\tau}\, J^{\sigma\tau} \phi(x) \overset{(4.56)}{=} R_{\mathrm{INF}} \phi(x) \ . \end{aligned} \tag{5.83}$$

Thus (5.74), and therefore (5.73), is proved.

After these preparations we turn to the actual subject of this section: The transformation of vector fields. No new representation of the Lorentz group is needed to transform fields of Lorentz vectors, like e.g. the electromagnetic potential

$$A(x) = \begin{pmatrix} \Phi/c\,(x) \\ \boldsymbol{A}(x) \end{pmatrix} = \begin{pmatrix} A^0(x) \\ A^1(x) \\ A^2(x) \\ A^3(x) \end{pmatrix} \ , \tag{5.84}$$

which was introduced as a gauge field in section 4.5. The transformation into a primed coordinate system′ is effected by combining the already known four-dimensional representation $\{\Lambda\}$ and the infinite-dimensional representation $\{R\}$. However a clarification is needed with regard to the combined application of the both transformations. We must make sure that space-time vectors will not be transformed twice:

$$\widetilde{\Lambda} \equiv \Lambda \quad , \quad \text{but } \widetilde{\Lambda} x = x \tag{5.85a}$$

$$\widetilde{B}^{\sigma\tau} \equiv B^{\sigma\tau} \quad , \quad \text{but } \widetilde{B}^{\sigma\tau} x = 0 \tag{5.85b}$$

The special rule applies *only* to space-time position vectors x.

Attention: It is not common practice in the literature to indicate by a tilde or some other mark, whether a transformation operator is acting only onto the amplitude of a vector field, or onto it's argument as well. Usually it is

left to the reader's thoughtfulness to guess in any case what is meant. We as well will sometimes skip the tilde in later chapters of this book.

First we consider a transformation, which rotates the coordinates by the infinitesimal small angle ω. It doesn't matter whether the combined transformation is written as $R_{\text{INF}}\widetilde{\Lambda}_{\text{INF}}$ or as $\widetilde{\Lambda}_{\text{INF}}R_{\text{INF}}$.

$$\begin{aligned} A^\mu(x) &\xrightarrow{\ell(\omega)} A'^\mu(x') \\ A'^\mu(x') &= \underbrace{\Big(I + \frac{i}{2\hbar}\omega_{\sigma\tau}J^{\sigma\tau}\Big)}_{R_{\text{INF}}} \underbrace{\Big(g^\mu{}_\nu + \frac{i}{2\hbar}\omega_{\sigma\tau}\widetilde{B}^{\sigma\tau\mu}{}_\nu\Big)}_{\widetilde{\Lambda}_{\text{INF}}} A^\nu(x) \end{aligned} \tag{5.86}$$

The transformation $A'^\mu(x) = \widetilde{\Lambda}_{\text{INF}}{}^\mu{}_\nu A^\nu(x)$ rotates the vector field's amplitude $A(x)$ exactly like it would rotate a vector A. By definition it does not affect the vector field's argument. The transformation $A'^\mu(x') = R_{\text{INF}}A'^\mu(x)$ rotates the argument x of the vector field's μ-component exactly as it would rotate the argument of a scalar field $\phi(x)$. It does not touch the vector field's amplitude.

The transformation's two steps may as well be performed in the sequence $A^\mu(x') = R_{\text{INF}}A^\mu(x)$ and $A'^\mu(x') = \widetilde{\Lambda}_{\text{INF}}{}^\mu{}_\nu A^\nu(x')$. In any case it's essential to execute *both* steps. It would be nonsense to evaluate vector fields like $A^\mu(x')$ or $A'^\mu(x)$, whose amplitudes and arguments are defined in different coordinate systems. Only $A^\mu(x)$ and $A'^\mu(x')$ are sensible terms. Remark: Under coordinate rotations, for scalar fields holds $\phi'(x') = \phi(x')$. Therefore we dropped the prime from ϕ.

We now will compute the product of the both infinitesimal transformations. With regard to the discussion subsequent to (5.18) we note, that the term with $\omega_{\sigma\tau}^2$ – different from a term $\omega_{\sigma\tau}\omega_{\alpha\beta}$ with $\sigma\tau \neq \alpha\beta$ – may be neglected.

$$A'^\mu(x') = \Big(g^\mu{}_\nu + \frac{i}{2\hbar}\omega_{\sigma\tau}(\widetilde{B}^{\sigma\tau\mu}{}_\nu + J^{\sigma\tau}g^\mu{}_\nu)\Big)A^\nu(x) \tag{5.87}$$

This is the μ-th row of the matrix equation

$$A'(x') = \Big(\mathbb{1} + \frac{i}{2\hbar}\omega_{\sigma\tau}\big(\underbrace{\widetilde{B}^{\sigma\tau}}_{(\widetilde{B}^{\sigma\tau\mu}{}_{\nu})} + J^{\sigma\tau}\mathbb{1}\big)\Big)A(x)\ . \tag{5.88}$$

$\mathbb{1}$ is the unit matrix in four-dimensional space-time. $J^{\sigma\tau}\mathbb{1}$ is the angular momentum operator, inflated to a four-dimensional diagonal matrix in space-time.

Finite transformations are achieved by concatenation of infinitely many infinitesimal small transformations:

$$A'(x') = \exp\Big\{\frac{i}{2\hbar}\Omega_{\sigma\tau}\Big(\widetilde{B}^{\sigma\tau} + J^{\sigma\tau}\mathbb{1}\Big)\Big\}A(x) \tag{5.89}$$

Summary: A vector field's transformation under rotation of the space-time coordinates can be written in three equivalent forms:

$$A^{\mu}(x) \xrightarrow{\ell(\Omega)} A'^{\mu}(x') = R\,\widetilde{\Lambda}^{\mu}{}_{\nu}A^{\nu}(x) \tag{5.90a}$$

$$= \widetilde{\Lambda}^{\mu}{}_{\nu}\,RA^{\nu}(x) \tag{5.90b}$$

$$= \widetilde{\Lambda}^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x)\ . \tag{5.90c}$$

5.7 Angular Momentum of Vector Fields

In section (4.3) we evaluated the angular momentum conservation of scalar fields. The extension to vector fields does not cause significant new problems.

The field equation of the vector field $A(x)$ is derived from the Lagrangian $\mathcal{L}$. The necessary (but not sufficient) condition for the transformation Γ with generator γ to be a symmetry of the vector field $A(x)$, is given by

$$(\partial_{\rho}\mathcal{L})w\gamma x^{\rho} \overset{(4.15)}{=} 0\ . \tag{5.91}$$

With (5.87) the condition becomes

$$0 = (\partial_{\rho}\mathcal{L})(w\gamma)^{\rho}{}_{\nu}x^{\nu} = (\partial_{\rho}\mathcal{L})\frac{1}{2}\omega_{\sigma\tau}(\widetilde{B}^{\sigma\tau\rho}{}_{\nu} + J^{\sigma\tau}g^{\rho}{}_{\nu})x^{\nu}$$
$$\overset{(5.85)}{=} (\partial_{\rho}\mathcal{L})\frac{1}{2}\omega_{\sigma\tau}J^{\sigma\tau}x^{\rho}\ . \tag{5.92}$$

This is identical to the necessary condition (4.66) for angular momentum conservation of a scalar field. Thus – as discussed at (4.66) – also for a vector field the condition is fulfilled, if and only if the Lagrangian at least does not depend explicitly on the directions σ and τ, in which the rotation takes place. Then $\partial_\sigma \mathcal{L} = \partial_\tau \mathcal{L} = 0$.

The sufficient symmetry condition (4.11)

$$\exists\, \mathcal{G} : \quad \mathcal{L} \xrightarrow{I + \frac{i}{\hbar} w\, \gamma} \mathcal{L}' = \mathcal{L} + \frac{i}{\hbar} w\, \gamma \mathcal{L} = \mathcal{L} + \mathrm{d}_\rho \mathcal{G}^\rho$$

can be met due to

$$\mathrm{d}_\rho \mathcal{G}^\rho = \frac{i}{\hbar} w\, \gamma \mathcal{L} = \frac{i}{\hbar} \frac{1}{2} \omega_{\sigma\tau} (\widetilde{B}^{\sigma\tau} + J^{\sigma\tau}) \mathcal{L} = \frac{i}{2\hbar} \omega_{\sigma\tau} J^{\sigma\tau} \mathcal{L}\ .$$

Lagrangians are — by construction — scalars. Therefore

$$\widetilde{B}^{\sigma\tau} \mathcal{L} = B^{\sigma\tau} \mathcal{L} = 0\ . \tag{5.93}$$

The sufficient condition for the vector field's angular momentum conservation thus is fulfilled by the same

$$\mathcal{G}^\rho \overset{(4.68)}{=} -\frac{1}{2} \omega_{\sigma\tau} (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathcal{L} \tag{5.94}$$

as the sufficient condition for the scalar field's angular momentum conservation.

The conserved current density's components are

$$\begin{aligned} j^\rho &\overset{(4.16)}{=} C \Big(\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)} \frac{1}{2} \omega_{\sigma\tau} (\widetilde{B}^{\sigma\tau} + J^{\sigma\tau}) A^\mu - i\hbar \mathcal{G}^\rho \Big) \\ &= \sum_{\sigma\tau = 10,20,30,23,31,12} C i\hbar \omega_{\sigma\tau} \Big(\\ &\qquad \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)} \Big(\frac{1}{i\hbar} \widetilde{B}^{\sigma\tau} + x^\sigma \mathrm{d}^\tau - x^\tau \mathrm{d}^\sigma \Big) A^\mu - (x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho}) \mathcal{L} \Big)\ . \end{aligned} \tag{5.95}$$

According to (4.16), $\sum_r$ is the sum of all fields ϕ_r, which are contained in

the Lagrangian. Thereby each component of a vector field is to be handled as an independent field. We have assumed that $A(x)$ is the only vector field within the Lagrangian, and did sum over it's four components $A^\mu(x)$ according to Einstein's sum convention. It doesn't matter whether A is inserted in covariant or in contravariant form, but at both places consistently the same form must be used. A contravariant factor in the denominator is equivalent to a covariant factor in the nominator, as is visible at the index ρ.

Furthermore in the last line the factor $1/2$ was dropped, because the sum is only running over the six linearly independent components of the skew-symmetric tensor $\omega_{\sigma\tau}$. As these components are independent, there are six independent conserved current densities j. With $C \equiv 1/(ci\hbar\omega_{\sigma\tau})$, their components are

$$j^\rho = x^\sigma \Big(\underbrace{\frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}\mathrm{d}^\tau A^\mu - g^{\tau\rho}\frac{\mathcal{L}}{c}}_{\mathcal{T}^{\rho\tau}/c} \Big) - x^\tau \Big(\underbrace{\frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}\mathrm{d}^\sigma A^\mu - g^{\sigma\rho}\frac{\mathcal{L}}{c}}_{\mathcal{T}^{\rho\sigma}/c} \Big) +$$
$$+ \underbrace{\frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}\frac{1}{i\hbar}\widetilde{B}^{\sigma\tau}A^\mu}_{\mathcal{S}^{\rho\sigma\tau}} \qquad \text{with } \sigma\tau = 10, 20, 30, 23, 31, 12\ . \tag{5.96}$$

The same definition as in (4.32) was applied for the energydensity-stress tensor $\mathcal{T}$. Merely the sum over the scalar fields ϕ_r was replaced by the sum over the four components A^μ of the vector field $A(x)$.

Compared to the conserved current density of scalar fields, here the spin density $\mathcal{S}$ shows up as an additional summand. With

$$\widetilde{B}^{\sigma\tau\mu}{}_\nu = B^{\sigma\tau\mu}{}_\nu \overset{(5.33)}{=} i\hbar(g^{\sigma\mu}g^\tau{}_\nu - g^{\tau\mu}g^\sigma{}_\nu) \tag{5.97}$$

the spin density can be written as

$$\mathcal{S}^{\rho\sigma\tau} = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}\frac{1}{i\hbar}\widetilde{B}^{\sigma\tau}A^\mu = \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}\frac{1}{i\hbar}B^{\sigma\tau\mu}{}_\nu A^\nu = \tag{5.98}$$
$$= \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}g^{\sigma\mu}A^\tau - \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A^\mu)}g^{\tau\mu}A^\sigma =$$

$$\boxed{\mathcal{S}^{\rho\sigma\tau} = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho A_\sigma)}A^\tau - \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho A_\tau)}A^\sigma} \,. \tag{5.99}$$

The angular momentum density tensor's components are

$$\boxed{\mathcal{M}^{\rho\sigma\tau} \equiv x^\sigma \frac{\mathcal{T}^{\rho\tau}}{c} - x^\tau \frac{\mathcal{T}^{\rho\sigma}}{c} + \mathcal{S}^{\rho\sigma\tau}} \,. \tag{5.100}$$

The dimension of this tensor is

$$\left[\mathcal{M}^{\rho\sigma\tau}\right] = \frac{\text{action}}{\text{volume}} \,. \tag{5.101}$$

There are six equations of continuity:

$$\mathrm{d}_\rho \mathcal{M}^{\rho\sigma\tau} = 0 \quad \text{with } \sigma\tau = 10, 20, 30, 23, 31, 12 \tag{5.102}$$

Integrating the null-components over the complete position space, the six conserved angular momenta are found:

$$M^{\sigma\tau} \equiv \int_\Omega \mathrm{d}^3x \Big(x^\sigma \frac{\mathcal{T}^{0\tau}}{c} - x^\tau \frac{\mathcal{T}^{0\sigma}}{c}\Big) + \int_\Omega \mathrm{d}^3x\, \mathcal{S}^{0\sigma\tau} \tag{5.103}$$
$$\text{with } \sigma\tau = 10, 20, 30, 23, 31, 12$$

We discussed them already at equation (4.74). There we have found, that only the purely space-like angular momenta with $\sigma\tau = jl = 23, 31, 12$ are useful for practical applications:

$$M^{jl} \equiv \underbrace{\int_\Omega \mathrm{d}^3x \Big(x^j\mathcal{P}^l - x^l\mathcal{P}^j\Big)}_{\text{orbital angular momentum}} + \underbrace{\int_\Omega \mathrm{d}^3x\, \mathcal{S}^{0jl}}_{\text{spin}} \tag{5.104}$$
$$\text{with } jl = 23, 31, 12$$

Here $\mathcal{P}^j \overset{(4.34)}{=} \mathcal{T}^{0j}/c$ was used. Note, that conservation laws only hold for the total angular momenta M^{jl}, but not for orbital angular momenta or

spins separately.

6 Covering Groups

In the preceding chapters we established how scalar fields, vectors, and vector fields are transformed under rotations of the coordinate system. I.e. we figured out the representations of the Lorentz group, whose bases are scalar fields resp. vectors. Now we must tackle our last and most difficult task: To find out, how spinors are transformed under rotations of the coordinate system. This is a tougher challenge, because spinors are not bases of representations of the Lorentz group. Instead they are bases of representations of another group – namely the group $\{\mathcal{B}\}$, which is the Lorentz group's "universal covering group".

For the following reason, here a new group comes into play: Obviously we expect that the description of no observable physical phenomenon should change under a space-like rotation of the coordinate system (i.e. under a coordinate rotation in the planes 23, 31, or 12 of four-dimensional space-time, or an arbitrary linear combination of these rotations) by an angle 2π . But now we should note, that the state functions of quantum theory in the descriptions of arbitrary observable quantities always appear as bi-linear combinations. If the state functions would change by a factor $\exp\{i\frac{\alpha}{2}\}$ whenever the coordinate axes are rotated by an angle α, then – because of $\exp\{i\frac{4\pi}{2}\} = 1$ – they would be identical to the not-rotated state functions after a coordinate rotation by $z \cdot 4\pi$ with $z \in \mathbb{Z}$. Thus the state functions would have the periodicity 4π under space-like coordinate rotations. Under a coordinate rotation by an odd multiple of 2π the state functions then would change by a factor $\exp\{i\frac{2\pi}{2}\} = -1$. Because of

$$\langle U(2\pi)\phi | U(2\pi)\psi\rangle = \langle -\phi | -\psi\rangle = \langle \phi|\psi\rangle \ , \tag{6.1}$$

these two factors -1 would compensate in all computations of observables, and the theory again would describe all observable quantities with a periodicity of 2π under space-like rotations.

Experience indeed has shown, that state functions with 4π periodicity are indispensable for a complete quantum theoretical description. Such state functions are called *spinors*.

It will be easier to penetrate this somewhat complex construction, if we first recall the analogous construction in elementary non-relativistic quantum mechanics: The transformations of non-relativistic spinors under coordinate rotations of three-dimensional position space. We write $\{\overline{\ell}\}$ for the group of coordinate rotations of three-dimensional position space. Non-relativistic spinors are not bases of representations of $\{\overline{\ell}\}$. Instead they are bases of representations of the group $\{\overline{\mathcal{B}}\}$. This will be explained immediately.

Upfront in a table we compile the nomenclature of some matrix groups. In the majority of cases we indicate transformation groups, which constitute representations of abstract groups, in form of matrix groups. Remember e.g. the Lorentz transformations $\{\Lambda\}$, which are 4×4 matrices. Such representations are called matrix representations. As the matrices of any matrix representation form a group, the product of two matrices again must be one of the group's matrices. And for each matrix there must exist the inverse matrix in the group. Therefore the matrices in any representation are quadratic and invertible. This is the systematic nomenclature of matrix groups:

U(n)	Unitary matrices with dimension $n \times n$. They leave invariant the scalar product (and thus particularly the norm) of the vectors onto which they are acting. Their determinant is ± 1. The matrix elements are complex numbers.
O(n)	Subgroup of U(n), consisting of all matrices with exclusively real elements. The O stands for orthogonal.
SU(n)	Subgroup of U(n), consisting of all matrices with determinant $+1$. The S stands for special, synonymic sometimes the notion unimodular is used.
SO(n)	Subgroup of O(n), consisting of all matrices with determinant $+1$.

SO(m,n) Pseudo-orthogonal group of matices with dimension $(n+m)\times(n+m)$. It leaves invariant the scalar product of vectors a, b in the form $a\cdot b = +a_1b_1+\ldots+a_nb_n-a_{n+1}b_{n+1}-\ldots-a_{n+m}b_{n+m}$. SO(3,1) is the systemic name of the group $\{\Lambda\}$ of Lorentz transformations.

SL(n,$\mathbb{C}$) Group consisting of linear matrices of dimension $n\times n$ with determinant +1. The matrices have complex elements. SU(n) is a subgroup of SL(n,$\mathbb{C}$).

6.1 The Non-Relativistic Covering Group $\{\overline{\mathcal{B}}\}$

6.1.1 The Groups $\{\overline{\ell}\}$ and $\{\overline{\mathcal{B}}\}$

$\{\overline{\ell}\}$ is the group of coordinate rotations of three-dimensional position space. It is a subgroup of the Lorentz group $\{\ell\}$. Both are abstract groups. They are defined solely by their group structure, not by their effect when acting on a base.

The matrix group SO(3) is a three-dimensional true representation of $\{\overline{\ell}\}$. It's base are the vectors $\boldsymbol{V}$, which are defined in three-dimensional position space. It's elements $R(\boldsymbol{\theta}) \in$ SO(3) transform the vectors into a primed coordinate system$'$, whose axes are rotated by the angle $\boldsymbol{\theta}$ relatively to the un-primed system:

$$\boldsymbol{V}' = R(\boldsymbol{\theta})\,\boldsymbol{V} \tag{6.2}$$

For the rotation angle we apply the nomenclature introduced in (5.64):

$$\Theta^j = \epsilon^{jkl}\Omega^{kl} \tag{6.3}$$

As $\{\overline{\ell}\}$ is a subgroup of $\{\ell\}$, it's easy to extract the representation SO(3) of $\{\overline{\ell}\}$ out of the representation $\{\Lambda\}$ of $\{\ell\}$. From the four-dimensional generator

$$L^j \overset{(5.69)}{=} B^{kl} \quad \text{generates rotations in } x^kx^l\text{-plane} \tag{6.4}$$

we derive the three-dimensional generator

$$\bar{L}^j \quad \text{generates rotations in } x^k x^l\text{-plane}\,, \tag{6.5}$$

and mark it by a horizontal bar. The three generators of group SO(3) are:
Rotation in x^2-x^3-plane:

$$\bar{L}^1 \overset{(5.38d)}{=} -i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{6.6}$$

Rotation in x^3-x^1.plane:

$$\bar{L}^2 \overset{(5.38e)}{=} -i\hbar \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.7}$$

Rotation in x^1-x^2-plane:

$$\bar{L}^3 \overset{(5.38f)}{=} -i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.8}$$

The finite transformations $R \in \mathrm{SO}(3)$ are

$$R(\boldsymbol{\Theta}) = \exp\left\{\frac{i}{\hbar}\Theta_k \bar{L}^k\right\}. \tag{6.9}$$

For the single rotation planes this is identical to

$$\text{Rotation}^{23}\text{:}\quad R(\Theta^1) \overset{(5.54b)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta^1 & \sin\Theta^1 \\ 0 & -\sin\Theta^1 & \cos\Theta^1 \end{pmatrix} \tag{6.10a}$$

$$\text{Rotation}^{31}\text{:}\quad R(\Theta^2) \overset{(5.54c)}{=} \begin{pmatrix} \cos\Theta^2 & 0 & -\sin\Theta^2 \\ 0 & 1 & 0 \\ \sin\Theta^2 & 0 & \cos\Theta^2 \end{pmatrix} \tag{6.10b}$$

$$\text{Rotation}^{12}\text{:}\quad R(\Theta^3) \overset{(5.54a)}{=} \begin{pmatrix} \cos\Theta^3 & \sin\Theta^3 & 0 \\ -\sin\Theta^3 & \cos\Theta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.10c}$$

We already know the Lie algebra of the representation SO(3). It is identical

with the Lie algebra of the abstract group $\{\overline{\ell}\}$:

$$[\bar{L}^j, \bar{L}^k] \overset{(5.71c)}{=} +i\hbar\epsilon^{jkl}\bar{L}^l \tag{6.11}$$

The state functions, which form the bases of representations of the group $\{\overline{\ell}\}$, are not sufficient for a complete description of the observed phenomena. According to experience, in addition we need state functions, which are the basis of representations of the group $\{\overline{\mathcal{B}}\}$. We implicitly define the group $\{\overline{\mathcal{B}}\}$ as follows:

Definition: The matrix group SU(2) is a two-dimensional true representation of the abstract group $\{\overline{\mathcal{B}}\}$. (6.12)

The elements of the group SU(2) are matrices, which transform two-component vectors. The group $\{\overline{\mathcal{B}}\}$ has the same group structure as the group SU(2), but the elements of $\{\overline{\mathcal{B}}\}$ are not acting onto anything outside the group. $\{\overline{\mathcal{B}}\}$ is an abstract group, no transformation group.

The 4 elements of a matrix $U \in \mathrm{SU}(2)$ are complex numbers. Thus each matrix has from start 8 degrees of freedom. Due to side conditions these are reduced to 3 degrees of freedom. Firstly the determinant must equal +1:

$$\det U = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = +1 \,. \tag{6.13}$$

This is a complex equation, which reduces the matrices' degrees of freedom from 8 to 6. The matrices must be unitary. That causes further conditions:

$$UU^+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$aa^* + bb^* = 1 \tag{6.14a}$$

$$ac^* + bd^* = 0 \tag{6.14b}$$

$$ca^* + db^* = 0 \tag{6.14c}$$

$$cc^* + dd^* = 1 \tag{6.14d}$$

These conditions reduce the remaining degrees of freedom from 6 to 3,

because (6.14b) and (6.14c) are complex conjugates and thus are not independent from another.

Considering the side conditions we find (see attachment A.4 for computational details) the following most general form of a matrix $U \in \mathrm{SU}(2)$:

$$\boxed{U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with} \quad aa^* + bb^* \overset{(6.14a)}{=} 1} \quad . \tag{6.15}$$

The complex numbers' a and b total 4 degrees of freedom are reduced to 3 by the real equation $aa^* + bb^* = 1$.

Thus the matrices $U \in \mathrm{SU}(2)$ are uniquely determined by 3 real parameters, which can be combined in a three-vector $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \Theta_3)$. We now are going to prove, that with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.16}$$

all matrices $U \in \mathrm{SU}(2)$ can be written in the form

$$\boxed{U(\boldsymbol{\Theta}) = \exp\left\{\frac{i}{\hbar}\Theta_k \frac{\hbar\sigma^k}{2}\right\}} \, . \tag{6.17}$$

From

$$\sigma^1\sigma^1 = \sigma^2\sigma^2 = \sigma^3\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.18}$$

follows with $\Theta \equiv |\boldsymbol{\Theta}|$

$$\begin{aligned} U(\boldsymbol{\Theta}) &\overset{(6.17)}{=} \sum_{n=0,2,4,\ldots}^{\infty} \frac{1}{n!}\Big(i\frac{\Theta}{2}\Big)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n!}\Big(i\frac{\Theta}{2}\Big)^n \frac{\boldsymbol{\Theta}}{\Theta} \cdot \boldsymbol{\sigma} \\ &= \cos(\Theta/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(\Theta/2) \frac{\boldsymbol{\Theta}}{\Theta} \cdot \boldsymbol{\sigma} \, . \end{aligned} \tag{6.19}$$

In this equation it is clearly visible, that the matrices $U \in SU(2)$ are periodic

with the real parameter 4π, but not with 2π. For arbitrary integers z holds

$$U\Big(\boldsymbol{\Theta} + z \cdot 4\pi \frac{\boldsymbol{\Theta}}{\Theta}\Big) = U\Big(\boldsymbol{\Theta}\Big) \tag{6.20a}$$

$$U\Big(\boldsymbol{\Theta} + (2z+1) \cdot 2\pi \frac{\boldsymbol{\Theta}}{\Theta}\Big) = U\Big(\boldsymbol{\Theta}\Big) \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.20b}$$

$$\text{with arbitrary } z \in \mathbb{Z} \ .$$

In particular, independent from the rotation angle's direction holds

$$U(0) = U(\pm 4\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad U(\pm 2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{6.20c}$$

We already concluded in (6.1) that this is acceptable, because in all computations of observable quantities the spinors occur bilinear, so that observable quantities have periodicity 2π, even if they are being computed by means of spinors whose periodicity is 4π.

We write (6.19) by components:

$$U(\Theta_1) = \begin{pmatrix} \cos(\Theta_1/2) & i\sin(\Theta_1/2) \\ i\sin(\Theta_1/2) & \cos(\Theta_1/2) \end{pmatrix} \tag{6.21a}$$

$$U(\Theta_2) = \begin{pmatrix} \cos(\Theta_2/2) & \sin(\Theta_2/2) \\ -\sin(\Theta_2/2) & \cos(\Theta_2/2) \end{pmatrix} \tag{6.21b}$$

$$U(\Theta_3) = \begin{pmatrix} \exp\{i\Theta_3/2\} & 0 \\ 0 & \exp\{-i\Theta_3/2\} \end{pmatrix} \tag{6.21c}$$

From these three equations we can directly read off, that any matrix $U \in \mathrm{SU}(2)$ can be written in the form

$$U(\boldsymbol{\Theta}) \overset{(6.15)}{=} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \overset{(6.17)}{=} \exp\Big\{\frac{i}{\hbar}\Theta_k \frac{\hbar\sigma^k}{2}\Big\} \ .$$

It's always possible to choose the three real parameters $\boldsymbol{\Theta}$ such, that

$$\begin{aligned}
\mathrm{Re}(b) &= \sin(\Theta_2/2) \\
\mathrm{Im}(b) &= \sin(\Theta_1/2) \\
a &= \cos(\Theta_1/2) + \cos(\Theta_2/2) + \exp\{i\Theta_3/2\} \qquad (6.22)\\
&\text{with } aa^* + bb^* \overset{(6.14a)}{=} 1 \ .
\end{aligned}$$

The generators $\hbar\sigma^k/2$ of the transformation (6.17) have the commutation relations

$$\begin{aligned}
\left[\frac{\hbar\sigma^1}{2}, \frac{\hbar\sigma^2}{2}\right] &= \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = i\hbar\,\frac{\hbar\sigma^3}{2} \qquad (6.23a)
\end{aligned}$$

$$\begin{aligned}
\left[\frac{\hbar\sigma^2}{2}, \frac{\hbar\sigma^3}{2}\right] &= \frac{\hbar^2}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = i\hbar\,\frac{\hbar\sigma^1}{2} \qquad (6.23b)
\end{aligned}$$

$$\begin{aligned}
\left[\frac{\hbar\sigma^3}{2}, \frac{\hbar\sigma^1}{2}\right] &= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = i\hbar\,\frac{\hbar\sigma^2}{2} \ . \qquad (6.23c)
\end{aligned}$$

The commutator's general form is

$$\left[\frac{\hbar\sigma^j}{2}, \frac{\hbar\sigma^k}{2}\right] = i\hbar\epsilon^{jkl}\frac{\hbar\sigma^l}{2} \quad , \qquad (6.24)$$

which is complying with (6.11).

The Lie algebra of the groups $\{\overline{\ell}\}$ and SU(2) respectively $\{\overline{\mathcal{B}}\}$ is identical.	(6.25)

As $\{\overline{\mathcal{B}}\}$ was defined as the abstract group represented by SU(2), both groups are isomorph and have the same Lie algebra (6.24).

6.1.2 The Manifold's Connectivity

In this section we will consider the formal relation between the groups $\{\overline{\mathcal{B}}\}$ and $\{\overline{\ell}\}$ under a further point of view taken from the theory of Lie groups: $\{\overline{\mathcal{B}}\}$ is the universal covering group of the group $\{\overline{\ell}\}$.

In section 5.3.1 the notion "parameter manifold" was introduced. The parameter manifold of the group $\{\overline{\ell}\}$ is identical in the three purely space-like parameters to the parameter manifold of the Lorentz group $\{\ell\}$: It is a sphere with radius π around the origin. Any two points which are located diametrical opposite on the manifold sphere's surface are considered to be one single topological point. The parameter manifold of the group $\{\overline{\mathcal{B}}\}$ resp. it's true representation SU(2) again is a sphere around the origin, however with radius 2π, as the elements of SU(2) have 4π-periodicity. As – independent from the direction of the rotation axis – for the elements $U \in \mathrm{SU}(2)$

$$U(\pm 2\pi \tfrac{\boldsymbol{\Theta}}{\Theta}) \overset{(6.20c)}{=} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

the complete surface of the parameter-sphere of SU(2) constitutes one single topological point.

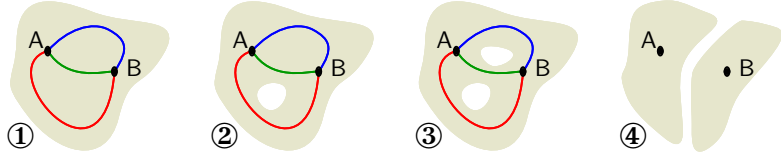


Fig. 6.1 : Several parameter manifolds

To clarify the notion "connectivity" of a parameter manifold, the two-dimensional parameter manifolds of four phantasy groups are sketched in figure 6.1 as gray areas.

Definition: A topological manifold is called *connected*, if any two points of the manifold can be connected by minimum one continuous path, which nowhere is leaving the manifold. (6.26)

Obviously the three manifolds ①,②,③ in figure 6.1 are connected. Only the manifold ④ is not connected, as points A and B can not be connected by a continuous path which does nowhere leave the manifold.

Definition: A topological manifold is called *n-fold connected*, if between any two points of the manifold exactly n standard paths exist, which are characterized by the following property: Any path between the two points, which nowhere leaves the manifold, can be continuously shifted, without anywhere leaving the manifold, onto exactly one standard path. (6.27)

The manifold ① in figure 6.1 is simply connected. For example the red path may be declared a standard path. The green path and the blue path, and any other path between points A and B can be continuously shifted within the manifold onto the red standard path. The same holds for arbitrary other two points in the manifold.

The manifold ② is two-fold connected. We may declare the red and the green path standard paths. The blue path can – without leaving anywhere the manifold — be continuously shifted onto the green path, but not onto the red one because of the hole in the manifold. Any arbitrary further path between the arbitrary points A and B can be continuously shifted within the manifold either onto the red or onto the green standard path, but not onto both.

Finally the manifold ③ is three-fold connected, because arbitrary paths between arbitrary points A and B can be continuously shifted within the manifold onto one of the three sketched standard paths. But no path can be shifted onto a second or a third standard path, because the shifts are blocked by the holes in the manifold.

Theorem: If several groups with different parameter manifolds have the identical Lie algebra, then exactly one of them has a simply connected parameter manifold. (6.28)

I spare the readers (and myself) the proof of this important theorem.

Definition: If several groups with different parameter manifolds have the identical Lie algebra, then that one with the singly connected parameter manifold is called the *universal covering group* of the other groups. (6.29)

The name "universal covering group" is intuitively plausible with regard to figure 6.1: If one places the parameter manifolds ②,③,④ onto the parameter manifold ①, then parts of ① remain visible through the holes and gaps of ②,③,④. But if the simply connected manifold ① is placed on top, then ②,③,④ are completely covered.

The parameter manifold of group $\{\overline{\ell}\}$ is a three-dimensional sphere with radius π. The sphere's center point is representing the parameter triplet $\mathbf{\Theta} = (0,0,0)$. The sphere's surface consists of the points, which are representing the parameter triplets $\mathbf{\Theta} = \pm\pi\,\mathbf{\Theta}/\Theta$. After a cursory glance one might guess that this parameter manifold is simply connected, because there are no holes or gaps in it. But we must read definition (6.27) carefully: Neither "holes" nor "gaps" are mentioned there. Instead we must check, how many different standard paths can be defined between any two arbitrary points.

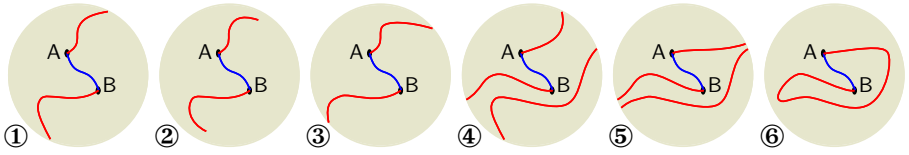


Fig. 6.2: The parameter manifold of group $\{\overline{\ell}\}$

In sketch ① of figure 6.2 two paths are indicated, which connect the points A and B of the parameter manifold of group $\{\overline{\ell}\}$. The blue path is a direct connection, which does not touch the manifold's surface. The red path connects A with the manifold's surface at approx. 1 o'clock. Der diametrical opposite point on the parameter manifold's surface at approx. 7 o'clock is defining the same rotation as the point at approx. 1 o'clock, i.e. the two points are topological identical. Generally holds, that any "two" points, which are positioned diametrically opposite on the manifold sphere's surface, are defining the same element of group $\{\overline{\ell}\}$, and thus must be

considered as one single topological point. From the surface point at approx. 7 o'clock the red path continues to point B. Seen through the glasses of the parameter manifold topology of group $\{\overline{\ell}\}$, the red path is as continuous as the blue one.

We declare the blue path a standard path, and try to shift the red path continuously onto the blue one. As soon as we pull back (even if only infinitesimally) the red path from the surface point at approx. 1 o'clock into the manifold's interior, it is being pulled back from the surface point at approx. 7 o'clock as well, because these "two" surface points actually are just one single topological point. Either the path is running across this point, or not. If the path doesn't run across this point any more, then a gap opens in the path, as visible in ②. By this method, evidently no continuous shift of the red path is possible.

Alternatively we may try to shift the path along the sphere's surface, e.g. from approx. 1 o'clock in direction 2 o'clock. But this is identical to a shift of the path from approx. 7 o'clock in direction 8 o'clock, see ③. By this method again we can't shift the red path onto the blue one. If in a path there is a topological point on the surface of the parameter manifold of group $\{\overline{\ell}\}$, then it's impossible to detach it from the surface by any continuous shift.

Things are quite different if – as sketched in ④ – a path is touching the manifold's surface at two topological points: The red path leads from point A to the manifold's surface at approx. 1 o'clock resp. 7 o'clock, then continues through the manifold's interior to the topological point on the surface at approx. half past 2 o'clock resp. half past 8 o'clock, and then continues to point B. Now the path can be shifted continuously from the topological point 1 o'clock resp. 7 o'clock along the manifold's surface towards the other topological point at approx. half past 2 o'clock resp. half past 8 o'clock, see ⑤. Once the path's surface points touch, the path can be detached by continuous shifting from the manifold's surface – as visible in ⑥ – and can be shifted onto the blue standard path.

Generally holds: Any path connecting the points A and B, which touches the surface of the parameter manifold of group $\{\overline{\ell}\}$ in an even number (including zero) of topological points, can be shifted continuously onto the blue standard path. Any path connecting points A and B, which is touching

the manifold's surface in an odd number of topological points, can be shifted continuously onto the red path in ①, which we now declare the second standard path. Thus the parameter manifold of group $\{\overline{\ell}\}$ is connected two-fold.

Now we turn to group $\{\overline{\mathcal{B}}\}$ resp. it's true representation SU(2): It's parameter manifold is a three-dimensional sphere with radius 2π. The sphere's center point is representing the parameter-triplet $\mathbf{\Theta} = (0, 0, 0)$. The sphere's surface consists of those points, which are representing the parameter-triplet $\mathbf{\Theta} = \pm 2\pi\, \mathbf{\Theta}/\Theta$. Different from $\{\overline{\ell}\}$, not only the diametrically opposite points on the parameter sphere's surface, but *all* surface points of the parameter manifold of group $\{\overline{\mathcal{B}}\}$ are just one single topological point. In sketch ① of figure 6.3 two paths are indicated, which are connecting the

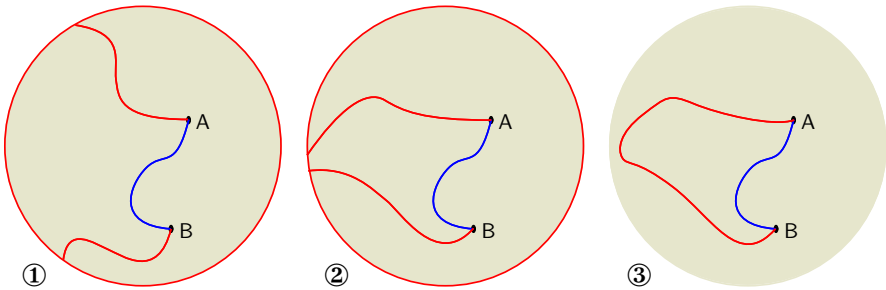


Fig. 6.3 : The parameter manifold of group $\{\overline{\mathcal{B}}\}$

two points A and B in the parameter manifold of group $\{\overline{\mathcal{B}}\}$. (Apropos: The scales of figures 6.3 and 6.2 are identical: The manifold of group $\{\overline{\mathcal{B}}\}$ is a sphere with radius 2π, the manifold of group $\{\overline{\ell}\}$ is a sphere with radius π.) The blue path is connecting A and B within the manifold's interior. The red path leads from point A to the manifold's surface at approx. 11 o'clock. The manifold's complete surface is one single topological point. It is painted red, because it is a point within the red path. From this topological point the path continues at approx. half past seven to point B. As the surface is one single point, the both legs between the surface and the points A and B can be shifted continuously towards one another, see sketch ②. Once the legs touch at the surface, the can by continuous shifting be detached from

the manifold's surface (see sketch ③), and eventually be shifted onto the blue path. Thus in the parameter manifold of group $\{\overline{\mathcal{B}}\}$ there is just one standard path in-beteen any two topological points. The manifold therefore is simple connected.

Therefore according to theorem (6.28), $\{\overline{\mathcal{B}}\}$ is the universal covering group of group $\{\overline{\ell}\}$.

6.1.3 The Homomorphy of the Groups

In (6.25) we found out, that the groups $\{\overline{\mathcal{B}}\}$ and $\{\overline{\ell}\}$ have the same Lie algebra. In this section we will furthermore define a map, which assigns to each element in $\{\overline{\ell}\}$ two elements in $\{\overline{\mathcal{B}}\}$. Overall this proves, that the group $\{\overline{\ell}\}$ of coordinate rotations of the three-dimensional postion space is homomorph to it's covering group $\{\overline{\mathcal{B}}\}$.

For arbitrary parameter triplets $\boldsymbol{\Theta}$ we define the following map:

$$\begin{aligned} A : \{\overline{\mathcal{B}}\} &\longrightarrow \{\overline{\ell}\} \\ \left.\begin{matrix} \overline{\mathcal{B}}(\boldsymbol{\Theta}) \\ \overline{\mathcal{B}}(\boldsymbol{\Theta} \pm 2\pi\frac{\boldsymbol{\Theta}}{\Theta}) \end{matrix}\right\} &\xrightarrow{A} \overline{\ell}(\boldsymbol{\Theta}) \end{aligned} \tag{6.30}$$

Here $\overline{\mathcal{B}}(\boldsymbol{\Theta} \pm 2\pi\frac{\boldsymbol{\Theta}}{\Theta}) = \overline{\mathcal{B}}(\boldsymbol{\Theta})\overline{\mathcal{B}}(\pm 2\pi)$ is exactly one element of the group $\{\overline{\mathcal{B}}\}$ for each parameter triplet $\boldsymbol{\Theta}$, but not two elements, as $\overline{\mathcal{B}}(+2\pi) = \overline{\mathcal{B}}(-2\pi)$ because of (6.20c).

The parameter manifold of group $\{\overline{\mathcal{B}}\}$ is a sphere with radius 2π around the point $\boldsymbol{\Theta} = (0,0,0)$. The parameter manifold of group $\{\overline{\ell}\}$ is a sphere with radius π around the point $\boldsymbol{\Theta} = (0,0,0)$. Therefore the map can be

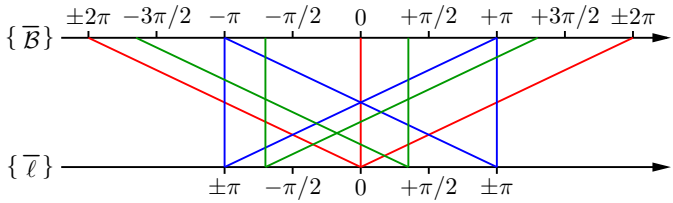


Fig. 6.4 : Map from group $\{\overline{\mathcal{B}}\}$ onto group $\{\overline{\ell}\}$

visualized as shown in figure 6.4. An arbitrary axis through the parameter manifolds center is shown. Each element in $\{\overline{\mathcal{B}}\}$ is uniquely mapped onto exactly one element in $\{\overline{\ell}\}$. The same holds for $\overline{\mathcal{B}}(+\pi\frac{\mathbf{\Theta}}{\Theta})$ and $\overline{\mathcal{B}}(-\pi\frac{\mathbf{\Theta}}{\Theta})$, because in the parameter manifold of $\{\overline{\ell}\}$, the point $\pm\pi$ on each axis is one single topological point, which parametrizes the unique element $\overline{\ell}(\pm\pi\frac{\mathbf{\Theta}}{\Theta})$.

Onto each element in $\{\overline{\ell}\}$ exactly 2 elements in $\{\overline{\mathcal{B}}\}$ are mapped. This holds as well for the element $\overline{\ell}\,(0)$, because onto this element the 2 elements $\overline{\mathcal{B}}(0)$ and $\overline{\mathcal{B}}(\pm 2\pi)$ are mapped, with $\overline{\mathcal{B}}(\pm 2\pi)$ being one single element for all possible rotation axes.

6.1.4 *n*-Dimensional Representations

Suggestion: As this section is about details which are not needed to understand the following chapters, you should skip it when working through this book first time, and continue immediately with section 6.2 on page 147.

6.1.4.1 Tensor Products

In non-relativistic quantum theory, spinors are – by definition – state functions which are the basis of representations of the group $\{\overline{\mathcal{B}}\}$. The matrix group SU(2) by definition is a true two-dimensional representation of the abstract group $\{\overline{\mathcal{B}}\}$. Symbolically we write for this two-dimensional representation

$$\mathbf{2} \equiv \mathrm{SU}(2)\ . \tag{6.31}$$

It's basis are two-component spinors ϕ:

$$\phi' = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}' = U\phi \overset{(6.15)}{=} \begin{pmatrix}a & b\\ -b^* & a^*\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}a\phi_1 + b\phi_2\\ -b^*\phi_1 + a^*\phi_2\end{pmatrix}$$

Matrix representations of $\{\overline{\mathcal{B}}\}$ with arbitrary dimension can be created by choosing as basis the tensor products of two-component spinors. To demonstrate the method, we now will construct a four-dimensional representation. The four-dimensional spinors χ, which are the basis of this representation,

can be found as the direct products of two-component spinors:

$$\chi = \begin{pmatrix} \chi_{11} \\ \chi_{12} \\ \chi_{21} \\ \chi_{22} \end{pmatrix} \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \otimes \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ \\ \phi_2 \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \phi_1\psi_1 \\ \phi_1\psi_2 \\ \phi_2\psi_1 \\ \phi_2\psi_2 \end{pmatrix} \tag{6.32}$$

χ is transformed by the 4×4 matrix W. Using

$$U \overset{(6.15)}{=} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \in \mathrm{SU}(2) \tag{6.33}$$

we find

$$\chi' = W\chi = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \otimes U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{6.34}$$

$$= \begin{pmatrix} (a\phi_1 + b\phi_2)(a\psi_1 + b\psi_2) \\ (a\phi_1 + b\phi_2)(-b^*\psi_1 + a^*\psi_2) \\ (-b^*\phi_1 + a^*\phi_2)(a\psi_1 + b\psi_2) \\ (-b^*\phi_1 + a^*\phi_2)(-b^*\psi_1 + a^*\psi_2) \end{pmatrix}$$

$$= \begin{pmatrix} aa & ab & ba & bb \\ -ab^* & aa^* & -bb^* & ba^* \\ -b^*a & -b^*b & a^*a & a^*b \\ b^*b^* & -b^*a^* & -a^*b^* & a^*a^* \end{pmatrix} \begin{pmatrix} \phi_1\psi_1 \\ \phi_1\psi_2 \\ \phi_2\psi_1 \\ \phi_2\psi_2 \end{pmatrix} \tag{6.35}$$

$$= (U \otimes U)\, \chi \ . \tag{6.36}$$

The matrices $W = U \otimes U$ constitute a four-dimensional matrix representation of $\{\overline{\mathcal{B}}\}$ on the basis of the four-component spinors χ. The representation W is *reducible*. To explain this notion, we apply an arbitrary unitary transformation T onto the equation $\chi' \overset{(6.34)}{=} W\chi$, and insert the unit matrix $\mathbb{1} = T^{-1}T$:

$$\widetilde{\chi}' \equiv T\chi' = \underbrace{TWT^{-1}}_{\widetilde{W}} \underbrace{T\chi}_{\widetilde{\chi}} \equiv \widetilde{W}\widetilde{\chi} \tag{6.37}$$

$\{\widetilde{W}\}$ is a further representation — which is equivalent to $\{W\}$ — of the group $\{\overline{\mathcal{B}}\}$ on the basis of the four-component spinors $\widetilde{\chi}$. By appropriate choice of T one can achieve, that the 4×4-matrices $\{\widetilde{W}\}$ get the form of a direct sum of a one-dimensional and a three-dimensional block (the detailed computation can be found in A.5):

$$\widetilde{W} = TWT^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & aa & ab\sqrt{2} & bb \\ 0 & -ab^*\sqrt{2} & aa^* - bb^* & a^*b\sqrt{2} \\ 0 & b^*b^* & -a^*b^*\sqrt{2} & a^*a^* \end{pmatrix} . \tag{6.38}$$

In all matrices of the representation $\{\widetilde{W}\}$, only the elements $\widetilde{W}_{11}$ and $\widetilde{W}_{jk}$ with $j,k = 2,3,4$ are different from zero. The three bottom spinor components $\widetilde{\chi}_{12}$, $\widetilde{\chi}_{21}$, $\widetilde{\chi}_{22}$ are only mixed among themselves, but not with the top spinor component $\widetilde{\chi}_{11}$. The top spinor component is "mixed" only with itself. The 4×4-dimensional matrices $\widetilde{W}$ therefore are the direct sum of 1×1-dimensional matrices and 3×3-dimensional matrices:

$$\widetilde{W} = (1) \oplus \begin{pmatrix} aa & ab\sqrt{2} & bb \\ -ab^*\sqrt{2} & aa^* - bb^* & a^*b\sqrt{2} \\ b^*b^* & -a^*b^*\sqrt{2} & a^*a^* \end{pmatrix} \tag{6.39}$$

The transformed spinors have the form

$$\widetilde{\chi} = T\chi = \begin{pmatrix} \sqrt{\frac{1}{2}}(\phi_1\psi_2 - \phi_2\psi_1) \\ \phi_1\psi_1 \\ \sqrt{\frac{1}{2}}(\phi_1\psi_2 + \phi_2\psi_1) \\ \phi_2\psi_2 \end{pmatrix} . \tag{6.40}$$

From the three bottom components, the three-component spinors

$$S \equiv \begin{pmatrix} \phi_1\psi_1 \\ \sqrt{\frac{1}{2}}(\phi_1\psi_2 + \phi_2\psi_1) \\ \phi_2\psi_2 \end{pmatrix} \tag{6.41}$$

are built. They are the basis of a three-dimensional representation of the group $\{\overline{\mathcal{B}}\}$. The 3×3-matrices Q of this representation have according to (6.39) the form

$$Q \equiv \begin{pmatrix} aa & ab\sqrt{2} & bb \\ -ab^*\sqrt{2} & aa^* - bb^* & a^*b\sqrt{2} \\ b^*b^* & -a^*b^*\sqrt{2} & a^*a^* \end{pmatrix} \tag{6.42}$$

$$\text{with } aa^* + bb^* \overset{(6.15)}{=} 1 \quad .$$

For this representation we define the symbolic notation

$$\mathbf{3} \equiv \{Q\} \; . \tag{6.43}$$

Note that the spinor S is symmetric under exchange of ϕ and ψ:

$$\begin{pmatrix} \phi_1\psi_1 \\ \sqrt{\frac{1}{2}}(\phi_1\psi_2 + \phi_2\psi_1) \\ \phi_2\psi_2 \end{pmatrix} = \begin{pmatrix} +\psi_1\phi_1 \\ +\sqrt{\frac{1}{2}}(\psi_1\phi_2 + \psi_2\phi_1) \\ +\psi_2\phi_2 \end{pmatrix} \tag{6.44}$$

From the top component of (6.40) the one-component spinor

$$A \equiv \sqrt{\frac{1}{2}}(\phi_1\psi_2 - \phi_2\psi_1) \tag{6.45}$$

is built. It is the basis of a one-dimensional representation of the group $\{\overline{\mathcal{B}}\}$. This is a trivial representation, as all 1×1-matrices P of this representation have according to (6.39) the form

$$P \equiv (1) \; . \tag{6.46}$$

For this representation we define the symbolic notation

$$\mathbf{1} \equiv \{P\} \ . \tag{6.47}$$

Note that the spinor A is antisymmetric under exchange of ϕ and ψ:

$$\sqrt{\frac{1}{2}}(\phi_1\psi_2 - \phi_2\psi_1) = -\sqrt{\frac{1}{2}}(\psi_1\phi_2 - \psi_2\phi_1) \tag{6.48}$$

These results can be combined into the symbolic equation

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \ . \tag{6.49}$$

In words: The reducible four-dimensional representation of the group $\{\overline{\mathcal{B}}\}$, which emerged from the direct product of two two-dimensional representations, can be decomposed into the direct sum of an irreducible one-dimensional representation and an irreducible three-dimensional representation.

We want to give the non-relativistic spinors still another, more pictorial form, which may be more familiar from undergraduate studies. For this purpose we define

$$\uparrow_\phi \equiv \phi_1 \qquad \downarrow_\phi \equiv \phi_2 \qquad \uparrow_\psi \equiv \psi_1 \qquad \downarrow_\psi \equiv \psi_2 \ . \tag{6.50}$$

The one- and three-component spinors, which are the bases of the irreducible representations, in this notation are

$$A \overset{(6.45)}{=} \sqrt{\tfrac{1}{2}}(\uparrow_\phi \downarrow_\psi \ - \downarrow_\phi \uparrow_\psi) \tag{6.51}$$

$$S \overset{(6.41)}{=} \begin{pmatrix} \uparrow_\phi \uparrow_\psi \\ \sqrt{\frac{1}{2}}(\uparrow_\phi \downarrow_\psi \ + \downarrow_\phi \uparrow_\psi) \\ \downarrow_\phi \downarrow_\psi \end{pmatrix} \ . \tag{6.52}$$

These equations allow a pictorial reading: The spinors ϕ and ψ each are representing a particle with spin $\frac{1}{2}$. The arrow $\uparrow$ signifies, that the spin projection onto a certain axis of position space is $+\frac{\hbar}{2}$. The arrow $\downarrow$ signifies, that the spin projection onto that axis is $-\frac{\hbar}{2}$. $\uparrow_\phi \uparrow_\psi$ signifies, that the composed system's total spin projection onto that axis is $+\hbar$, while $\downarrow_\phi \downarrow_\psi$ signifies, that the composed system's total spin projection onto that axis is

$-\hbar$.

The composed system, whose spin projection onto the reference axis is 0, remarkably is not described by $\uparrow_\phi \downarrow_\psi$ nor by $\downarrow_\phi \uparrow_\psi$, but either by the antisymmetric combination $\sqrt{\frac{1}{2}}(\uparrow_\phi \downarrow_\psi - \downarrow_\phi \uparrow_\psi)$, or by the symmetric combination $\sqrt{\frac{1}{2}}(\uparrow_\phi \downarrow_\psi + \downarrow_\phi \uparrow_\psi)$ of the two particles. In the composed system, the two particles have merged into a single entity, and the one or the other spin orientation can't any longer be assigned to the one or the other particle.

6.1.4.2 Symmetric-Product Representations

Irreducible representations of arbitrary dimensions can be found by constructing tensor products of representations with low dimensions, and then tensor products of tensor products, and so on, and finally decomposing these representations into a direct sum of irreducible representations. At higher dimensions, this brute-force method soon becomes quite tedious. In this section we present a method, by which the detour via the reducible representations can be avoided, and straightaway irreducible representations of arbitrary dimensions can be built.

Why at all do reducible representations appear in the method of tensor products? To understand this, we compute the direct product of a two-component spinor with itself:

$$\begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uu \\ uv \\ vu \\ vv \end{pmatrix} = \begin{pmatrix} u^2 \\ uv \\ uv \\ v^2 \end{pmatrix} \tag{6.53}$$

There are two identical components in this product. If we again multiply this product directly with the spinor $\binom{u}{v}$, we get a spinor with the components u^3,u^2v,u^2v,uv^2,u^2v,uv^2,uv^2,v^3. Three components in this spinor equal u^2v, and three components equal uv^2. In total, only four of the eight components, namely u^3,u^2v,uv^2,v^3, are linearly independent. It is the multiple appearance of identical components, which makes the representations reducible. It seems plausible that a representation will be irreducible if and only if it's basis

is consisting of spinors, whose components are linearly independent from another. We won't bother to prove that.

To construct spinors of arbitrary dimensions with linearly independent components, we start from the two-dimensional true representation of the group $\{\overline{\mathcal{B}}\}$ by the matrix group SU(2):

$$\begin{pmatrix} u' \\ v' \end{pmatrix} \overset{(6.15)}{=} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ -b^*u + a^*v \end{pmatrix} \tag{6.54}$$

We code a representation's dimension by means of a parameter j:

$$\begin{aligned} \text{dimension} = 2j+1 = 1\,,2\,,3\,,4\,,5\,,6\,,\ldots \\ \Longrightarrow \quad j = 0\,,\tfrac{1}{2}\,,1\,,\tfrac{3}{2}\,,2\,,\tfrac{5}{2}\,,\ldots \end{aligned} \tag{6.55}$$

The spinors $\phi^{(j)}$, which constitute the basis of a $2j+1$-dimensional representation of $\{\overline{\mathcal{B}}\}$, are marked by (j). The elements $D^{(j)}$ of the representation $\{D^{(j)}\}$ are marked identically. The round brackets are a reminder, that multiply appearing (j) are *not* automatically to be summed-up according to the sum convention.

We define the component $\phi_r^{(j)}$ of the $2j+1$-component spinor $\phi^{(j)}$ by

$$\boxed{\phi_r^{(j)} \equiv N_r^{(j)} u^{2j+1-r} v^{r-1}} \tag{6.56a}$$

with the normalization factor

$$N_r^{(j)} \equiv \sqrt{\frac{(2j)!}{(r-1)!\,(2j+1-r)!}}\,. \tag{6.56b}$$

Thus the $2j+1$-component spinor is

$$\phi^{(j)} = \begin{pmatrix} N_1^{(j)} u^{2j} v^0 \\ N_2^{(j)} u^{2j-1} v^1 \\ \vdots \\ N_{2j}^{(j)} u^1 v^{2j-1} \\ N_{2j+1}^{(j)} u^0 v^{2j} \end{pmatrix}. \tag{6.57}$$

In the sequel some spinors with few components, and their absolute squares, are explicitly listed:

$$\phi^{(0)} = \left(N_1^{(0)} u^0 v^0\right) = (1) \tag{6.58a}$$

$$|\phi^{(0)}|^2 = 1 \tag{6.58b}$$

$$\phi^{(\frac{1}{2})} = \begin{pmatrix} N_1^{(\frac{1}{2})} u^1 v^0 \\ N_2^{(\frac{1}{2})} u^0 v^1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{6.59a}$$

$$|\phi^{(\frac{1}{2})}|^2 = |u|^2 + |v|^2 \tag{6.59b}$$

$$\phi^{(1)} = \begin{pmatrix} N_1^{(1)} u^2 v^0 \\ N_2^{(1)} u^1 v^1 \\ N_3^{(1)} u^0 v^2 \end{pmatrix} = \begin{pmatrix} u^2 \\ \sqrt{2} uv \\ v^2 \end{pmatrix} \tag{6.60a}$$

$$|\phi^{(1)}|^2 = |u|^4 + 2|u|^2|v|^2 + |v|^4 = (|u|^2 + |v|^2)^2 \tag{6.60b}$$

$$\phi^{(\frac{3}{2})} = \begin{pmatrix} N_1^{(\frac{3}{2})} u^3 v^0 \\ N_2^{(\frac{3}{2})} u^2 v^1 \\ N_3^{(\frac{3}{2})} u^1 v^2 \\ N_4^{(\frac{3}{2})} u^0 v^3 \end{pmatrix} = \begin{pmatrix} u^3 \\ \sqrt{3} u^2 v \\ \sqrt{3} u v^2 \\ v^3 \end{pmatrix} \tag{6.61a}$$

$$|\phi^{(\frac{3}{2})}|^2 = |u|^6 + 3|u|^4|v|^2 + 3|u|^2|v|^4 + |v|^6 = (|u|^2 + |v|^2)^3 \tag{6.61b}$$

Obviously all spinors are normalized to 1, if the two-component spinor $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ is normalized to 1 .

The $2j+1$-dimensional basis vectors are computed by means of the two-dimensional basis vectors:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} \overset{(6.54)}{=} \begin{pmatrix} au + bv \\ -b^* u + a^* v \end{pmatrix}$$

$$\begin{aligned}\phi_r^{(j)} &\overset{(6.56a)}{=} N_r^{(j)} u^{2j+1-r} v^{r-1} \\ \phi_r^{(j)\prime} &= N_r^{(j)} (u')^{2j+1-r} (v')^{r-1} \\ &= N_r^{(j)} (au + bv)^{2j+1-r} (-b^* u + a^* v)^{r-1}\end{aligned} \tag{6.62}$$

The elements $D^{(j)}$ of the representation $\{D^{(j)}\}$ are specified as linear transformations, which are acting onto the representation's basis vectors (6.62):

$$\phi^{(j)\prime} \equiv D^{(j)} \phi^{(j)} \tag{6.63}$$

resp. in component notation

$$\phi_{r'}^{(j)\prime} \equiv D_{r'r}^{(j)} \phi_r^{(j)} \tag{6.64}$$

The row index of ϕ' is named r', to make it distinguishable from the row index of ϕ. In the equation's right-hand side the index r is summed-up from 1 to $2j+1$ according to the sum convention. The transformations $D_{r'r}^{(j)}$ are (see appendix A.1 for computational details):

$$\boxed{\begin{aligned} D_{r'r}^{(j)} = \sum_{k=0}^{2j+1-r'} & \frac{\sqrt{(2j+1-r')!(r'-1)!(r-1)!(2j+1-r)!}}{k!(2j+1-r'-k)!(-k+r-1)!(r'+k-r)!} \cdot \\ & \cdot a^{2j+1-r'-k} b^k (-b^*)^{r'+k-r} a^{*-k+r-1} \end{aligned}} \tag{6.65}$$

Attention: In the denominator (but not in the numerator) of this formula the factorials of negative integers may occur. The factorial of integers is defined by

$$\left.\begin{aligned} z! &\equiv 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (z-1) \cdot z && \text{if } z > 0 \\ z! &\equiv 1 && \text{if } z = 0 \\ z! &\equiv \pm\infty && \text{if } z < 0 \end{aligned}\right\} z \in \mathbb{Z} \,. \tag{6.66}$$

Thus all summands with the factorial of a negative integer in their denominator are zero.

As the representations (6.65) have been constructed directly from the

two-dimensional representation SU(2), we can be sure that they really are representations of $\{\overline{\mathcal{B}}\}$, which meet all formal representation requirements, and in particular that they are homomorph to $\{\overline{\mathcal{B}}\}$.

Some representation matrices with low j-values, computed by means of (6.65), are displayed in the sequel. See appendix A.2 for computational details:

$$D^{(0)} \overset{(A.10)}{=} (1) \tag{6.67a}$$

$$D^{(\frac{1}{2})} \overset{(A.11)}{=} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{6.67b}$$

$$D^{(1)} \overset{(A.12)}{=} \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}ab^* & aa^* - bb^* & \sqrt{2}ba^* \\ b^{*2} & -\sqrt{2}b^*a^* & a^{*2} \end{pmatrix} \tag{6.67c}$$

$$D^{(\frac{3}{2})} \overset{(A.13)}{=} \tag{6.67d}$$

$$= \begin{pmatrix} a^3 & \sqrt{3}a^2b & \sqrt{3}ab^2 & b^3 \\ -\sqrt{3}a^2b^* & a^2a^* - 2abb^* & 2aba^* - b^2b^* & \sqrt{3}b^2a^* \\ \sqrt{3}ab^{*2} & -2ab^*a^* + bb^{*2} & aa^{*2} - 2bb^*a^* & \sqrt{3}ba^{*2} \\ -b^{*3} & \sqrt{3}b^{*2}a^* & -\sqrt{3}b^*a^{*2} & a^{*3} \end{pmatrix}$$

By construction, the two-dimensional representation $\{D^{(\frac{1}{2})}\} = \mathbf{2}$ clearly is identical to (6.15).

We compare this result to the representations, which we created in the previous section by means of tensor products of $D^{(\frac{1}{2})}$. The one-dimensional representation $\{D^{(0)}\}$ is – see (6.46) – identical to the one-dimensional representation $\{P\} = \mathbf{1}$, and the three-dimensional representation $\{D^{(1)}\}$ is – see (6.42) – identical to the three-dimensional representation $\{Q\} = \mathbf{3}$. In contrast the irreducible representation $\{D^{(\frac{3}{2})}\} = \mathbf{4}$ differs from the both equivalent reducible four-dimensional representations $\{W\}$ – see (6.35) – and $\{\widetilde{W}\}$ – see (6.38).

Only for irreducible representations we sometimes are using the boldface dimension numbers as shortcut representation names. Attention: Some authors code by the boldface numbers not the representation's dimension, but the value of j. They name the two-dimensional representation $\mathbf{\frac{1}{2}}$, the

three-dimensional representation **1**, and instead of our equation $\mathbf{2}\otimes\mathbf{2} \overset{(6.49)}{=} \mathbf{1}\oplus\mathbf{3}$ they are writing $\frac{\mathbf{1}}{\mathbf{2}}\otimes\frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0}\oplus\mathbf{1}$.

In the following sections, we also will need the generators and the exponential form of the transformations $D^{(j)}$. To find them, we use infinitesimal small parameters

$$\Theta_k^{\text{INF}} = \lim_{n\to\infty}\frac{\Theta_k}{n} \quad \text{with } n\in\mathbb{N}\ , \tag{6.68}$$

and expand the representation $D^{(j)}(\boldsymbol{\Theta}^{\text{INF}})$ in a Taylor series around the argument $\Theta = 0$). Because the parameters are infinitesimally small, the evolution can be ended after the linear term:

$$D^{(j)}(\boldsymbol{\Theta}_{\text{INF}}) = I + \Theta_k^{\text{INF}}\frac{\mathrm{d}\,D^{(j)}}{\mathrm{d}\,\Theta_k}\Big|_{\boldsymbol{\Theta}=0} = I + \frac{i}{\hbar}\Theta_k^{\text{INF}}\Sigma^{(j)k} \tag{6.69}$$

$$\Sigma^{(j)k} \equiv -i\hbar\frac{\mathrm{d}\,D^{(j)}}{\mathrm{d}\,\Theta_k}\Big|_{\boldsymbol{\Theta}=0} \tag{6.70}$$

$\Sigma^{(j)k}$ are the three generators of the $2j+1$-dimensional representation $\{D^{(j)}\}$ of $\{\overline{\mathcal{B}}\}$. The transformation with finite parameters is found by concatenation of infinitely many infinite small transformations:

$$\begin{aligned} D^{(j)}(\boldsymbol{\Theta}) &= \lim_{n\to\infty}\Big(I + \frac{i}{\hbar}\frac{\Theta_k}{n}\Sigma^{(j)k}\Big)^n \\ &= \exp\Big\{\frac{i}{\hbar}\Theta_k\Sigma^{(j)k}\Big\} \end{aligned} \tag{6.71}$$

6.2 The Relativistic Covering Group $\{\mathcal{B}\}$

In the previous section we have seen, that non-relativistic spinors are bases of representations of the group $\{\overline{\mathcal{B}}\}$, which is the universal covering group of the rotation group $\{\overline{\ell}\}$ of three-dimensional position space. The parameter manifold of $\{\overline{\ell}\}$ is a two-fold connected three-dimensional sphere with radius π and center point $(0,0,0)$. The parameter manifold of $\{\overline{\mathcal{B}}\}$ is a simply connected three-dimensional sphere with radius 2π and center point $(0,0,0)$.

The parameter manifold $(\eta_1, \eta_2, \eta_3, \Theta_1, \Theta_2, \Theta_3)$ of the Lorentz group $\{\ell\}$ is in three of it's six dimensions identical with the parameter manifold of $\{\overline{\ell}\}$, i.e. a two-fold connected three-dimensional sphere with radius π. Consequently the parameter manifold of the Lorentz group's universal covering group – which we name $\{\mathcal{B}\}$– must be in three of it's six dimensions identical to the parameter manifold of $\{\overline{\mathcal{B}}\}$, i.e. a simply connected sphere with radius 2π and center point $(0, 0, 0, 0, 0, 0)$.

The Lorentz group's remaining three parameters η_1, η_2, η_3 can assume any value $-\infty \leq \eta \leq +\infty$. This part of the manifold is simply connected, because no path between any two points can touch the manifold's surface. Therefore the manifold of the tree parameters $\boldsymbol{\eta}$ is identical for $\{\ell\}$ and for it's covering group $\{\mathcal{B}\}$. Thus the parameter manifold of the group $\{\mathcal{B}\}$ is completely fixed.

Relativistic spinors are the bases of representations of the group $\{\mathcal{B}\}$. Our next task therefore is to find representations of $\{\mathcal{B}\}$ with arbitrary dimensions. Some indications will be helpful in this search:

* The matrix group SU(2) is a two-dimensional representation of $\{\overline{\mathcal{B}}\}$. It's three generators are $\frac{\hbar}{2}\sigma^k$, with the Pauli-matrices σ^k, $k = 1, 2, 3$.
* The elements of all matrix representations of the groups $\{\overline{\ell}\}$ and $\{\overline{\mathcal{B}}\}$ have determinant $+1$.
* The elements of all matrix representations of the Lorentz group $\{\ell\}$ have determinant $+1$. Therefore the elements of all matrix representations of the group $\{\mathcal{B}\}$ must as well have determinant $+1$.
* $\{\mathcal{B}\}$ and all it's representations must have the same Lie algebra as the Lorentz group $\{\ell\}$.

6.2.1 Two-Dimensional Representations of $\{\mathcal{B}\}$

First we will look for a true matrix representation of $\{\mathcal{B}\}$ with dimension as low as possible. 1×1-matrices won't do, because $\{\mathcal{B}\}$ has 6 real parameters. We try 2×2-matrices with complex elements:

$$\begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ ce^{i\gamma} & de^{i\delta} \end{pmatrix} \quad \text{with } a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R} \tag{6.72}$$

The matrices' 8 real parameters are constraint by the condition for the determinant:

$$\det \begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ ce^{i\gamma} & de^{i\delta} \end{pmatrix} = ade^{i(\alpha+\delta)} - bce^{i(\beta+\gamma)} = +1 \tag{6.73}$$

In attachment A.3 it is shown, that this condition reduces the number of free matrix parameters to exactly six. The matrices (6.72) with condition (6.73) constitute the matrix group SL(2,$\mathbb{C}$). The S in the group's name signifies "special" or "unimodular". It refers to the condition that the representation's elements must have determinant = +1. (2,$\mathbb{C}$) signifies, that it is a group of 2×2-matrices with complex elements (there also exists a group SL(2,$\mathbb{R}$) with real elements). The L signifies "linear". Different from the elements of SU(2), the elements of SL(2,$\mathbb{C}$) are not unitary. Actually we did not expect that, as we already know from (5.56), that a Lie group can have a finite dimensional unitary representations only if it's parameter manifold is compact.

By means of the six real numbers, which parameterize the elements of SL(2,$\mathbb{C}$), a bijective map between $\{\mathcal{B}\}$ and SL(2,$\mathbb{C}$) can be defined. If in addition the Lie algebra of SL(2,$\mathbb{C}$) is identical to the Lie algebra of $\{\mathcal{B}\}$, then SL(2,$\mathbb{C}$) is a true two-dimensional representation of $\{\mathcal{B}\}$.

We now want to find the generators of SL(2,$\mathbb{C}$). They must be 2×2-dimensional matrices with the same Lie algebra as the Lorentz group. For comparision we consider the four-dimensional generators of the Lorentz transformations $\{\Lambda\}$, which are listed in (5.69). Their Lie algebra is

$$[K^k, K^l] \overset{(5.71a)}{=} -i\hbar\epsilon^{klm}L^m \tag{6.74a}$$

$$[K^k, L^l] \overset{(5.71b)}{=} +i\hbar\epsilon^{klm}K^m \tag{6.74b}$$

$$[L^k, L^l] \overset{(5.71c)}{=} +i\hbar\epsilon^{klm}L^m \quad . \tag{6.74c}$$

To find two-dimensional generators with the same Lie algebra, we recall that the generators of SU(2) have the same comutation relations as (6.74c):

$$\left[\frac{\hbar\sigma^k}{2}, \frac{\hbar\sigma^l}{2}\right] \overset{(6.24)}{=} +i\hbar\epsilon^{klm}\frac{\hbar\sigma^m}{2} \tag{6.75}$$

Now it isn't difficult any more, to guess the six generators

$$\pm i\,\frac{\hbar\sigma^k}{2}\ ,\ \frac{\hbar\sigma^k}{2} \quad \text{with } k = 1, 2, 3\ . \tag{6.76}$$

The Lie algebra of these generators in fact is identical to (6.74):

$$\left[\pm\, i\,\frac{\hbar\sigma^k}{2}, \pm i\,\frac{\hbar\sigma^l}{2}\right] = -i\hbar\epsilon^{klm}\frac{\hbar\sigma^m}{2} \tag{6.77a}$$

$$\left[\pm\, i\,\frac{\hbar\sigma^k}{2}, \frac{\hbar\sigma^l}{2}\right] = +i\hbar\epsilon^{klm}\Big(\pm\, i\,\frac{\hbar\sigma^m}{2}\Big) \tag{6.77b}$$

$$\left[\frac{\hbar\sigma^k}{2}, \frac{\hbar\sigma^l}{2}\right] = +i\hbar\epsilon^{klm}\frac{\hbar\sigma^m}{2} \tag{6.77c}$$

As the Pauli-matrices are hermitean, the three rotation generators $\frac{\hbar\sigma^k}{2}$ are hermitean as well. In contrast the three boost generators $\pm i\frac{\hbar\sigma^k}{2}$ are not hermitean. Caused by their double signs, there are two different two-dimensional representations of $\{\mathcal{B}\}$, which we will call $\{{}^{L}D\}$ and $\{{}^{R}D\}$ respectively. The spinors, which are the basis of the representation $\{{}^{L}D\}$, are named L. The spinors, which are the basis of the representation $\{{}^{R}D\}$, are named R:

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}' = {}^{L}D\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \equiv \exp\left\{\frac{i}{\hbar}\Big(\Theta_k\frac{\hbar\sigma^k}{2} + \eta_k i\frac{\hbar\sigma^k}{2}\Big)\right\}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \tag{6.78a}$$

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}' = {}^{R}D\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \equiv \exp\left\{\frac{i}{\hbar}\Big(\Theta_k\frac{\hbar\sigma^k}{2} - \eta_k i\frac{\hbar\sigma^k}{2}\Big)\right\}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \tag{6.78b}$$

In the notation with double-indexed angular matrices

$$(\Omega_{\sigma\tau}) \overset{(5.64a)}{=} \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ -\eta_1 & 0 & \theta_3 & -\theta_2 \\ -\eta_2 & -\theta_3 & 0 & \theta_1 \\ -\eta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \tag{6.79}$$

and double-indexed generator matrices

$$({}^{L}S^{\sigma\tau}) \equiv \frac{\hbar}{2}\begin{pmatrix} 0 & i\sigma^1 & i\sigma^2 & i\sigma^3 \\ -i\sigma^1 & 0 & \sigma^3 & -\sigma^2 \\ -i\sigma^2 & -\sigma^3 & 0 & \sigma^1 \\ -i\sigma^3 & \sigma^2 & -\sigma^1 & 0 \end{pmatrix} \tag{6.80}$$

$$({}^{R}S^{\sigma\tau}) \equiv \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sigma^1 & -i\sigma^2 & -i\sigma^3 \\ i\sigma^1 & 0 & \sigma^3 & -\sigma^2 \\ i\sigma^2 & -\sigma^3 & 0 & \sigma^1 \\ i\sigma^3 & \sigma^2 & -\sigma^1 & 0 \end{pmatrix}, \tag{6.81}$$

which are defined analogously to (5.64b), these representations can also be written in the form

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}' = {}^{L}D\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \equiv \exp\left\{\frac{i}{2\hbar}\Omega_{\sigma\tau}{}^{L}S^{\sigma\tau}\right\}\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \tag{6.82a}$$

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}' = {}^{R}D\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \equiv \exp\left\{\frac{i}{2\hbar}\Omega_{\sigma\tau}{}^{R}S^{\sigma\tau}\right\}\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}. \tag{6.82b}$$

In honor of their discoverer[1], these representations are called Weyl-representations. Attention: This is not a bivalent map. Instead these are two uniquely defined maps from $\{\mathcal{B}\}$ onto the two groups $\{{}^{R}D\}$ and $\{{}^{L}D\}$, which differ by the sign of their boost generators. We have mentioned that some authors consider the representations of $\{\mathcal{B}\}$ to be representations of the Lorentz group $\{\ell\}$. Thereby then an ambiguity appears, because the Lorentz group's parameter manifold is two-fold connected in the three parameters Θ^k, while the parameter manifold of it's covering group $\{\mathcal{B}\}$ is simply connected in these parameters. The appearance of different signs in the two representations (6.78) has nothing at all to do with that ambiguity. This is evident already from the fact that the differing signs in (6.78) are associated with the boost-generators, i.e. with the parameters η^k, whose manifold is simply connected for both the Lorentz group and it's covering group $\{\mathcal{B}\}$.

[1] Hermann Klaus Hugo Weyl (1885-1955)

Due to $\tanh\eta \overset{(5.43)}{=} v/c$, the three boost-parameters $\boldsymbol{\eta}$ are related to the relative velocity $\boldsymbol{v}$ of two coordinate systems. Therefore the spinors, which are the bases of the two representations with opposite rapidity parameters, are well suited – as we will see – to describe fields with opposite helicity. For this reason both variants of (6.78) are kept, even though the difference seemingly is marginal. In section 8.10 we will prove, that the spinors R are describing fields with right-handed helicity, while the spinors L are describing fields with left-handed helicity. Thus the indices R and L should be interpreted as "right" and "left" respectively.

6.2.2 n-Dimensional Representations of $\{\mathcal{B}\}$

Suggestion: As this section is about details which are not needed to understand the following chapters, you should skip it when working through this book first time, and continue immediately with section 6.3 on page 154.

In section 6.1.4 we have deployed techniques for the construction of arbitrary-dimensional representations $D^{(j)}$ of the group $\{\overline{\mathcal{B}}\}$, and in (6.70) we constructed their generators. The dimension of representation $D^{(j)}$ is $2j+1$. The parameter j can assume the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The (j) in brackets is *not* automatically summed-up according to the summation convention. Now we want to construct representations of the group $\{\mathcal{B}\}$ with arbitrary dimension $2j+1$. At (6.76) it was not difficult to guess the six generators of the two two-dimensional representations of $\{\mathcal{B}\}$, because we already knew the generators $\frac{\hbar\sigma^k}{2}$ of the two-dimensional representations of the group $\{\overline{\mathcal{B}}\}$. All we had to do for finding the boost-generators was to add the factor $\pm i$. This is no special feature of the two-dimensional representations. Instead the following theorem holds:

Theorem: If $\Sigma^{(j)k}$ with $k = 1, 2, 3$ are the three generators of a $2j+1$-dimensional representation of $\{\overline{\mathcal{B}}\}$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, then

$$+\Sigma^{(j)k}\ ,\ +i\Sigma^{(j)k} \quad \text{with } k = 1, 2, 3$$

are the six generators of a $2j+1$-dimensional left-handed representation of the group $\{\mathcal{B}\}$, and

$$+\Sigma^{(j)k}\ ,\ -i\Sigma^{(j)k} \quad \text{with } k = 1, 2, 3$$

are the six generators of a $2j+1$-dimensional right-handed representation of the group $\{\mathcal{B}\}$. (6.83)

Proof: Being the generators of a representation of the group $\{\overline{\mathcal{B}}\}$, the operators $\Sigma^{(j)k}$ with $k = 1, 2, 3$ have the Lie algebra

$$[\Sigma^{(j)k}, \Sigma^{(j)l}] \overset{(6.24)}{=} +i\hbar\epsilon^{klm}\Sigma^{(j)m}\ . \tag{6.84}$$

Therefore the Lie algebra of the six generators $\Sigma^{(j)k}, \pm i\Sigma^{(j)k}$ is:

$$[\pm i\Sigma^{(j)k}, \pm i\Sigma^{(j)l}] = -i\hbar\epsilon^{klm}\Sigma^{(j)m} \tag{6.85a}$$

$$[\pm i\Sigma^{(j)k}, \Sigma^{(j)l}] = +i\hbar\epsilon^{klm}(\pm i\Sigma^{(j)m}) \tag{6.85b}$$

$$[\Sigma^{(j)k}, \Sigma^{(j)l}] = +i\hbar\epsilon^{klm}\Sigma^{(j)m} \tag{6.85c}$$

This again is identical to (6.74). Theorem and proof are valid for generators of arbitrary dimension. With theorem (6.83) we can construct left- and right-handed representations of $\{\mathcal{B}\}$:

$${}^{L}D^{(j)} \equiv \exp\left\{\frac{i}{\hbar}\left(\Theta_k\Sigma^{(j)k} + \eta_k i\Sigma^{(j)k}\right)\right\} \tag{6.86a}$$

$${}^{R}D^{(j)} \equiv \exp\left\{\frac{i}{\hbar}\left(\Theta_k\Sigma^{(j)k} - \eta_k i\Sigma^{(j)k}\right)\right\} \tag{6.86b}$$

${}^{L}D^{(j)}$ and ${}^{R}D^{(j)}$ both are of dimension $2j+1$.

Sometimes the direct products of ${}^{R}D^{(j_R)}$ and ${}^{L}D^{(j_L)}$ are useful, in which $j_R = j_L$ or $j_R \neq j_L$ is possible. The $2j_R+1$-dimensional spinors, which

constitute the basis of the representation $\{{}^{R}D^{(j_R)}\}$, may be named $R^{(j_R)}$. And the $2j_L + 1$-dimensional spinors, which constitute the basis of the representation $\{{}^{L}D^{(j_L)}\}$, may be named $L^{(j_L)}$. The basis of the direct-product representations are the direct products of the spinors $R^{(j_R)}$ and $L^{(j_L)}$:

$$D^{(j_L,j_R)}\left(L^{(j_L)} \otimes R^{(j_R)}\right) \equiv \left({}^{L}D^{(j_L)}L^{(j_L)}\right) \otimes \left({}^{R}D^{(j_R)}R^{(j_R)}\right) \tag{6.87}$$

The dimension of the representation $\{D^{(j_L,j_R)}\}$ is $(2j_L + 1) \cdot (2j_R + 1)$.

6.3 Spinor Field Transformations

Due to the formal similarity to the transformation of vector fields, the formulas from section 5.6 can be carried over after some slight modifications. If the n-dimensional spinor ϕ under a rotation of the four-dimensional space-time coordinates by an infinitesimally small angle ω is transformed by

$$\phi'_a = D^{\text{INF}}_{ab}\phi_b = \Big(\delta_{ab} + \frac{i}{2\hbar}\,\omega_{\sigma\tau}S^{\sigma\tau}_{ab}\Big)\phi_b \ , \tag{6.88}$$

then under the same rotation of coordinates the n-dimensional spinor field $\phi(x)$ is transformed by

$$\phi'_a(x') \overset{(5.87)}{=} \Big(\delta_{ab} + \frac{i}{2\hbar}\omega_{\sigma\tau}(S^{\sigma\tau}_{ab} + J^{\sigma\tau}\delta_{ab})\Big)\phi_b(x) \ . \tag{6.89}$$

This is the a-th row of the matrix equation

$$\phi'(x') = \Big(1 + \frac{i}{2\hbar}\omega_{\sigma\tau}(S^{\sigma\tau} + J^{\sigma\tau})\Big)\phi(x) \ . \tag{6.90}$$

The transformation under a rotation of the four--dimensional space-time coordinates by the finite angle Ω becomes

$$\phi'(x') = \exp\Big\{\frac{i}{2\hbar}\Omega_{\sigma\tau}(S^{\sigma\tau} + J^{\sigma\tau})\Big\}\phi(x) \ . \tag{6.91}$$

As the generators S are acting onto the spinor field's amplitude components ϕ_n, but not onto their arguments x, and as the generators J are acting onto the spinor field's argument x, but not onto it's amplitude components ϕ_n, the generators S and J commute. The sequence of transformations therefore doesn't matter:

$$\phi'(x') = DR\phi(x) \tag{6.92a}$$
$$= RD\phi(x) \tag{6.92b}$$

As a third alternative the spinor field's argument can be transformed – as proved in section 5.6 – by the inverse Lorentz transformation:

$$\phi'(x') = \exp\left\{\frac{i}{2\hbar}\Omega_{\sigma\tau}S^{\sigma\tau}\right\}\phi(\Lambda^{-1}x) = D\phi(\Lambda^{-1}x) \tag{6.92c}$$

$(S^{\sigma\tau}_{ab})$ is the generator of the spinor transformation D. The space-time indices σ and τ are running from 0 to 3. We did not specify the possible values of the spinor indices a, b. Until now we only stated explicitly the representations (6.82), which are defined on the basis of two-component spinors. Of minimum same importance are the transformations of four-component Dirac-spinors. We postpone the explicit construction of a four-dimensional representation of $\{\mathcal{B}\}$ until section 8.3.

Warning: Some authors define $\phi'(x') = D\phi(x)$, i.e. they integrate the transformation of the argument x into the transformation D. Definitions can't be right or wrong, they just can be more or less convenient. It's important however, once a definition is made, to stick to it consistently. Unfortunately this is not always respected in the textbook literature.

6.4 Angular Momentum of Spinor Fields

In section (4.3) the conservation of angular momentum of scalar fields was considered, in section (5.7) the conservation of angular momentum of vector fields. We now extend the investigation to spinor fields.

The field equation of a spinor field $\phi(x)$ is derived from the Lagrangian $\mathcal{L}$. According to (4.15), the necessary (but not sufficient) condition for a transformation Γ with the generator γ to be a symmetry of ϕ, is given by

$$(\partial_\rho \mathcal{L}) w\gamma x^\rho = 0 \ . \tag{6.93}$$

Insertion of $\frac{1}{2}\omega_{\sigma\tau}(S^{\sigma\tau} + J^{\sigma\tau})$ for $w\gamma$ results in

$$0 = (\partial_\rho \mathcal{L})\frac{1}{2}\omega_{\sigma\tau}(S^{\sigma\tau} + J^{\sigma\tau})x^\rho = (\partial_\rho \mathcal{L})\frac{1}{2}\omega_{\sigma\tau}J^{\sigma\tau}x^\rho \ , \tag{6.94}$$

because x^ρ is a scalar with regard to spinor space, and therefore $S^{\sigma\tau}x^\rho = 0$. We recognize that (6.94) is identical to the necessary condition (4.66) for the conservation of angular momentum of a scalar field. Thus also for a spinor field the condition is fulfilled – as discussed at (4.66) – if and only if the Lagrangian is not explicitly dependent at least from those directions of space-time σ and τ, in which the rotation takes place, so that $\partial_\sigma \mathcal{L} = \partial_\tau \mathcal{L} = 0$.

The sufficient symmetry condition (4.11)

$$\exists\, \mathcal{G} : \quad \mathcal{L} \xrightarrow{I + \frac{i}{\hbar} w\gamma} \mathcal{L}' = \mathcal{L} + \frac{i}{\hbar} w\gamma \mathcal{L} = \mathcal{L} + \mathrm{d}_\rho \mathcal{G}^\rho$$

results by insertion of $\frac{1}{2}\,\omega_{\sigma\tau}(S^{\sigma\tau} + J^{\sigma\tau})$ for $w\gamma$ into

$$\frac{i}{\hbar} w\gamma \mathcal{L} = \mathrm{d}_\rho \mathcal{G}^\rho = \frac{i}{2\hbar}\omega_{\sigma\tau}(S^{\sigma\tau} + J^{\sigma\tau})\mathcal{L} = \frac{i}{2\hbar}\omega_{\sigma\tau}J^{\sigma\tau}\mathcal{L} \ . \tag{6.95}$$

All Lagrangians which occur in this book are scalars with respect to spinor space, i.e. in $\mathcal{L}$ all spinor indices are contracted. Thus

$$S^{\sigma\tau}\mathcal{L} = 0 \ . \tag{6.96}$$

The sufficient condition for conservation of the spinor field's angular momentum thus is met with the same function

$$\mathcal{G}^\rho \overset{(4.68)}{=} -\frac{1}{2}\omega_{\sigma\tau}(x^\sigma g^{\tau\rho} - x^\tau g^{\sigma\rho})\mathcal{L} \ , \tag{6.97}$$

with which the the sufficient condition for the scalar field's conservation of angular momentum could be met. In the same manner as in section 4.3 the conserved current's components are found:

$$j^\rho \overset{(4.16)}{=} C\Big(\frac{\sum_r \partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\frac{1}{2}\omega_{\sigma\tau}(S^{\sigma\tau}+J^{\sigma\tau})\phi_r + i\hbar\mathcal{G}^\rho\Big)$$
$$= \sum_{\sigma\tau=10,20,30,23,31,12} C i\hbar\omega_{\sigma\tau}\cdot$$
$$\cdot\Big(\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\Big(\frac{1}{i\hbar}S^{\sigma\tau}+x^\sigma\mathrm{d}^\tau - x^\tau\mathrm{d}^\sigma\Big)\phi_r - (x^\sigma g^{\tau\rho}-x^\tau g^{\sigma\rho})\mathcal{L}\Big)$$

The sum is over all spinor fields ϕ_r, which are included in the Lagrangian. In the last line the factor $1/2$ was dropped, because only the six linearly independent components of the skew-symmetric tensor $\omega_{\sigma\tau}$ are summed up. As these components are independent, there are six different conserved current densities j. By means of the energy-stress tensor

$$\mathcal{T}^{\rho\sigma} \overset{(4.32)}{=} \sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\mathrm{d}^\sigma\phi_r - g^{\rho\sigma}\mathcal{L} \tag{6.98}$$

(which was defined in section 4.2), and by means of the spin density

$$\mathcal{S}^{\rho\sigma\tau} \equiv \frac{1}{i\hbar c}\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}S^{\sigma\tau}\phi_r \tag{6.99}$$

(which significantly differs from the spin density (5.99) of the vector field considered in section 5.7), and with the definition $C \equiv 1/(ci\hbar\omega_{\sigma\tau})$, the components of the conserved current density can be written in the simple form

$$j^\rho \overset{(4.70)}{=} x^\sigma\frac{\mathcal{T}^{\rho\tau}}{c} - x^\tau\frac{\mathcal{T}^{\rho\sigma}}{c} + \mathcal{S}^{\rho\sigma\tau} \qquad \text{with } \sigma\tau = 10,20,30,23,31,12\ . \tag{6.100}$$

The conserved current densities are combined into the angular momentum density tensor

$$\boxed{\mathcal{M}^{\rho\sigma\tau} \equiv x^\sigma\frac{\mathcal{T}^{\rho\tau}}{c} - x^\tau\frac{\mathcal{T}^{\rho\sigma}}{c} + \mathcal{S}^{\rho\sigma\tau}}\ . \tag{6.101}$$

It's dimension is action/volume. The tensor $\mathcal{M}$ fulfills the six linearly independent equations of continuity

$$\mathrm{d}_\rho j^\rho = \mathrm{d}_\rho \mathcal{M}^{\rho\sigma\tau} = 0 \quad \text{for } \sigma\tau = 10, 20, 30, 23, 31, 12 \ . \tag{6.102}$$

Integrating the null-components over the complete position space we find the six conserved angular momenta

$$M^{\sigma\tau} \equiv \underbrace{\int\limits_\Omega \mathrm{d}^3x \Big(x^\sigma \, \frac{\mathcal{T}^{0\tau}}{c} - x^\tau \, \frac{\mathcal{T}^{0\sigma}}{c} \Big)}_{\text{orbital angular momentum}} + \underbrace{\int\limits_\Omega \mathrm{d}^3x \, \mathcal{S}^{0\sigma\tau}}_{\text{spin}} \tag{6.103}$$
$$\text{with } \sigma\tau = 10, 20, 30, 23, 31, 12 \ ,$$

which we discussed already at (4.73). There we had noticed, that for practical applications only the purely space-like angular momenta with $\sigma\tau = 23, 31, 12$ are useful. Note, that conservation laws only hold for the total angular momenta, but not separately for orbital angular momenta or spins.

7 Normalization, Delta Function, Kronecker Symbol

The solutions of free field equations will be our starting point for field quantization. Important formal topics of free fields are the normalization of state vectors, and — related to normalization — different methods to implement Dirac's delta function and the Kronecker symbol. We now will address these general topics, and then turn to the individual fields.

If a system is described by the state vector $\psi(x)$, then

$$\langle K \rangle \equiv \frac{\int\limits_{\Omega} \mathrm{d}^3x\, \psi^*(t,\boldsymbol{x}) K \psi(t,\boldsymbol{x})}{\int\limits_{\Omega} \mathrm{d}^3x\, \psi^*(t,\boldsymbol{x}) \psi(t,\boldsymbol{x})} \tag{7.1}$$

is the expectation value of an observable, which is represented by the operator K. To make this method of computation possible, the value of the normalization integral

$$\int\limits_{\Omega} \mathrm{d}^3x\, \psi^*(t,\boldsymbol{x}) \psi(t,\boldsymbol{x}) = N \tag{7.2}$$
$$\text{with } N \in \mathbb{R}\ ,\ N \neq 0\ ,\ -\infty < N < +\infty$$

compulsory must be real, finite, and different from zero. It's often convenient to have $N = 1$, or at least lorentz-invariant. But neither of both is inevitable. In many cases we will apply other normalization conditions, to simplify equations or make contexts transparent.

Should the normalization volume Ω better be finite or infinite? Only a finite normalization volume is physically reasonable. The visible universe is finite. Even in case that there should be anything beyond the visible universe, we can by no means get informations about it. Nothing, which is

beyond the visible universe, can execute any action which can be observed by us. Our interpretation of

$$\frac{1}{N}\int_{\Omega} \mathrm{d}^3x\, \psi^*(t,\boldsymbol{x})\psi(t,\boldsymbol{x}) \overset{(7.2)}{=} 1 \tag{7.3}$$

is, that e.g. a particle, of whose existence we have positive knowledge, but whose location in space we don't know, can be detected with probability = 1 somewhere in the normalization volume Ω. It's impossible that a particle, of whose existence we have positive knowledge, is located outside the visible universe. In most cases the space volume, in which it must be localized with certainty, is by far smaller. If for example we want to describe what is happening in a high-energy experiment in an electron-positron collider, then we know for sure, that one nanosecond after the collision, all generated fields with no exception can be detected inside a sphere with 30 cm radius around the collision point. A reasonable normalization volume can not exceed the finite size of the visible universe. In almost all cases of practical relevance, it may be chosen much smaller.

But the finite normalization volume is associated with a drawback: It is incompatible with plane waves. Strictly speaking, plane waves don't exist. At (and the more so beyond) the normalization volume's boundary, any field's amplitude must be zero. Only wave packets, but not plane waves, can have finite extension. On the other hand, if a wave packet's amplitude does not change significantly within the observed region – which in this case is much smaller than the normalization volume – then in good approximation it can be described as a plane wave. We would like to avoid the immense paperwork, to describe such quasi-plane waves as wave packets. Instead we prefer to handle them as simple plane waves $\psi(x) \sim \exp\{-i(\omega t - \boldsymbol{kx})\}$.

Therefore we formally relax the normalization requirement by two methods: Either we define the normalization volume $\Omega = S^1 \cdot S^2 \cdot S^3$ as a box with edge lengths S^1,S^2,S^3, and postulate periodic boundary conditions

$$\psi(t, x^1 + 2z_1S^1, x^2 + 2z_2S^2, x^3 + 2z_3S^3) = \psi(t,\boldsymbol{x}) \qquad \text{with } z_1, z_2, z_3 \in \mathbb{Z}\ . \tag{7.4}$$

This normalization condition allows for plane waves. As a second method, we formally consider the limit $\Omega \to \infty$, notwithstanding the mentioned physical objections. In doing so, we often will apply the formula

$$\lim_{\Omega\to\infty} \frac{1}{\Omega} \sum_{\boldsymbol{k}} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^3 k}{(2\pi)^3} \ . \tag{7.5}$$

We now consider a field $\psi(\boldsymbol{x}) \equiv \psi(t=0,\boldsymbol{x})$ at time $t=0$. Due to the finite normalization volume, $\psi(\boldsymbol{x})$ can be expanded in a Fourier series

$$\psi(\boldsymbol{x}) = \frac{1}{\Omega} \sum_{\boldsymbol{k}} w_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}\boldsymbol{x}\} \tag{7.6a}$$

$$w_{\boldsymbol{k}} = \int_{\Omega} \mathrm{d}^3 x \, \psi(\boldsymbol{x}) \exp\{-i\boldsymbol{k}\boldsymbol{x}\} \ . \tag{7.6b}$$

Remark: Some authors define (7.6b) as $w_{-\boldsymbol{k}}$ and/or distribute the normalization factor $\sqrt{1/\Omega}$ onto both lines of (7.6).

The sum in (7.6a) is over all wave numbers $\boldsymbol{k}$ which are compatible with the boundary condition (7.4). If the normalization volume Ω is large, the spectrum of wave numbers is to dense to be resolved by observations, and seems to be continuous. Still we consider the wave numbers as a discrete and countable infinite set which can be summed-up, because Ω is finite.

Authors, who choose an infinitely large normalization volume, get a Fourier integral instead of the Fourier series:

$$\psi(\boldsymbol{x}) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^3 k}{(2\pi)^3} \, w(\boldsymbol{k}) \exp\{+i\boldsymbol{k}\boldsymbol{x}\} \tag{7.7a}$$

$$w(\boldsymbol{k}) = \int_{-\infty}^{+\infty} \mathrm{d}^3 x \, \psi(\boldsymbol{x}) \exp\{-i\boldsymbol{k}\boldsymbol{x}\} \tag{7.7b}$$

The delta function is defined by

$$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}) = 0 \quad \text{if } \boldsymbol{x} \neq \boldsymbol{z} \tag{7.8a}$$

$$\int \mathrm{d}^3x\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{z})\phi(\boldsymbol{x}) = \phi(\boldsymbol{z})\ . \tag{7.8b}$$

Here the integration volume must cover the point $\boldsymbol{z}$. With finite normalization volume Ω, the delta function can be written as

$$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}) = \frac{1}{\Omega}\sum_{\boldsymbol{k}} \exp\{\pm i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{z})\}\ , \tag{7.9}$$

with infinite normalization volume as

$$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{z}) = \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}^3k}{(2\pi)^3}\ \exp\{\pm i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{z})\}\ . \tag{7.10}$$

Using an infinite normalization volume, the three-dimensional delta function of wave numbers

$$\delta^{(3)}(\boldsymbol{k}-\boldsymbol{f}) = 0 \quad \text{if } \boldsymbol{k} \neq \boldsymbol{f} \qquad\qquad \int\limits_{-\infty}^{+\infty} \mathrm{d}^3k\, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{f}) = 1$$

can be written as

$$\delta^{(3)}(\boldsymbol{k}-\boldsymbol{f}) = \frac{1}{(2\pi)^3} \int\limits_{-\infty}^{+\infty} \mathrm{d}^3x\ \exp\{\pm i(\boldsymbol{k}-\boldsymbol{f})\boldsymbol{x}\}\ . \tag{7.11}$$

With finite normalization volume, the spectrum of wave numbers is discrete, and the delta function $\delta^{(3)}(\boldsymbol{k}-\boldsymbol{f})$, whose dimension is volume, is replaced by the dimension-less Kronecker symbol $\delta_{\boldsymbol{kf}}$:

$$\delta_{\boldsymbol{kf}} = \frac{1}{\Omega}\int\limits_{\Omega} \mathrm{d}^3x\ \exp\{\pm i(\boldsymbol{k}-\boldsymbol{f})\boldsymbol{x}\} \tag{7.12}$$

We also will need the extensions of Fourier expansions and delta functions to four space-time dimensions. With infinite normalization volume, from (7.7) we get

$$\psi(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, w(k) \exp\{-ikx\} \tag{7.13a}$$

$$w(k) = \int_{-\infty}^{+\infty} \mathrm{d}^4x\, \psi(x) \exp\{+ikx\}\ , \tag{7.13b}$$

and the delta functions become

$$\delta^{(4)}(x-z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\, \exp\{\pm ik(x-z)\} \tag{7.14a}$$

$$\delta^{(4)}(k-f) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4x}{(2\pi)^4}\, \exp\{\pm i(k-f)x\}\ . \tag{7.14b}$$

The finite normalization volume Ω only refers to the three space-like dimensions. The time is considered infinite. Therefore with finite normalization volume, the four-dimensional Fourier transformation must be constructed with a combination of (7.6) and (7.13):

$$\psi(x) = \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, w_{\boldsymbol{k}}(k^0) \exp\{-i(k^0x^0 - \boldsymbol{kx})\} \tag{7.15a}$$

$$w_{\boldsymbol{k}}(k^0) = \int_{\Omega} \mathrm{d}^3x \int_{-\infty}^{+\infty} \mathrm{d}x^0\, \psi(x) \exp\{+i(k^0x^0 - \boldsymbol{kx})\} \tag{7.15b}$$

With finite normalization volume, the four-dimensional delta functions are constructed by combining (7.9), (7.12), and (7.14):

$$\delta^{(4)}(x-z) = \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \exp\{\pm ik(x-z)\} \tag{7.16a}$$

$$\delta(k^0 - f^0)\, \delta_{\boldsymbol{kf}} = \frac{1}{2\pi\Omega} \int_{\Omega} \mathrm{d}^3x \int_{-\infty}^{+\infty} \mathrm{d}x^0\, \exp\{\pm i(k-f)x\} \tag{7.16b}$$

In equations (7.13) to (7.16a), the integration variable k^0 can assume any value between $-\infty$ and $+\infty$. But as null-component of the wave vector of a plane wave $\exp\{-i(\omega_{\boldsymbol{k}}t - \boldsymbol{kx})\}$, the frequency $\omega_{\boldsymbol{k}} = ck^0$ is connected to the wave number's space components $\boldsymbol{k}$ due to the invariant wavenumber square

$$k_\nu k^\nu = (k^0)^2 - \boldsymbol{k}^2 = \left(\frac{\omega_{\boldsymbol{k}}}{c}\right)^2 - \boldsymbol{k}^2 = \frac{m^2c^2}{\hbar^2} \tag{7.17}$$

of Special Relativity theory, thus being no independent variable. We fix the sign of $k^0 = \omega_{\boldsymbol{k}}/c$ by definition:

We stipulate, that

$$k^0 = \frac{\omega_{\boldsymbol{k}}}{c} = +\sqrt{\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}} > 0 \tag{7.18}$$

always is the positive root. Therefore a negative frequency must be written as $-\omega_{\boldsymbol{k}} = -ck^0 < 0$.

(The case $\omega_{\boldsymbol{k}} = 0$ is impossible for fields with finite mass $m \neq 0$. And a massless field with frequency zero would simply not be existent.) To include all possible waves, the Fourier-expansion to the space-like wave numbers $\boldsymbol{k}$ then must incorporate for each wave number one summand with $+\omega_{\boldsymbol{k}} > 0$ and one summand with $-\omega_{\boldsymbol{k}} < 0$:

$$\psi(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}}\Big(a_{\boldsymbol{k}} \exp\{-ikx\} + b^*_{\boldsymbol{k}} \exp\{+ikx\}\Big) \tag{7.19}$$

$$= \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}}\Big(a_{\boldsymbol{k}} \exp\{-i(\omega_{\boldsymbol{k}}t - \boldsymbol{kx})\} + b^*_{\boldsymbol{k}} \exp\{+i(\omega_{\boldsymbol{k}}t - \boldsymbol{kx})\}\Big)$$

The sum runs over all positive and all negative wave numbers $\boldsymbol{k}$, which are compatible with the condition (7.4). But $k^0 = \omega_{\boldsymbol{k}}/c > 0$ always is positive. We introduced the normalization factor $1/\sqrt{N\Omega}$ with temporarily indefinite N, to make the normalization flexible for each particular application case. It is an additional advantage of the notation with the Fourier-coefficients $a_{\boldsymbol{k}}$ and $b^*_{\boldsymbol{k}}$, that we can easily make sure that the field $\psi(x)$ is real, if desired: For that purpose we just need to specify $b_{\boldsymbol{k}} = a_{\boldsymbol{k}}$ for all wave numbers.

The finite normalization volume is not only physically better justified than an infinite one. It also spares us a complication regarding the normalization integral's Lorentz invariance. Let the plain wave

$$|\boldsymbol{k}\rangle \equiv \frac{1}{\sqrt{\Omega}} \exp\{-ikx\} = \frac{1}{\sqrt{\Omega}} \exp\{-i(\omega_{\boldsymbol{k}} t - \boldsymbol{kx})\} \tag{7.20}$$

be a solution of a field equation. With finite normalization volume Ω,

$$\langle \boldsymbol{q}|\boldsymbol{k}\rangle = \frac{\exp\{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{q}})t\}}{\Omega} \int\limits_{\Omega} \mathrm{d}^3x \, \exp\{i(\boldsymbol{k}-\boldsymbol{q})\boldsymbol{x}\} \overset{(7.12)}{=} \delta_{\boldsymbol{qk}} \tag{7.21}$$

because of $\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{q}}$ at $\boldsymbol{k} = \boldsymbol{q}$. Observed from a coordinate system$'$, which is moving with velocity v relatively to the unprimed system, the normalization volume, which is at rest in the unprimed system, is shrunk by the factor

$$\sqrt{1-(v/c)^2} \equiv \gamma^{-1} \ . \tag{7.22}$$

Therefore [32, (45a)]

$$\Omega' = \gamma^{-1}\Omega < \Omega \ . \tag{7.23}$$

Due to relativistic length contraction, the normalization volume is smaller in the primed system (in which it is moving) than in the unprimed system (in which it is at rest). Exactly the same holds for the infinitesimal volume d^3x, which is summed-up in (7.21). The contraction of d^3x and the contraction of Ω compensate mutually. Therefore, provided that the probability density $\psi^*(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$ is Lorentz-invariant, the normalization integral

$$\frac{1}{\Omega} \int\limits_{\Omega} \mathrm{d}^3x \, \psi^*(t,\boldsymbol{x})\psi(t,\boldsymbol{x}) \tag{7.24}$$

is generally Lorentz-invariant as well.

Using an infinite normalization volume, one gets instead for plane waves

$$|\boldsymbol{k}\rangle \equiv \psi(x) = \frac{1}{\sqrt{N}} \exp\{-ikx\} = \frac{1}{\sqrt{N}} \exp\{-i(\omega_{\boldsymbol{k}} t - \boldsymbol{kx})\}$$

the normalization integral

$$\langle \boldsymbol{q}|\boldsymbol{k}\rangle = \frac{\exp\{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{q}})t\}}{N} \int_{-\infty}^{+\infty} \mathrm{d}^3x \, \exp\{i(\boldsymbol{k}-\boldsymbol{q})\boldsymbol{x}\} =$$

$$\overset{(7.11)}{=} \frac{(2\pi)^3}{N} \delta^{(3)}(\boldsymbol{q}-\boldsymbol{k}) \, . \tag{7.25}$$

If the normalization factor N is Lorentz-invariant, then this integral is not Lorentz-invariant due to $\mathrm{d}^3x' = \gamma^{-1}\mathrm{d}^3x < \mathrm{d}^3x$. If we want to compensate this contraction, then we must apply a normalization factor with same contraction. As an alternative, the time-like component of an arbitrary Lorentz vector can be inserted as the reciprocal normalization factor $1/N$, because this component is stretched by the factor γ under a rotation of the space-time-coordinates. Many authors choose the energy $\hbar\omega$ for this purpose. Using an infinite normalization volume, they define the Lorentz-invariant normalization integral

$$\langle \boldsymbol{q}|\boldsymbol{k}\rangle = \sqrt{\hbar\omega_{\boldsymbol{k}}\hbar\omega_{\boldsymbol{q}}} \exp\{-i(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{q}})t\} \int_{-\infty}^{+\infty} \mathrm{d}^3x \, \exp\{i(\boldsymbol{k}-\boldsymbol{q})\boldsymbol{x}\} =$$

$$\overset{(7.11)}{=} \hbar\omega_{\boldsymbol{k}} (2\pi)^3 \delta^{(3)}(\boldsymbol{q}-\boldsymbol{k}) \, . \tag{7.26}$$

While in this book we will stick to the finite normalization volume (7.21), the reader should as well be familiar with normalization to infinite volume, as this is often encountered in the literature. Normalization factors like $\hbar\omega_{\boldsymbol{k}}$ might seem quite perplexing to somebody, who does not know that they are arbitrarily inserted for the purpose of Lorentz-invariance of the normalization integral (7.26).

8 The classical Dirac Field

8.1 Dirac Equation

From the energy-momentum vector

$$p = (p^\nu) = \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \tag{8.1}$$

of special relativity theory one gets

$$\sqrt{c^2 p_\nu p^\nu} = \sqrt{E^2 - (c\boldsymbol{p})^2} = mc^2 \tag{8.2}$$

as the rest energy, which is invariant under Lorentz transformations. Therefore a relativistically invariant field equation with energy operator $i\hbar \mathrm{d}_t$ and momentum operator $-i\hbar\boldsymbol{\nabla}$ should be of the form

$$\sqrt{\left(i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\right)^2 - (-i\hbar c\boldsymbol{\nabla})^2}\,\psi = mc^2\psi \ . \tag{8.3}$$

Due to the square root, this equation is extremely unwieldy and difficult to handle. If the square root is expanded into an infinite series, one arrives – because infinite powers of the differential operators are occurring – at a non-local theory, with which a plethora of new problems are associated. It was Dirac's idea, to convert (8.3) into a linear equation by means of (initially indefinite) coefficients γ^μ:

$$(i\hbar c\gamma^0 \frac{\mathrm{d}}{\mathrm{d}ct} + i\hbar c\boldsymbol{\gamma}\boldsymbol{\nabla})\psi = i\hbar c\gamma^\mu \mathrm{d}_\mu \psi = mc^2\psi \tag{8.4}$$

The result is the Dirac equation

$$\boxed{(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\psi = 0}\,, \tag{8.5}$$

which turned out to be a remarkably successful concept.

To determine the coefficients γ^μ, we take the square of equation (8.2)

$$E^2 - (c\boldsymbol{p})^2 = (mc^2)^2\,, \tag{8.6}$$

and iterate both sides of (8.4):

$$\begin{aligned} i\hbar c\gamma^\mu \mathrm{d}_\mu i\hbar c\gamma^\sigma \mathrm{d}_\sigma \psi &= (mc^2)^2\psi \\ (i\hbar c)^2\Big(\gamma^0\gamma^0\mathrm{d}_0\mathrm{d}_0 + (\gamma^0\gamma^k + \gamma^k\gamma^0)\mathrm{d}_0\mathrm{d}_k + \gamma^k\gamma^j\mathrm{d}_k\mathrm{d}_j\Big)\psi &= (mc^2)^2\psi \end{aligned} \tag{8.7}$$

Comparing this to (8.6) results into the condition

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{8.8}$$

The stronger condition $\gamma^\mu\gamma^\nu = g^{\mu\nu}$ would make both equations compatible as well. But it would be a too severe constriction. The weaker condition (8.8) is sufficient, because (8.8) can be fulfilled with imaginary parts $\mathrm{Im}(\gamma^\mu\gamma^\nu) = -\mathrm{Im}(\gamma^\nu\gamma^\mu)$, while the stronger condition $\gamma^\mu\gamma^\nu = g^{\mu\nu}$ can only be met with $\mathrm{Im}(\gamma^\mu\gamma^\nu) = 0$.

(8.8) can not be solved by numbers. For this reason, Dirac proposed to interpret the coefficients γ^ν as matrices. Thus condition (8.8) turns into the condition

$$\boxed{\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1}}\,. \tag{8.9}$$

$\mathbb{1}$ is a n-dimensional unit matrix in the n-dimensional spinor space, which is spanned by the spinors ψ. The spinor space is an abstract functional space of the theory. It must not be confused with the four-dimensional time-position space.

By now we don't know the γ-matrices explicitly. We don't even have fixed their dimension. To find out details of these matrices, we multiply

(8.5) from left by γ^0:

$$(i\hbar c \underbrace{\gamma^0\gamma^0}_{=\mathbb{1} \text{ because of (8.9)}} \mathrm{d}_0 + i\hbar c\gamma^0\gamma^j \mathrm{d}_j - \gamma^0 mc^2)\psi = 0 \tag{8.10}$$

As any quantum-theoretical operator, which is representing an observable quantity, the energy operator $i\hbar c\mathrm{d}_0$ is hermitean (= self-adjoint). The same holds for the momentum operator $-i\hbar\mathrm{d}_j$ and the operator mc^2 (which is simply a real factor). Therefore $\gamma^0\gamma^j$ and γ^0 must be hermitean as well:

$$\gamma^{0\dagger} = \gamma^0 \tag{8.11a}$$

$$(\gamma^0\gamma^j)^\dagger = \gamma^0\gamma^j \overset{(8.9)}{=} (-\gamma^j\gamma^0)^\dagger = -\gamma^{0\dagger}\gamma^{j\dagger} = -\gamma^0\gamma^{j\dagger}$$

$$\Longrightarrow \quad \gamma^{j\dagger} = -\gamma^j \tag{8.11b}$$

This can be combined into the equation

$$\boxed{\gamma^{\mu\dagger} = g_{\mu\nu}\gamma^\nu}\,. \tag{8.12}$$

This is the second and final condition which the γ-matrices must meet. Besides (8.9) and (8.12) there are no further constraints.

In appendix A.8 it is proved, that the γ-matrices must have even dimension. For simplicity we try to find appropriate matrices of dimension as low as possible. The matrices

$$\gamma^k \equiv i\sigma^k \qquad \text{preliminary only!} \tag{8.13}$$

with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.14}$$

are complying with both (8.9) and (8.12). But there does not exist a fourth 2×2 matrix γ^0, which fulfills these conditions as well. Therefore the γ-matrices' dimension must minimum be 4×4.

The partial success with the Pauli matrices at least is indicating that

we are searching in the right direction. Therefore we try to construct the four-dimensional matrices as the direct product $\rho \otimes \sigma$ of the Pauli matrices and appropriate 2×2-matrices ρ. In (8.13) we made the γ^k anti-hermitean due to multiplication of σ^k by i. Now for the same purpose we define ρ^k as $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For σ^0 usually simply $\mathbb{1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is chosen. This leads to the desired result with $\rho^0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ or with $\rho^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Dirac decided for the second alternative, which often is called the "standard form". As this form of the γ-matrices is in widespread use, we will shortly present it in section 8.7. But in this book we prefer the other, so-called "chiral" form of the matrices, because it makes transparent the structure of the Dirac equation and it's relationship with the Weyl-equations. The Dirac matrices in chiral form are:

$$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ g_{00} & 0 \end{pmatrix} \otimes \mathbb{1} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8.15a}$$

$$\gamma^1 \equiv \begin{pmatrix} 0 & 1 \\ g_{11} & 0 \end{pmatrix} \otimes \sigma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{8.15b}$$

$$\gamma^2 \equiv \begin{pmatrix} 0 & 1 \\ g_{22} & 0 \end{pmatrix} \otimes \sigma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{8.15c}$$

$$\gamma^3 \equiv \begin{pmatrix} 0 & 1 \\ g_{33} & 0 \end{pmatrix} \otimes \sigma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8.15d}$$

It is not difficult to prove by straight-forward computation, that these matrices indeed are solutions of the equations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \overset{(8.9)}{=} 2g^{\mu\nu}\mathbb{1} \tag{8.16}$$

$$\gamma^{\mu\dagger} \overset{(8.12)}{=} g_{\mu\nu}\gamma^{\nu} \tag{8.17}$$

with $\mu, \nu = 0, 1, 2, 3$. Attention: The spinor space of Dirac-theory only by chance has the same dimension — namely four — as the time-position space. The spinor space is an abstract space of functions, which is not identical with time-position space.

The four γ-matrices are constants, i. e. Lorentz-scalars. They don't change when the coordinates of time-position space are rotated. In other words: They do *not* constitute a four-dimensional Lorentz vector $\gamma = (\gamma^{\mu})$. Therefore the factor $\gamma^{\mu}\mathrm{d}_{\mu}$ in the Dirac equation $(i\hbar c\gamma^{\mu}\mathrm{d}_{\mu} - mc^2)\psi \overset{(8.5)}{=} 0$ is no Lorentz scalar. Thus the Dirac equation's Lorentz invariance is not at all evident, but needs an accurate check, which we will undertake in section 8.3.

For some computations a fifth matrix will be useful, which is defined as follows:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{8.18}$$

Attention: Authors, who choose a different form of the γ-matrices (8.15), consequently get a different form of γ^5.

Due to our chosen chiral form of the γ-matrices, the four-dimensional Dirac spinors become – as we will see – the direct sum of two-dimensional Weyl spinors, which we already know from (6.78). The both top components of any Dirac spinor constitute a left-handed Weyl spinor, and it's both bottom components constitute a right-handed Weyl spinor:

$$\psi(x) = \begin{pmatrix} L_1(x) \\ L_2(x) \\ R_1(x) \\ R_2(x) \end{pmatrix} = L(x) \oplus R(x) \tag{8.19}$$

By means of the matrix γ^5, projection operators ${}^{L}P$ and ${}^{R}P$ can be con-

structed, which project the left- and right-handed components out of Dirac spinors:

$$ {}^{L}P\psi \equiv \frac{1}{2}(\mathbb{1} - \gamma^5)\begin{pmatrix} L \\ R \end{pmatrix} = \begin{pmatrix} L \\ 0 \end{pmatrix} \tag{8.20a} $$

$$ {}^{R}P\psi \equiv \frac{1}{2}(\mathbb{1} + \gamma^5)\begin{pmatrix} L \\ R \end{pmatrix} = \begin{pmatrix} 0 \\ R \end{pmatrix} \tag{8.20b} $$

This transparent relation in-between the two-dimensional Weyl-representations and the four-dimensional Dirac-representation of the group $\{\mathcal{B}\}$ is the reason, why we prefer the chiral form of the γ-matrices.

8.2 Lagrangian

The Dirac field's Lagrangian was already indicated in section 4.5:

$$ \boxed{\mathcal{L} \overset{(4.120)}{=} \overline{\psi}_a \left(i\hbar c \gamma^{\nu}_{ab} (\mathrm{d}_\nu + \frac{i}{\hbar} q A_\nu) - mc^2 \delta_{ab} \right) \psi_b - \frac{1}{4\mu_0} F_{\sigma\tau} F^{\sigma\tau}} \tag{8.21} $$

The indices a, b are spinor indices. Whenever they show up twice in a product, according to the summation convention they are to be summed up automatically, in this case from 1 to 4. The indices ν, σ, τ are space-time indices, which automatically are to be summed up from 0 to 3 whenever they show up twice in a product. We carefully must distinguish superscript contravariant space-time indices from subscript covariant indices. In contrast the spinor space is an euclidean space, and we place spinor indices always bottom. We emphasize once more, that time-position space and spinor space are different spaces, which only by chance both are four-dimensional in Dirac theory.

The Dirac-matrices $(\gamma_{ab})^{\nu}$ are most notable. They are four 4×4-matrices in spinor space. In (8.15) these matrices are explicitly displayed. They are *not* the components of a Lorentz vector, but constants (Lorentz-scalars). Still they are — not without reason — suggestively marked by a space-time index. We will turn back to this curiosity in the investigation of the Lagrangian's Lorentz invariance in section 8.3.

The field $\overline{\psi}(x)$ is closely related to the field $\psi(x)$ by it's definition

$$\overline{\psi} \equiv \psi^\dagger \gamma^0 \ . \tag{8.22}$$

$\psi^\dagger$ is the adjoint, i.e. transposed complex-conjugate spinor of ψ. Abbreviating — though slightly imprecise — also $\overline{\psi}$ is often called the adjoint spinor of ψ. As we already decided in (8.15) for a concrete form of the Dirac matrices, we can indicate $\overline{\psi}$ explicitly:

$$\begin{aligned} \psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad\qquad \psi^\dagger = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{pmatrix} \\ \overline{\psi} &= \psi^\dagger \gamma^0 \overset{(8.15a)}{=} \begin{pmatrix} \psi_3^* & \psi_4^* & \psi_1^* & \psi_2^* \end{pmatrix} \end{aligned} \tag{8.23}$$

The unit of ψ and of $\overline{\psi}$ is volume$^{-1/2}$, the unit of the Lagrangian $\mathcal{L}$ is energy/volume.

In section 4.5 the "covariant derivative" $\mathrm{d}_\nu + \frac{i}{\hbar} q A_\nu$, which is replacing the "normal" derivative d_ν, has been motivated by the demand for invariance of the charged Dirac field under gauge transformations. There also the term $-\frac{1}{4\mu_0} F_{\sigma\tau} F^{\sigma\tau}$, which exclusively is related to the gauge field, was justified. The other ingredients of the Lagrangian have not been somehow derived from basic principles. Only after Dirac had found the field equation (8.5), the Lagrangian was constructed such, that this field equation results from it due to variation of it's variables. Still the Lagrangian is helpful to achieve a systematic understanding of the theory's symmetries and conservation laws. In section 8.3 we will use the Lagrangian to investigate the Dirac equation's Lorentz invariance.

When we quantize the fields, we will in a first step separately quantize the "free", not interacting Dirac field, and the "free", not interacting electromagnetic field. Only after that we will in a second step analyze the interactions in-between quantized fields. Therefore it is reasonable to define the Lagrangian of the free Dirac field, in which the gauge field is switched off:

$$\mathcal{L} \overset{(8.21)}{=} \overline{\psi}_a \left(i\hbar c \gamma^\nu_{ab} \mathrm{d}_\nu - mc^2 \delta_{ab} \right) \psi_b \tag{8.24}$$

Concluding this section, we now will check that the field equations indeed can be derived from the Lagrangian. $\psi(x)$, $\overline{\psi}\,(x)$, and the gauge field's four components $A^\mu(x)$ are the six variables, with respect to which the Lagrangian (8.21) must be varied to derive the six field equations. In the lines after (3.37c) we explained, why $\overline{\psi}\,(x)$ and $\psi(x)$ must be considered independent variables of $\mathcal{L}$ for derivation of the field equations. The field equations for $A^\mu(x)$ were already derived and analyzed in section 4.5. Therefore we now will only consider the field equations for $\psi(x)$ and for $\overline{\psi}\,(x)$.

Variation of $\mathcal{L}$ with respect to $\psi(x)$ leads to the field equation

$$\begin{aligned} 0 &\overset{(3.37b)}{=} \mathrm{d}_\nu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} \\ &= \mathrm{d}_\nu \, \overline{\psi}_a \, i\hbar c \gamma^\nu_{ab} - \overline{\psi}_a \left(i\hbar c \gamma^\nu_{ab} (\frac{i}{\hbar} q A_\nu) - mc^2 \delta_{ab} \right) \\ &= i\hbar c (\mathrm{d}_\nu - \frac{i}{\hbar} q A_\nu) \, \overline{\psi}_a \, \gamma^\nu_{ab} + mc^2 \delta_{ab} \, \overline{\psi}_a = 0 \, . \end{aligned} \tag{8.25a}$$

Variation of $\mathcal{L}$ to $\overline{\psi}\,(x)$ leads to the field equation

$$\begin{aligned} 0 &\overset{(3.37b)}{=} \mathrm{d}_\nu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu \overline{\psi})} - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} \\ &= 0 - \left(i\hbar c \gamma^\nu_{ab} (\mathrm{d}_\nu + \frac{i}{\hbar} q A_\nu) - mc^2 \delta_{ab} \right) \psi_b \, . \end{aligned} \tag{8.25b}$$

(8.25b) is the a-th row of the spinor equation

$$\left(i\hbar c \gamma^\nu (\mathrm{d}_\nu + \frac{i}{\hbar} q A_\nu) - mc^2 \right) \psi = 0 \, , \tag{8.26a}$$

which – apart from replacement of the normal derivative d_ν by the covariant derivative $\mathrm{d}_\nu + \frac{i}{\hbar} q A_\nu$, see section 4.5 – is identical to the Dirac equation (8.5). (8.25a) is the b-th column of the spinor equation

$$-i\hbar c(\mathrm{d}_\nu - \frac{i}{\hbar}qA_\nu)\,\overline{\psi}\,\gamma^\nu - mc^2\,\overline{\psi} = 0\,. \tag{8.26b}$$

We now will prove, that this is not an independent equation, but the adjoint of (8.26a): With the rule $(AB)^\dagger = B^\dagger A^\dagger$, which is valid for arbitrary matrices, and with $i^\dagger = -i$, we get

$$\begin{aligned} 0 &\overset{(8.26a)}{=} \Big((i\hbar c\gamma^\nu(\mathrm{d}_\nu + \frac{i}{\hbar}qA_\nu) - mc^2)\psi\Big)^\dagger \\ &= -i\hbar c(\mathrm{d}_\nu - \frac{i}{\hbar}qA_\nu)\psi^\dagger\gamma^{\nu\dagger} - mc^2\psi^\dagger\,. \end{aligned} \tag{8.27}$$

Multiplication from right by γ^0 results into

$$0 = -i\hbar c(\mathrm{d}_\nu - \frac{i}{\hbar}qA_\nu)\psi^\dagger \underbrace{\gamma^{\nu\dagger}\gamma^0}_{\gamma^0\gamma^\nu} - mc^2\psi^\dagger\gamma^0\,. \tag{8.28}$$

$$\begin{aligned} &\text{case } \nu = 0: \quad \gamma^{\nu\dagger}\gamma^0 \overset{(8.12)}{=} \gamma^0\gamma^0 = \gamma^0\gamma^\nu \\ &\text{case } \nu \neq 0: \quad \gamma^{\nu\dagger}\gamma^0 \overset{(8.12)}{=} -\gamma^\nu\gamma^0 \overset{(8.9)}{=} \gamma^0\gamma^\nu\,. \end{aligned}$$

With $\psi^\dagger\gamma^0 \overset{(8.22)}{=} \overline{\psi}$ (8.28) thus becomes

$$-i\hbar c(\mathrm{d}_\nu - \frac{i}{\hbar}qA_\nu)\,\overline{\psi}\,\gamma^\nu - mc^2\,\overline{\psi} = 0\,, \tag{8.29}$$

which is identical to (8.26b).

8.3 Lorentz Invariance

In section 6.3 we indicated, how spinor fields are transformed under a rotation of the space-time coordinates:

$$\psi'(x') \overset{(6.92c)}{=} D\psi(\Lambda^{-1}x) = \exp\Big\{\frac{i}{2\hbar}\Omega_{\sigma\tau}S^{\sigma\tau}\Big\}\psi(\Lambda^{-1}x) \tag{8.30a}$$

In section 8.4 we will prove that the field $\overline{\psi}$ is transformed by D^{-1}:

$$\overline{\psi}'(x') = \overline{\psi}(\Lambda^{-1}x)D^{-1} = \overline{\psi}(\Lambda^{-1}x)\exp\Big\{-\frac{i}{2\hbar}\Omega_{\sigma\tau}S^{\sigma\tau}\Big\} \tag{8.30b}$$

The group $\{D\}$ of spinor transformations $D = (8.30)$ is a representation of the group $\{\mathcal{B}\}$, which is the covering group of the Lorentz group $\{\ell\}$. Thus the Lie algebra of $\{D\}$ and $\{\mathcal{B}\}$ resp. $\{\ell\}$ must be identical. This condition does not uniquely fix the transformations D respectively their generators S. Some margin is left, which we will use to define appropriate generators. For that purpose we will apply the spinor transformation (8.30) with indefinite generator S, when we check the Lorentz invariance of the Lagrangian (8.21). This evaluation will result into an exact condition, which the generators S of the spinor transformations D must meet. Subsequently we will check whether the generators, which were defined by that method, also have the correct Lie algebra.

The Dirac equation is invariant under Lorentz transformations, if the Lagrangian (8.21), from which it was derived in (8.26a), is invariant under Lorentz transformations. The term $-\frac{1}{4\mu_0}F_{\sigma\tau}F^{\sigma\tau}$ obviously is Lorentz-invariant, because all it's space-time indices are contracted (i. e. it is a Lorentz-scalar). Thus we only need to consider the other terms. When the Lagrangian is transformed into a primed coordinate system′, it reads

$$\mathcal{L}'(x') \overset{(8.21)}{=} \overline{\psi}'_a(x')\Big(i\hbar c\gamma^{\nu}_{ab}(\mathrm{d}_{\nu}' + \frac{i}{\hbar}qA'_{\nu}(x')) - mc^2\delta_{ab}\Big)\psi'_b(x') \ .$$

To avoid an outgrowing number of indices, in the following we will only write those spinor indices explicitly, which really are mandatory.

$$\begin{aligned}
\mathcal{L}'(x') &\overset{(8.21)}{=} \overline{\psi}'(x')\Big(i\hbar c\gamma^{\nu}(\mathrm{d}_{\nu}' + \frac{i}{\hbar}qA'_{\nu}(x')) - mc^2\Big)\psi'(x') \\
\mathcal{L}'(\Lambda^{-1}x) &= \overline{\psi}(\Lambda^{-1}x)D^{-1}\Big(i\hbar c\gamma^{\nu}\,\Lambda^{-1\,\mu}{}_{\nu}(\mathrm{d}_{\mu} + \frac{i}{\hbar}qA_{\mu}(\Lambda^{-1}x)) - mc^2\Big)D\psi(\Lambda^{-1}x) \\
\mathcal{L}'(x) &= \overline{\psi}(x)\Big(i\hbar cD^{-1}\gamma^{\nu}D\,\Lambda^{-1\,\mu}{}_{\nu}(\mathrm{d}_{\mu} + \frac{i}{\hbar}qA_{\mu}(x)) - mc^2\Big)\psi(x)
\end{aligned} \tag{8.31}$$

In the last step, the space-time variables $\Lambda^{-1}x$ have been renamed to x. Note firstly, that the matrices γ^{ν} didn't get a prime′ under the transformation, because they are not a Lorentz vector's components, but constants, i. e.

Lorentz scalars. Note secondly, that the matrix operators in time-position space (namely the Lorentz transformations Λ^{-1}) commute with the matrix operators in spinor space (namely the spinor transformations D and D^{-1}).

The Lagrangian is invariant under a rotation of space-time coordinates, if

$$D^{-1}\gamma^\nu D\,\Lambda^{-1}{}^\mu{}_\nu = \gamma^\mu$$
$$\boxed{D^{-1}\gamma^\nu D = \Lambda^\nu{}_\mu\gamma^\mu}\ . \tag{8.32}$$

We once again should clearly point out, what we really are doing here: We don't aim to prove the Lorentz-invariance of a completely specified theory. Instead at this stage we are *constructing* a Lorentz-invariant theory. The spinor transformations D are not yet fixed, but we are free to define them. We are trying to define them in such a way, that firstly the group $\{D\}$ becomes a representation of the Lorentz group's covering group $\{\mathcal{B}\}$, and that secondly the elements D of this group are solutions of equation (8.32). This equation is a central point in Dirac's spinor theory, we will often turn back to it. It is describing, how the space-time transformations $\Lambda^\nu{}_\mu$ and the spinor transformations D must match, so that a Lorentz-invariant theory will emerge from their cooperation. The γ-matrices are the interface, where transformations in spinor space and transformations in time-position space interfere.

From (8.32) it becomes clear, why the γ-matrices, though they are constants, are indexed like the components of a Lorentz vector: In section 5.6 we found out, that a vector field $A(x)$ under a rotation of space-time coordinates is transformed by

$$A^\mu(x) \xrightarrow{\ell(\Omega)} A'^\mu(x') \overset{(5.90c)}{=} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)\ . \tag{8.33}$$

The field

$$K^\mu(x) \equiv \overline{\psi}(x)\gamma^\mu\psi(x) \tag{8.34}$$

under a coordinate rotation is transformed as

$$K'^\mu(x') \overset{(8.30)}{=} \overline{\psi}(\Lambda^{-1}x)D^{-1}\gamma^\mu D\psi(\Lambda^{-1}x)\ . \tag{8.35}$$

If (8.32) holds, then

$$K'^{\mu}(x') = \overline{\psi}(\Lambda^{-1}x)\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\psi(\Lambda^{-1}x) = \Lambda^{\mu}{}_{\nu}K^{\nu}(\Lambda^{-1}x) \tag{8.36}$$

follows. Not γ^{μ}, but $\overline{\psi}\gamma^{\mu}\psi$ therefore is the μ-component of a contravariant Lorentz four-vector.

Theorem: If under rotations of the space-time coordinates the spinor field $\psi(x)$ incorrectly is transformed like a scalar field, and if at the same time the constant spinor-matrices γ^{μ} incorrectly are transformed like the components of a Lorentz vector, then both errors compensate exactly in products of the form $\overline{\psi}(x)\gamma^{\mu}\psi(x)$ or $\overline{\psi}(x)\gamma^{\mu}A_{\mu}\psi(x)$ with arbitrary Lorentz vectors A_{μ}, and in total a correct result is achieved. (8.37)

Therefore it's also reasonable to apply to the γ-matrices the method of pulling indices up and down by means of the metric tensor:

$$\overline{\psi}\gamma_{\mu}\psi = g_{\mu\nu}\overline{\psi}\gamma^{\nu}\psi \quad \Longrightarrow \quad \gamma_{\mu} = g_{\mu\nu}\gamma^{\nu} \tag{8.38}$$

For the construction of appropriate generators $S^{\sigma\tau}$, we write equation (8.32) in infinitesimal form:

$$(\mathbb{1} - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\,S^{\sigma\tau})\gamma^{\nu}(\mathbb{1} + \frac{i}{2\hbar}\,\omega_{\sigma\tau}\,S^{\sigma\tau}) = (g^{\nu}{}_{\mu} + \frac{i}{2\hbar}\,\omega_{\sigma\tau}\,B^{\sigma\tau\nu}{}_{\mu})\gamma^{\mu} \tag{8.39}$$

Neglecting terms $\mathcal{O}(\omega^2)$, the following equation must hold:

$$\begin{aligned}\gamma^{\nu} + \frac{i}{2\hbar}\,\omega_{\sigma\tau}\Big(\gamma^{\nu}S^{\sigma\tau} - S^{\sigma\tau}\gamma^{\nu}\Big) &= \gamma^{\nu} + \frac{i}{2\hbar}\,\omega_{\sigma\tau}B^{\sigma\tau\nu}{}_{\mu}\gamma^{\mu} \\ [\gamma^{\nu}, S^{\sigma\tau}] &= B^{\sigma\tau\nu}{}_{\mu}\gamma^{\mu}\end{aligned} \tag{8.40}$$

The computation, which is documented in appendix A.6, results in

$$[\gamma^{\nu}, S^{\sigma\tau}] \overset{(A.43)}{=} \Big[\gamma^{\nu}, \frac{i\hbar}{4}[\gamma^{\sigma}, \gamma^{\tau}]\Big]\,. \tag{8.41}$$

Thus the Lagrangian (8.21) – and consequently the Dirac equation (8.26a) – is Lorentz-invariant, if firstly the spinor transformation's generators have

the form

$$\boxed{S^{\sigma\tau} = \frac{i\hbar}{4}\,[\gamma^\sigma, \gamma^\tau]}\,, \tag{8.42}$$

and if secondly the spinor transformations, which were defined by this method, have the Lie algebra of the Lorentz group. The

Theorem: A group of transformations with the six generators

$$S^{\sigma\tau} \equiv \frac{i\hbar}{4}\,[\gamma^\sigma, \gamma^\tau]$$

has the Lie algebra of the Lorentz group, if the four n-dimensional matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ comply with the relations

$$\{\gamma^\sigma, \gamma^\tau\} \equiv \gamma^\sigma\gamma^\tau + \gamma^\tau\gamma^\sigma \overset{(8.9)}{=} 2g^{\sigma\tau}\mathbb{1}\,. \tag{8.43}$$

says, that this indeed is the case. (By the way, note that the theorem is not limited to $n = 4$ dimensional spinor spaces.) To prove the theorem, we directly compute the Lie algebra (see appendix A.7 for details of the computation):

$$\begin{aligned}[S^{\alpha\beta}, S^{\eta\delta}] = -\frac{\hbar^2}{16}\Big((\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)(\gamma^\eta\gamma^\delta - \gamma^\delta\gamma^\eta) \\ - (\gamma^\eta\gamma^\delta - \gamma^\delta\gamma^\eta)(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\Big)\end{aligned}$$

$$[S^{\alpha\beta}, S^{\eta\delta}] \overset{(A.44)}{=} i\hbar(g^{\beta\eta}S^{\alpha\delta} - g^{\beta\delta}S^{\alpha\eta} - g^{\alpha\eta}S^{\beta\delta} + g^{\alpha\delta}S^{\beta\eta}) \tag{8.44}$$

This Lie algebra indeed is identical to the Lorentz group's Lie algebra (5.24b). The group $\{D\}$ of transformations

$$D = \exp\Big\{\frac{i}{2\hbar}\,\Omega_{\sigma\tau}\,S^{\sigma\tau}\Big\} \tag{8.45}$$

with generators according to (8.43) is a true representation of the Lorentz group, if it's parameter manifold is identical to the Lorentz group's parameter manifold. If the parameter manifold of the group $\{D\}$ is chosen identical to the parameter manifold of the group $\{\mathcal{B}\}$ — the Lorentz group's universal

covering group — , then $\{D\}$ is a true representation of $\{\mathcal{B}\}$, because the Lorentz group $\{\ell\}$ and it's covering group $\{\mathcal{B}\}$ have the same Lie algebra, and differ only by their parameter manifolds.

8.4 Probability Density

By analogy to non-relativistic quantum theory, we would guess the product $\psi^\dagger\psi$ to be the probability density. But actually it is the product $\overline{\psi}\psi$, which is defined in Dirac theory as probability density. This is the reason for that definition:

It is absolutely mandatory, that the state functions of quantum theory can be normalized, i.e. that there exists a real number N (possibly multiplied by some physical units) with the property

$$\frac{1}{N}\int_\Omega \mathrm{d}^3x\ \overline{\psi}(x)\psi(x) = 1 \quad \text{with } N \in \mathbb{R}\ ,\ -\infty < N < +\infty\ . \tag{8.46}$$

In case of the Dirac field, N is dimensionless, because we defined $[\psi] = \text{volume}^{-1/2}$. We now will prove, that this normalization condition can always be met with the probability density $\overline{\psi}\psi$, but that it could not be met in certain cases, if we would define $\psi^\dagger\psi$ instead of $\overline{\psi}\psi$ as probability density.

In section 8.3 we constructed a representation of the group $\{\mathcal{B}\}$ on the basis of Dirac-spinors. The elements of that representation are

$$D(\Omega) = \exp\left\{\frac{i}{2\hbar}\Omega_{\sigma\tau}S^{\sigma\tau}\right\} \tag{8.47}$$

$$\text{with} \quad S^{\sigma\tau} \overset{(8.43)}{=} \frac{i\hbar}{4}[\gamma^\sigma, \gamma^\tau]\ .$$

The generators, which are adjoint (i.e. transposed complex-conjugate) to $S^{\sigma\tau}$, are

$$S^{\sigma\tau\dagger} = -\frac{i\hbar}{4}(\gamma^{\tau\dagger}\gamma^{\sigma\dagger} - \gamma^{\sigma\dagger}\gamma^{\tau\dagger}) =$$

$$\overset{(8.12)}{=} -\frac{i\hbar}{4}(g^{\beta\tau}\gamma^\beta g^{\alpha\sigma}\gamma^\alpha - g^{\alpha\sigma}\gamma^\alpha g^{\beta\tau}\gamma^\beta) =$$

$$= -g^{\alpha\sigma} g^{\beta\tau} S^{\beta\alpha} = +g^{\alpha\sigma} g^{\beta\tau} S^{\alpha\beta} =$$
$$= \begin{cases} +S^{\sigma\tau} \text{ if } (\sigma = 0\,,\ \tau = 0) \text{ or } (\sigma \neq 0\,,\ \tau \neq 0) \\ -S^{\sigma\tau} \text{ if } (\sigma = 0\,,\ \tau \neq 0) \text{ or } (\sigma \neq 0\,,\ \tau = 0)\ . \end{cases} \tag{8.48}$$

The generators $S^{\sigma\tau}$ of spinor rotations are not self-adjoint in case of the index combinations of Lorentz boosts. Consequently the representation D of the boosts is not unitary. This is no surprise. The Lorentz group and it's covering group $\{\mathcal{B}\}$ according to theorem (5.56) have no finite-dimensional unitary representations, because their parameter manifolds are not compact. To clarify the impact of this fact onto the probability density in Dirac theory, we first investigate an infinitesimal transformation of the product $\psi^\dagger\psi$. Terms $\mathcal{O}(\omega^2)$ may be neglected:

$$\psi'^\dagger\,\psi' = (D_{\text{INF}}\psi)^\dagger\,(D_{\text{INF}}\psi) = \psi^\dagger\, D_{\text{INF}}^\dagger\, D_{\text{INF}}\,\psi =$$
$$\overset{(8.48)}{=} \psi^\dagger\Big(1 - \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, g^{\alpha\sigma} g^{\beta\tau} S^{\alpha\beta}\Big)\Big(1 + \frac{i}{2\hbar}\,\omega_{\sigma\tau}\, S^{\sigma\tau}\Big)\psi\ =$$
$$= \psi^\dagger\Big(1 - \frac{i}{2\hbar}\,\omega_{\sigma\tau}(\, g^{\alpha\sigma} g^{\beta\tau} S^{\alpha\beta} - \, S^{\sigma\tau})\Big)\psi \tag{8.49}$$
$$\begin{cases} = \psi^\dagger\psi \text{ if } (\sigma = 0\,,\ \tau = 0) \text{ or } (\sigma \neq 0\,,\ \tau \neq 0) \\ \neq \psi^\dagger\psi \text{ if } (\sigma = 0\,,\ \tau \neq 0) \text{ or } (\sigma \neq 0\,,\ \tau = 0) \end{cases} \tag{8.50}$$

Now the matrix γ^0 is added due to the definition $\overline{\psi} \overset{(8.22)}{=} \psi^\dagger\gamma^0$. The commutation relation of γ^0 and $S^{\sigma\tau}$ is

$$S^{\alpha\beta}\gamma^0 \overset{(8.43)}{=} \frac{i\hbar}{4}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\gamma^0$$
$$\overset{(8.9)}{=} \begin{cases} +\gamma^0 S^{\alpha\beta} \text{ if } (\alpha = 0\,,\ \beta = 0) \text{ or } (\alpha \neq 0\,,\ \beta \neq 0) \\ -\gamma^0 S^{\alpha\beta} \text{ if } (\alpha = 0\,,\ \beta \neq 0) \text{ or } (\alpha \neq 0\,,\ \beta = 0) \end{cases}$$
$$= \gamma^0 g^{\mu\alpha} g^{\nu\beta} S^{\mu\nu}\ . \tag{8.51}$$

These are exactly the same cases as in (8.48). Thus the sign changes compensate in the transformation of $\bar{\psi}\psi$, which is done analogous to (8.50):

$$\begin{aligned}
\overline{\psi}\,'\psi' &= \psi'^{\dagger}\gamma^0\psi' = \psi^\dagger\, D_{\text{INF}}^\dagger\,\gamma^0 D_{\text{INF}}\,\psi \;= \\
&\overset{(8.48)}{=} \psi^\dagger\Big(1-\frac{i}{2\hbar}\,\omega_{\sigma\tau}\,g^{\alpha\sigma}g^{\beta\tau}S^{\alpha\beta}\Big)\gamma^0\Big(1+\frac{i}{2\hbar}\,\omega_{\sigma\tau}\,S^{\sigma\tau}\Big)\psi \;= \\
&= \psi^\dagger\gamma^0\Big(1-\frac{i}{2\hbar}\,\omega_{\sigma\tau}\,(\underbrace{g^{\alpha\sigma}g^{\mu\alpha}g^{\beta\tau}g^{\nu\beta}S^{\mu\nu}}_{S^{\sigma\tau}}-S^{\sigma\tau})\Big)\psi \\
&= \overline{\psi}\,\psi
\end{aligned} \tag{8.52}$$

We see that $\overline{\psi}$, but not $\psi^\dagger$, is transformed by D^{-1}:

$$\overline{\psi}\,' = \overline{\psi}\,D^{-1} \qquad\qquad \psi'^{\dagger} = \psi^\dagger D^\dagger \neq \psi^\dagger D^{-1} \tag{8.53}$$

As $\overline{\psi}\,\psi$ is Lorentz-invariant, the condition (8.46) can be fulfilled in any coordinate system. In contrast there is no number $\widetilde{N}\in\mathbb{R}$, which solves the equation

$$\frac{1}{N}\int\limits_\Omega \mathrm{d}^3x\;\overline{\psi}\,\psi = \int\limits_\Omega \mathrm{d}^3x\,\psi^\dagger\frac{\gamma^0}{N}\psi = \int\limits_\Omega \mathrm{d}^3x\,\psi^\dagger\frac{1}{\widetilde{N}}\psi\;. \tag{8.54}$$

Therefore the Dirac field is normalizable only if $\overline{\psi}\,\psi$, but not $\psi^\dagger\psi$, is defined as probability density.

8.5 The free Field

A Dirac field is "free", if it is not interacting with any other field. Any interaction of a Dirac field is conveyed by gauge fields. With the gauge fields switched off, the equation of the free Dirac field gets the simple form

$$(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\psi(x) \overset{(8.5)}{=} 0\;. \tag{8.55}$$

Based on (7.19), we try the general solution

$$\psi(x) = \sum_{\boldsymbol{k}}\frac{1}{\sqrt{N\Omega}}\Big(a_{\boldsymbol{k}}u^{\boldsymbol{k}}\exp\{-ikx\} + b^*_{\boldsymbol{k}}v^{\boldsymbol{k}}\exp\{+ikx\}\Big)\;. \tag{8.56}$$

We factored the four-component spinors

$$u^{\boldsymbol{k}} \equiv \begin{pmatrix} u_1^{\boldsymbol{k}} \\ u_2^{\boldsymbol{k}} \\ u_3^{\boldsymbol{k}} \\ u_4^{\boldsymbol{k}} \end{pmatrix} \quad \text{and} \quad v^{\boldsymbol{k}} \equiv \begin{pmatrix} v_1^{\boldsymbol{k}} \\ v_2^{\boldsymbol{k}} \\ v_3^{\boldsymbol{k}} \\ v_4^{\boldsymbol{k}} \end{pmatrix} \tag{8.57}$$

out from the Fourier coefficients. While the Dirac equation's solution $\psi(x)$ is a four-component spinor, we want the Fourier coefficients to stay spinor-scalars. Note that the wave numbers $\boldsymbol{k}$ in the products $a_{\boldsymbol{k}} u^{\boldsymbol{k}}$ and $b^*_{\boldsymbol{k}} v^{\boldsymbol{k}}$ are *not* automatically summed-up. Only space-time indices and spinor indices are automatically summed-up. As the $\boldsymbol{k}$ are no space-time indices, their up- or down-position is not related to contra- or covariant transformations. We placed them just where some free space was available.

We now will analyze in more detail those solutions of the classical (not quantized) Dirac equation, in which only one Fourier coefficient is $a_{\boldsymbol{k}} = 1$ or $b^*_{\boldsymbol{k}} = 1$, while all other Fourier coefficients are zero. They have the form

$$\left.\begin{aligned} \psi(x) &= \frac{1}{\sqrt{N\Omega}}\, u^{\boldsymbol{k}} \exp\{-ikx\} \\ \psi(x) &= \frac{1}{\sqrt{N\Omega}}\, v^{\boldsymbol{k}} \exp\{+ikx\} \end{aligned}\right\} \text{ with } \omega_{\boldsymbol{k}} > 0\,. \tag{8.58a}$$

The adjoint solutions $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$ are

$$\begin{aligned} \overline{\psi}(x) &= \frac{1}{\sqrt{N\Omega}}\, \bar{u}^{\boldsymbol{k}} \exp\{+ikx\} \\ \overline{\psi}(x) &= \frac{1}{\sqrt{N\Omega}}\, \bar{v}^{\boldsymbol{k}} \exp\{-ikx\} \\ \text{with} \quad \bar{u}^{\boldsymbol{k}} &\equiv u^{\boldsymbol{k}\dagger}\gamma^0 \quad , \quad \bar{v}^{\boldsymbol{k}} \equiv v^{\boldsymbol{k}\dagger}\gamma^0\,. \end{aligned} \tag{8.58b}$$

Interpreted as state functions of point-particle quantum mechanics, these solutions would have the energy

$$i\hbar\,\frac{\mathrm{d}\psi}{\mathrm{d}t} = E\psi = i\hbar(\mp i\omega_{\boldsymbol{k}})\psi = \pm\hbar\omega_{\boldsymbol{k}}\psi\,. \tag{8.59}$$

The solutions $\exp\{+ikx\}$ with negative frequency would have negative energy. Dirac invented his hole theory (see e. g. [33, chap. 5]) to prevent that the system would run by emission of energy into a state of infinitely negative energy. We don't need to worry about that issue. It will turn out that the solutions with negative energy simply disappear upon appropriate quantization of the Dirac field. In all states with no exception the quantized Dirac field has positive (or zero) energy.

By insertion of (8.58a) into (8.55) we get with the momentum four-vector $p \equiv \hbar k$ the two equations

$$\begin{aligned}(+c\gamma^\mu p_\mu - mc^2)u^{\boldsymbol{k}} &= 0\\ (-c\gamma^\mu p_\mu - mc^2)v^{\boldsymbol{k}} &= 0\,.\end{aligned} \tag{8.60}$$

Provided $m \neq 0$, there is a coordinate system in which the field is at rest ($\boldsymbol{p} = \hbar\boldsymbol{k} = 0$). Using the invariant wave-number square and the invariant momentum square

$$k_\nu k^\nu = \left(\frac{\omega_{\boldsymbol{k}}}{c}\right)^2 - \boldsymbol{k}^2 = \frac{m^2c^2}{\hbar^2} \tag{8.61}$$

$$\hbar k_\nu \hbar k^\nu = p_\nu p^\nu = \left(\frac{\hbar\omega_{\boldsymbol{k}}}{c}\right)^2 - \boldsymbol{p}^2 = m^2c^2\,, \tag{8.62}$$

(8.60) can be written in the field's rest system as

$$(c\gamma^0 mc - mc^2)u^{\mathbf{0}} \overset{(8.15a)}{=} \begin{pmatrix}-\mathbb{1} & \mathbb{1}\\ \mathbb{1} & -\mathbb{1}\end{pmatrix} mc^2 u^{\mathbf{0}} =$$

$$= mc^2 \begin{pmatrix}-u_1^{\mathbf{0}} + u_3^{\mathbf{0}}\\ -u_2^{\mathbf{0}} + u_4^{\mathbf{0}}\\ +u_1^{\mathbf{0}} - u_3^{\mathbf{0}}\\ +u_2^{\mathbf{0}} - u_4^{\mathbf{0}}\end{pmatrix} = 0 \quad\Longrightarrow\quad \begin{cases} u_3^{\mathbf{0}} = u_1^{\mathbf{0}}\\ u_4^{\mathbf{0}} = u_2^{\mathbf{0}}\end{cases} \tag{8.63a}$$

$$\text{resp.} \quad \begin{pmatrix}-\mathbb{1} & -\mathbb{1}\\ -\mathbb{1} & -\mathbb{1}\end{pmatrix} mc^2 v^{\mathbf{0}} = 0 \quad\Longrightarrow\quad \begin{cases} v_3^{\mathbf{0}} = -v_1^{\mathbf{0}}\\ v_4^{\mathbf{0}} = -v_2^{\mathbf{0}}\,.\end{cases} \tag{8.63b}$$

Note that $p^0 = mc^2 > 0$ was inserted into both equations, according to (7.18). Because of (8.63), a Dirac spinor in it's rest system has only two

independent components. There are two linearly independent spinors $u^{\mathbf{0}}$, and two linearly independent spinors $v^{\mathbf{0}}$, which we will call ${}^1u^{\mathbf{0}}$, ${}^2u^{\mathbf{0}}$, ${}^1v^{\mathbf{0}}$, ${}^2v^{\mathbf{0}}$:

$$ {}^r u^{\mathbf{0}} = \begin{pmatrix} {}^r u_1^{\mathbf{0}} \\ {}^r u_2^{\mathbf{0}} \\ {}^r u_1^{\mathbf{0}} \\ {}^r u_2^{\mathbf{0}} \end{pmatrix} \qquad {}^r v^{\mathbf{0}} = \begin{pmatrix} {}^r v_1^{\mathbf{0}} \\ {}^r v_2^{\mathbf{0}} \\ -{}^r v_1^{\mathbf{0}} \\ -{}^r v_2^{\mathbf{0}} \end{pmatrix} \qquad r = 1, 2 \tag{8.64} $$

In the rest system, the Dirac equation has the four solutions

$$ \left.\begin{aligned} \psi(x) &= \frac{1}{\sqrt{N\Omega}}\, {}^r u^{\mathbf{0}} \exp\Big\{ - i\, \frac{mc^2}{\hbar}\, t \Big\} \\ \psi(x) &= \frac{1}{\sqrt{N\Omega}}\, {}^r v^{\mathbf{0}} \exp\Big\{ + i\, \frac{mc^2}{\hbar}\, t \Big\} \end{aligned} \right\} \text{ with } r = 1, 2\,. \tag{8.65} $$

The field's wave number is $\boldsymbol{k} = 0$. Thus it's wave length is infinite, and it oscillates with the frequency $mc^2/\hbar > 0$, which is determined by it's mass.

The solutions with momentum $\boldsymbol{p} = \hbar\boldsymbol{k} \neq 0$ can be found by transformation into a coordinate system, which is moving relatively to the field (8.65). In the moving system, the four solutions of the free Dirac equation are

$$ \psi(x) = \frac{1}{\sqrt{N\Omega}}\, {}^r u^{\boldsymbol{k}} e^{-ikx} = \frac{1}{\sqrt{N\Omega}}\, D\, {}^r u^{\mathbf{0}} e^{-ikx} \tag{8.66a} $$

$$ \psi(x) = \frac{1}{\sqrt{N\Omega}}\, {}^r v^{\boldsymbol{k}} e^{+ikx} = \frac{1}{\sqrt{N\Omega}}\, D\, {}^r v^{\mathbf{0}} e^{+ikx} \tag{8.66b} $$

$$ \text{with } r = 1, 2\,. $$

For a pure boost (rapidity $\boldsymbol{\eta} \neq 0$, space-like rotation $\boldsymbol{\theta} = 0$), the transformation

$$ D \overset{(11.5)}{=} \exp\Big\{ \frac{i}{\hbar} \Big(\Theta_j \frac{\hbar}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} + \eta_j \frac{i\hbar}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \Big) \Big\} $$

can be written in the following form (see appendix A.9 for details of the computation):

$$D \overset{(A.50)}{=} \begin{pmatrix} \mathbb{1}\cosh\left(\frac{\eta}{2}\right) - \frac{\eta_j}{\eta}\sigma^j \sinh\left(\frac{\eta}{2}\right) & 0 \\ 0 & \mathbb{1}\cosh\left(\frac{\eta}{2}\right) + \frac{\eta_j}{\eta}\sigma^j \sinh\left(\frac{\eta}{2}\right) \end{pmatrix} \tag{8.67}$$

Alternatively, this transformation can be described as a function of the field's three momentum components p_j and it's energy E in the moving coordinate system (see appendix A.9 for details of the computation):

$$D \overset{(A.57)}{=} \sqrt{\frac{1}{2mc^2(E+mc^2)}} \cdot \cdot \begin{pmatrix} (E+mc^2)\mathbb{1} + cp_j\sigma^j & 0 \\ 0 & (E+mc^2)\mathbb{1} - cp_j\sigma^j \end{pmatrix} \tag{8.68}$$

In the limit of negligible rest mass m, the four-component Dirac equation decomposes into the two two-component Weyl equations, see section 8.9. As one normally would like to avoid the need for a new definition of the normalizing factor in that case, one chooses already for the Dirac equation the *two-component* spinors

$$\begin{pmatrix} {}^{r}u_1^{\mathbf{0}} \\ {}^{r}u_2^{\mathbf{0}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} {}^{r}v_1^{\mathbf{0}} \\ {}^{r}v_2^{\mathbf{0}} \end{pmatrix} \quad \text{with } r = 1, 2 \tag{8.69}$$

as orthogonal vectors, and assigns to them by definition the dimension $\sqrt{\text{energy}}$:

$$\begin{pmatrix} {}^{r}u_1^{\mathbf{0}*} & {}^{r}u_2^{\mathbf{0}*} \end{pmatrix} \begin{pmatrix} {}^{s}u_1^{\mathbf{0}} \\ {}^{s}u_2^{\mathbf{0}} \end{pmatrix} = \begin{pmatrix} {}^{r}v_1^{\mathbf{0}*} & {}^{r}v_2^{\mathbf{0}*} \end{pmatrix} \begin{pmatrix} {}^{s}v_1^{\mathbf{0}} \\ {}^{s}v_2^{\mathbf{0}} \end{pmatrix} \equiv mc^2\,\delta_{rs} \tag{8.70}$$
$$\text{with } r = 1, 2 \quad s = 1, 2$$

Using (8.64), we get for the adjoint *four-component* spinors

$$\left.\begin{aligned} {}^{r}\bar{u}^{\mathbf{0}} &= {}^{r}u^{\mathbf{0}+}\left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right) = {}^{r}u^{\mathbf{0}+} \\ {}^{r}\bar{v}^{\mathbf{0}} &= {}^{r}v^{\mathbf{0}+}\left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right) = -\,{}^{r}v^{\mathbf{0}+} \end{aligned}\right\} \quad \text{with } r = 1, 2\ . \tag{8.71}$$

If the coordinate system is rotated, ${}^{r}\bar{u}^{\boldsymbol{k}}$ and ${}^{r}\bar{v}^{\boldsymbol{k}}$ are transformed by D^{-1},

same as $\overline{\psi}$. Therefore the products

$$
\begin{aligned}
{}^{r}\bar{u}^{\boldsymbol{k}}\,{}^{s}u^{\boldsymbol{k}} &= {}^{r}\bar{u}^{\boldsymbol{0}} D^{-1} D\,{}^{s}u^{\boldsymbol{0}} = 2mc^2\,\delta_{rs} && \text{(8.72a)}\\
{}^{r}\bar{v}^{\boldsymbol{k}}\,{}^{s}v^{\boldsymbol{k}} &= {}^{r}\bar{v}^{\boldsymbol{0}} D^{-1} D\,{}^{s}v^{\boldsymbol{0}} = -2mc^2\,\delta_{rs} && \text{(8.72b)}\\
{}^{r}\bar{u}^{\boldsymbol{k}}\,{}^{s}v^{\boldsymbol{k}} &= {}^{r}u_1^{\boldsymbol{0}*}\,{}^{s}v_1^{\boldsymbol{0}} + {}^{r}u_2^{\boldsymbol{0}*}\,{}^{s}v_2^{\boldsymbol{0}} - {}^{r}u_1^{\boldsymbol{0}*}\,{}^{s}v_1^{\boldsymbol{0}} - {}^{r}u_2^{\boldsymbol{0}*}\,{}^{s}v_2^{\boldsymbol{0}} = 0 && \text{(8.72c)}\\
{}^{r}\bar{v}^{\boldsymbol{k}}\,{}^{s}u^{\boldsymbol{k}} &= -{}^{r}v_1^{\boldsymbol{0}*}\,{}^{s}u_1^{\boldsymbol{0}} - {}^{r}v_2^{\boldsymbol{0}*}\,{}^{s}u_2^{\boldsymbol{0}} + {}^{r}v_1^{\boldsymbol{0}*}\,{}^{s}u_1^{\boldsymbol{0}} + {}^{r}v_2^{\boldsymbol{0}*}\,{}^{s}u_2^{\boldsymbol{0}} = 0 && \text{(8.72d)}
\end{aligned}
$$

are lorentz-invariant.

With the transformation (8.68), and with the Pauli matrices (8.14), we can immediately write down the spinors $u^{\boldsymbol{k}}, v^{\boldsymbol{k}}$ with wave number $\boldsymbol{k}$:

$$
\begin{aligned}
{}^{r}u^{\boldsymbol{k}} = D\,{}^{r}u^{\boldsymbol{0}} = \sqrt{\frac{1}{2mc^2(E+mc^2)}}\cdot\\
\cdot\begin{pmatrix}
(E+mc^2+cp_3)\,{}^{r}u_1^{\boldsymbol{0}} + (cp_1 - icp_2)\,{}^{r}u_2^{\boldsymbol{0}}\\
(cp_1+icp_2)\,{}^{r}u_1^{\boldsymbol{0}} + (E+mc^2-cp_3)\,{}^{r}u_2^{\boldsymbol{0}}\\
(E+mc^2-cp_3)\,{}^{r}u_1^{\boldsymbol{0}} + (-cp_1+icp_2)\,{}^{r}u_2^{\boldsymbol{0}}\\
(-cp_1-icp_2)\,{}^{r}u_1^{\boldsymbol{0}} + (E+mc^2+cp_3)\,{}^{r}u_2^{\boldsymbol{0}}
\end{pmatrix}
\end{aligned}
\qquad \text{(8.73a)}
$$

$$
\begin{aligned}
{}^{r}v^{\boldsymbol{k}} = D\,{}^{r}v^{\boldsymbol{0}} = \sqrt{\frac{1}{2mc^2(E+mc^2)}}\cdot\\
\cdot\begin{pmatrix}
(E+mc^2+cp_3)\,{}^{r}v_1^{\boldsymbol{0}} + (cp_1 - icp_2)\,{}^{r}v_2^{\boldsymbol{0}}\\
(cp_1+icp_2)\,{}^{r}v_1^{\boldsymbol{0}} + (E+mc^2-cp_3)\,{}^{r}v_2^{\boldsymbol{0}}\\
(-E-mc^2+cp_3)\,{}^{r}v_1^{\boldsymbol{0}} + (cp_1-icp_2)\,{}^{r}v_2^{\boldsymbol{0}}\\
(cp_1+icp_2)\,{}^{r}v_1^{\boldsymbol{0}} + (-E-mc^2-cp_3)\,{}^{r}v_2^{\boldsymbol{0}}
\end{pmatrix}
\end{aligned}
\qquad \text{(8.73b)}
$$

$$
r = 1,2 \quad , \quad D = (8.68) \text{ with } \begin{cases} E = \hbar\omega_{\boldsymbol{k}} > 0 \\ p_j = \hbar k_j\ . \end{cases}
$$

In the following paragraphs, we can proceed with less efforts if we fix for the spinors with $\boldsymbol{k} = 0$ the following explicit form, which complies with the conditions (8.64) and (8.70):

$$
{}^1u^0 \equiv \sqrt{mc^2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad {}^2u^0 \equiv \sqrt{mc^2}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}
$$

$$
{}^1v^0 \equiv \sqrt{mc^2}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \qquad {}^2v^0 \equiv \sqrt{mc^2}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \tag{8.74}
$$

Thus the spinors become

$$
{}^1u^{\boldsymbol{k}} = \sqrt{\frac{1}{2(E+mc^2)}}\begin{pmatrix}E+mc^2+cp_3\\ cp_1+icp_2\\ E+mc^2-cp_3\\ -cp_1-icp_2\end{pmatrix} \tag{8.75a}
$$

$$
{}^2u^{\boldsymbol{k}} = \sqrt{\frac{1}{2(E+mc^2)}}\begin{pmatrix}cp_1-icp_2\\ E+mc^2-cp_3\\ -cp_1+icp_2\\ E+mc^2+cp_3\end{pmatrix} \tag{8.75b}
$$

$$
{}^1v^{\boldsymbol{k}} = \sqrt{\frac{1}{2(E+mc^2)}}\begin{pmatrix}E+mc^2+cp_3\\ cp_1+icp_2\\ -E-mc^2+cp_3\\ cp_1+icp_2\end{pmatrix} \tag{8.75c}
$$

$$
{}^2v^{\boldsymbol{k}} = \sqrt{\frac{1}{2(E+mc^2)}}\begin{pmatrix}cp_1-icp_2\\ E+mc^2-cp_3\\ cp_1-icp_2\\ -E-mc^2-cp_3\end{pmatrix} \tag{8.75d}
$$

$$
\text{with } E = \hbar\omega_{\boldsymbol{k}} > 0\ ,\ p_j = \hbar k_j = -p^j = -\hbar k^j\ .
$$

Using (8.75), in appendix A.10 the following spinor-relations are derived, which will be needed for the Dirac field's quantization. For arbitrary $\boldsymbol{k}$

$$
\sum_r \left({}^r u_a^{\boldsymbol{k}}\, {}^r u_b^{\boldsymbol{k}\dagger} + {}^r v_a^{-\boldsymbol{k}}\, {}^r v_b^{-\boldsymbol{k}\dagger} \right) \overset{(A.61)}{=} 2E\,\delta_{ab} = 2\hbar\omega_{\boldsymbol{k}}\,\delta_{ab} \tag{8.76a}
$$

$$ {}^{r}u^{\boldsymbol{k}\dagger}\,{}^{s}u^{\boldsymbol{k}} \overset{(A.62)}{=} {}^{r}v^{\boldsymbol{k}\dagger}\,{}^{s}v^{\boldsymbol{k}} \overset{(A.62)}{=} 2E\,\delta_{rs} = 2\hbar\omega_{\boldsymbol{k}}\,\delta_{rs} \tag{8.76b} $$

$$ {}^{r}u^{-\boldsymbol{k}\dagger}\,{}^{s}v^{\boldsymbol{k}} \overset{(A.63)}{=} {}^{r}v^{-\boldsymbol{k}\dagger}\,{}^{s}u^{\boldsymbol{k}} \overset{(A.63)}{=} 0\ . \tag{8.76c} $$

In appendix A.11 these relations are proved:

$$ \sum_{r=1}^{2} {}^{r}u^{\boldsymbol{k}}\,{}^{r}\bar{u}^{\boldsymbol{k}} \overset{(A.67a)}{=} c\gamma^{\mu}p_{\mu} + mc^{2} \tag{8.77a} $$

$$ \sum_{r=1}^{2} {}^{r}v^{\boldsymbol{k}}\,{}^{r}\bar{v}^{\boldsymbol{k}} \overset{(A.67b)}{=} c\gamma^{\mu}p_{\mu} - mc^{2} \tag{8.77b} $$

This is a 4×4 matrix in spinor space with $mc^2 \equiv \mathbb{1}mc^2$. Furthermore in appendix A.27 the relations

$$ {}^{r}\bar{u}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}u^{\boldsymbol{k}} \overset{(A.205)}{=} {}^{r}\bar{v}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}v^{\boldsymbol{k}} \overset{(A.205)}{=} 2c\hbar k^{\rho}\delta_{rs} \tag{8.78a} $$

$$ {}^{r}\bar{u}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}v^{\boldsymbol{k}} \overset{(A.205)}{\neq} 0 \quad , \quad {}^{r}\bar{v}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}u^{\boldsymbol{k}} \overset{(A.205)}{\neq} 0 \tag{8.78b} $$

are proved.

The probability density of the four solutions (8.66) is

$$ \bar{\psi}\psi = \frac{1}{N\Omega}\exp\{+ikx\}\,{}^{r}\bar{u}^{\boldsymbol{k}}\,{}^{r}u^{\boldsymbol{k}}\exp\{-ikx\} = \frac{2mc^{2}}{N\Omega} \tag{8.79a} $$

$$ \bar{\psi}\psi = \frac{1}{N\Omega}\exp\{-ikx\}\,{}^{r}\bar{v}^{\boldsymbol{k}}\,{}^{r}v^{\boldsymbol{k}}\exp\{+ikx\} = \frac{-2mc^{2}}{N\Omega}\ . \tag{8.79b} $$

With the definition

$$ N \equiv +2E = 2\hbar\omega_{\boldsymbol{k}}\ , \tag{8.80} $$

the probability density is lorentz-invariant, because the product $E\Omega$ is lorentz-invariant as was discussed at the end of section 7. The unusual negative probability density in (8.79b) is caused by the likewise unusual definition of $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ by means of the Dirac-matrix γ^{0}. Complying with the remarks we added to (7.1) and (7.2), the probability $W(V)$ to find

the field in volume V is – as necessary – always ≥ 0 and ≤ 1, even if the probability density is negative:

$$0 \leq W(V) = \frac{\int\limits_V \mathrm{d}^3x \;\; \overline{\psi}(t,\boldsymbol{x})\psi(t,\boldsymbol{x})}{\int\limits_\Omega \mathrm{d}^3x \;\; \overline{\psi}(t,\boldsymbol{x})\psi(t,\boldsymbol{x})} \leq +1 \tag{8.81}$$

The boundary conditions (8.63), which reduce the degrees of freedom of the Dirac spinors to 2, are valid as well for the boosted spinors. If a field's spin quantum number is s, then the number of it's spin degrees of freedom is $2s+1$. Therefore the Dirac spinors, even though they have 4 components, are describing fields with spin $s = 1/2$, but not $s = 3/2$.

$$\boxed{\text{The spin of Dirac-fields is } s = 1/2 \;\; .} \tag{8.82}$$

In this section we concentrated on those solutions of the Dirac equation, in which only one single Fourier-coefficient of the general solution (8.56) is 1, and all other Fourier-coefficients are 0. Concluding this section, we state the general solution. As there are two linearly independent variants $r = 1, 2$ of each spinor ${}^r u^{\boldsymbol{k}}$ and ${}^r v^{\boldsymbol{k}}$, there also are two ${}^r a_{\boldsymbol{k}}$ and two ${}^r b^*_{\boldsymbol{k}}$ Fourier-coefficients with $r = 1, 2$. Furthermore we insert the normalization factor (8.80):

$$\psi(x) \overset{(8.56)}{=} \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^r a_{\boldsymbol{k}} \, {}^r u^{\boldsymbol{k}} \exp\{-ikx\} + {}^r b^*_{\boldsymbol{k}} \, {}^r v^{\boldsymbol{k}} \exp\{+ikx\} \Big) \tag{8.83a}$$

$$\overline{\psi}(x) = \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^r a^*_{\boldsymbol{k}} \, {}^r \overline{u}^{\boldsymbol{k}} \exp\{+ikx\} + {}^r b_{\boldsymbol{k}} \, {}^r \overline{v}^{\boldsymbol{k}} \exp\{-ikx\} \Big) \tag{8.83b}$$

The momentum densities, which are canonical conjugate to $\psi(x)$ and $\overline{\psi}(x)$ respectively, are

$$\pi_\psi \overset{(4.40)}{\equiv} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}}{c\,\partial(\mathrm{d}_0\psi)} \overset{(8.24)}{=} i\hbar\,\overline{\psi}\gamma^0 \tag{8.84a}$$

$$\pi_{\overline{\psi}} \equiv \frac{\partial \mathcal{L}}{c\,\partial(\mathrm{d}_0\,\overline{\psi})} \overset{(8.24)}{=} 0 \;\; . \tag{8.84b}$$

We will comment in (8.99)ff on the peculiar asymmetry between (8.84a) and (8.84b). The dimension of the canonical conjugate momentum density of the Dirac field is action · volume$^{-1/2}$. Note that the momentum density, which is canonical conjugate to the column spinor ψ, is a row spinor. Inserting (8.83), we find the canonically conjugate momentum density

$$\pi(x) \overset{(8.84a)}{=} i\hbar\psi^\dagger = \sum_{\boldsymbol{k},r} \frac{i\hbar}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^r a^*_{\boldsymbol{k}}\, {}^r u^{\dagger \boldsymbol{k}} \exp\{+ikx\} + {}^r b_{\boldsymbol{k}}\, {}^r v^{\dagger \boldsymbol{k}} \exp\{-ikx\} \Big) \,. \tag{8.85}$$

(8.83) and (8.85) will be our starting point for the quantization of the Dirac field in chapter 16 .

8.6 Conserved Quantities

The components of the Dirac field's energydensity-stress tensor (ES-tensor) are

$$\begin{aligned} \mathcal{T}^{\rho\sigma} &\overset{(4.32)}{=} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho\psi)}\, \mathrm{d}^\sigma\psi + (\mathrm{d}^\sigma\overline{\psi})\, \underbrace{\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho\overline{\psi})}}_{0} - g^{\rho\sigma}\mathcal{L} \\ &\overset{(8.24)}{=} \overline{\psi}\, i\hbar c\gamma^\rho \mathrm{d}^\sigma\psi - g^{\rho\sigma}\, \overline{\psi} \left(i\hbar c\gamma^\nu \mathrm{d}_\nu - mc^2 \right) \psi \\ &\overset{(8.5)}{=} i\hbar c\, \overline{\psi}\gamma^\rho \mathrm{d}^\sigma\psi \,. \end{aligned} \tag{8.86}$$

It's dimension is energy/volume. In the first line, the factors have been arranged such that each term is a number, but not a spinor-matrix. In preparation of the evaluation of other fields, we make this a general rule:

> All products of fields and their derivatives in the Lagrangian $\mathcal{L}$ and in the ES-tensor's components $\mathcal{T}^{\rho\sigma}$ must be arranged such, that all terms are scalars in space-time, in spinor space, in weak iso-spin space, and in color space. (8.87)

(Weak iso-spin will be introduced in chapter 29, and color will be introduced in chapter 28 .)

In particular, the Dirac field's energy density (Hamiltonian) is

$$\mathcal{H} \overset{(4.34)}{=} \mathcal{T}^{00} = i\hbar c\, \overline{\psi}\gamma^0 \mathrm{d}_0 \psi \ , \tag{8.88}$$

and it's physical momentum density is

$$\mathcal{P}^j \overset{(4.35)}{=} \frac{1}{c}\mathcal{T}^{0j} = i\hbar\, \overline{\psi}\gamma^0 \mathrm{d}^j \psi \ . \tag{8.89}$$

Inserting the fields (8.83), we can work out the ES-tensor in more detail:

$$\begin{aligned}
\mathcal{T}^{\rho\sigma} = i\hbar c \sum_{\boldsymbol{f},\boldsymbol{k},s,r} \frac{ik^\sigma}{2\Omega\hbar\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}}\Big(\\
- {}^s a^*_{\boldsymbol{f}}\, {}^r a_{\boldsymbol{k}}\, {}^s\overline{u}^{\boldsymbol{f}} \gamma^\rho\, {}^r u^{\boldsymbol{k}} \exp\{+i(f-k)x\} \\
+ {}^s a^*_{\boldsymbol{f}}\, {}^r b^*_{\boldsymbol{k}}\, {}^s\overline{u}^{\boldsymbol{f}} \gamma^\rho\, {}^r v^{\boldsymbol{k}} \exp\{+i(f+k)x\} \\
- {}^s b_{\boldsymbol{f}}\, {}^r a_{\boldsymbol{k}}\, {}^s\overline{v}^{\boldsymbol{f}} \gamma^\rho\, {}^r u^{\boldsymbol{k}} \exp\{-i(f+k)x\} \\
+ {}^s b_{\boldsymbol{f}}\, {}^r b^*_{\boldsymbol{k}}\, {}^s\overline{v}^{\boldsymbol{f}} \gamma^\rho\, {}^r v^{\boldsymbol{k}} \exp\{-i(f-k)x\}\Big)
\end{aligned} \tag{8.90}$$

Now we will integrate this result in three-dimensional position space over the normalization volume Ω. We want to make use of the Kronecker symbol

$$\frac{1}{\Omega}\int_\Omega \mathrm{d}^3x\, \exp\{\pm i(\boldsymbol{k}-\boldsymbol{f})\boldsymbol{x}\} \overset{(7.12)}{=} \delta_{\boldsymbol{k}\boldsymbol{f}} \tag{8.91}$$

in all terms. As the sums are running symmetrically over all positive and negative wavenumbers $\boldsymbol{k}$ and $\boldsymbol{f}$, we can e. g. write:

$$\sum_{\boldsymbol{f},\boldsymbol{k},s,r} {}^{s}a^{*}_{\boldsymbol{f}}\, {}^{r}b^{*}_{\boldsymbol{k}}\, {}^{s}\bar{u}^{\boldsymbol{f}}\gamma^{\rho}\, {}^{r}v^{\boldsymbol{k}}\, \frac{1}{\Omega}\int_{\Omega} \mathrm{d}^3x\, \exp\{+i(f+k)x\} =$$
$$= \sum_{\boldsymbol{f},\boldsymbol{k},s,r} {}^{s}a^{*}_{\boldsymbol{f}}\, {}^{r}b^{*}_{-\boldsymbol{k}}\, {}^{s}\bar{u}^{\boldsymbol{f}}\gamma^{\rho}\, {}^{r}v^{-\boldsymbol{k}}\, \exp\{+i(f^0+k^0)x_0\}\cdot$$
$$\cdot \underbrace{\frac{1}{\Omega}\int_{\Omega} \mathrm{d}^3x\, \exp\{+i(f^j-k^j)x_j\}}_{\delta_{\boldsymbol{kf}}}$$

Thus we get

$$T^{\rho\sigma} \equiv \int_{\Omega} \mathrm{d}^3x\, \mathcal{T}^{\rho\sigma} = -\hbar c \sum_{\boldsymbol{k},s,r} \frac{k^{\sigma}}{2\hbar\omega_{\boldsymbol{k}}}\Big($$
$$- {}^{s}a^{*}_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}\bar{u}^{\boldsymbol{k}}\gamma^{\rho}\, {}^{r}u^{\boldsymbol{k}} + {}^{s}a^{*}_{-\boldsymbol{k}}\, {}^{r}b^{*}_{\boldsymbol{k}}\, {}^{s}\bar{u}^{-\boldsymbol{k}}\gamma^{\rho}\, {}^{r}v^{\boldsymbol{k}} \exp\{+i2k^0x_0\}$$
$$- {}^{s}b_{-\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}\bar{v}^{-\boldsymbol{k}}\gamma^{\rho}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i2k^0x_0\} + {}^{s}b_{\boldsymbol{k}}\, {}^{r}b^{*}_{\boldsymbol{k}}\, {}^{s}\bar{v}^{\boldsymbol{k}}\gamma^{\rho}\, {}^{r}v^{\boldsymbol{k}}\Big)\ . \qquad (8.92)$$

Here we applied our convention (7.18), that the null-component of the wavenumbers of free fields is chosen always ≥ 0. The terms with the exponential functions are describing rapid oscillations. Their frequency does depend on the energy of the field. Even for relatively light particles with rest energy 1 MeV, the frequency of these oscillations is minimum as high as

$$2k^0c > \frac{2\,\mathrm{MeV}}{\hbar} \approx 3\cdot 10^{21}\mathrm{Hz}\ . \qquad (8.93)$$

Schrödinger [34], who detected them (theoretically, these oscillations have never been observed experimentally), named them "Zitterbewegung" (trembling motion). The mean value of these terms is zero, and due to the high frequency, we certainly may neglect them, even though their coefficients according to (8.78b) are different from zero. Thereby the ES-tensor simplifies to

$$T^{\rho\sigma} \equiv \int_{\Omega} \mathrm{d}^3x\, \mathcal{T}^{\rho\sigma} = \sum_{\boldsymbol{k},s} \frac{c^2\hbar^2k^{\rho}k^{\sigma}}{\hbar\omega_{\boldsymbol{k}}}\Big({}^{s}a^{*}_{\boldsymbol{k}}\, {}^{s}a_{\boldsymbol{k}} - {}^{s}b_{\boldsymbol{k}}\, {}^{s}b^{*}_{\boldsymbol{k}}\Big)\ . \qquad (8.94)$$

In particular, the Hamilton function (energy) is

$$H \equiv T^{00} = \sum_{\boldsymbol{k},s} \hbar\omega_{\boldsymbol{k}} \left({}^{s}a^{*}_{\boldsymbol{k}}\, {}^{s}a_{\boldsymbol{k}} - {}^{s}b_{\boldsymbol{k}}\, {}^{s}b^{*}_{\boldsymbol{k}} \right) , \tag{8.95}$$

and the field's physical momentum is

$$P^{j} \equiv \frac{1}{c}\, T^{0j} = \sum_{\boldsymbol{k},s} \hbar k^{j} \left({}^{s}a^{*}_{\boldsymbol{k}}\, {}^{s}a_{\boldsymbol{k}} - {}^{s}b_{\boldsymbol{k}}\, {}^{s}b^{*}_{\boldsymbol{k}} \right) . \tag{8.96}$$

The Dirac field's electrical current density

$$j^{\nu} \overset{(4.87)}{=} qc\, \overline{\psi} \gamma^{\nu} \psi \ , \tag{8.97}$$

for which the continuity equation

$$c\mathrm{d}_0 \underbrace{q\overline{\psi}\gamma^0\psi}_{\text{charge density} = q\psi^{\dagger}\psi} \overset{(4.88b)}{=} -\mathrm{d}_k qc\overline{\psi}\gamma^{k}\psi \tag{8.98}$$

holds, was computed already in section 4.4. The current density differs from the Dirac field's Lorentz-invariant probability density $\overline{\psi}\psi$ by the constant factor q, and in addition by the factor γ^0, which interchanges the spinor components 1 and 2 with the spinor components 3 and 4. The charge, but not the charge density, is a conserved quantity, because under Lorentz transformations the size of the volume changes, in which the unchanged charge is enclosed.

The canonical conjugate momentum densities are quite "asymmetric": $\pi_{\psi} \overset{(8.84a)}{\neq} 0$, $\pi_{\overline{\psi}} \overset{(8.84b)}{=} 0$. Some authors therefore prefer to define an alternative Lagrangian

$$\mathcal{L}' = -\frac{i\hbar c}{2} (\mathrm{d}_{\nu}\overline{\psi})\gamma^{\nu}\psi + \frac{i\hbar c}{2}\, \overline{\psi}\gamma^{\nu}\mathrm{d}_{\nu}\psi - \overline{\psi} mc^2 \psi \ , \tag{8.99}$$

in which the fields $\overline{\psi}$ and ψ appear more symmetrical. This Lagrangian leads to the same field equations as $\mathcal{L} = (8.24)$:

$$0 = \mathrm{d}_\nu \frac{\partial \mathcal{L}'}{\partial(\mathrm{d}_\nu \psi)} - \frac{\partial \mathcal{L}'}{\partial \psi} = i\hbar c \mathrm{d}_\nu \overline{\psi}\gamma^\nu + \overline{\psi} m c^2 \tag{8.100a}$$

$$0 = \mathrm{d}_\nu \frac{\partial \mathcal{L}'}{\partial(\mathrm{d}_\nu \overline{\psi})} - \frac{\partial \mathcal{L}'}{\partial \overline{\psi}} = -i\hbar c \gamma^\nu \mathrm{d}_\nu \psi + m c^2 \psi \tag{8.100b}$$

From the Lagrangian (8.99) the canonically conjugate momentum densities

$$\pi'_\psi \equiv \frac{\partial \mathcal{L}'}{c\,\partial(\mathrm{d}_0 \psi)} = \frac{i\hbar}{2}\,\overline{\psi}\gamma^0 \tag{8.101a}$$

$$\pi'_{\overline{\psi}} \equiv \frac{\partial \mathcal{L}'}{c\,\partial(\mathrm{d}_0 \overline{\psi})} = -\frac{i\hbar}{2}\gamma^0 \psi \tag{8.101b}$$

are derived, which are significantly more symmetric than (8.84). But of course the Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ are completely equivalent only, if they also result into the same ES-tensor. With $\mathcal{L}' =$ (8.99) we get this ES-tensor:

$$\begin{aligned} \mathcal{T}'^{\rho\sigma} &\overset{(8.86)}{=} \frac{\partial \mathcal{L}'}{\partial(\mathrm{d}_\rho \psi)}\,\mathrm{d}^\sigma \psi + (\mathrm{d}^\sigma \overline{\psi})\,\frac{\partial \mathcal{L}'}{\partial(\mathrm{d}_\rho \overline{\psi})} - g^{\rho\sigma}\mathcal{L}' \\ &\overset{(8.99)}{=} \frac{i\hbar c}{2}\Big(\overline{\psi}\gamma^\rho\,\mathrm{d}^\sigma \psi - (\mathrm{d}^\sigma \overline{\psi})\,\gamma^\rho \psi\Big) - g^{\rho\sigma}\mathcal{L}' \end{aligned} \tag{8.102}$$

In particular we find

$$\begin{aligned} \mathcal{H}' &= \frac{i\hbar}{2}\,\overline{\psi}\gamma^0\, c\mathrm{d}_0 \psi - (c\mathrm{d}_0\,\overline{\psi})\,\frac{i\hbar}{2}\gamma^0\psi - \mathcal{L}' \\ &= \frac{i\hbar c}{2}\,(\mathrm{d}_j \overline{\psi})\gamma^j \psi - \frac{i\hbar c}{2}\,\overline{\psi}\gamma^j \mathrm{d}_j \psi + \overline{\psi} m c^2 \psi\ , \end{aligned} \tag{8.103}$$

which can be written as

$$\begin{aligned} \mathcal{H}' &= (8.103) - \underbrace{(8.100a)}_{0}\cdot\frac{\psi}{2} - \frac{\overline{\psi}}{2}\cdot\underbrace{(8.100b)}_{0} \\ &= -\frac{i\hbar c}{2}\,(\mathrm{d}_0 \overline{\psi})\gamma^0 \psi + \frac{i\hbar c}{2}\,\overline{\psi}\gamma^0 \mathrm{d}_0 \psi\ . \end{aligned} \tag{8.104}$$

The physical momentum density is found to be

$$\mathcal{P}'^{j} \equiv \frac{i\hbar}{2}\,\overline{\psi}\gamma^0 \mathrm{d}^j\psi - (\mathrm{d}^j\overline{\psi})\frac{i\hbar}{2}\gamma^0\psi\ . \tag{8.105}$$

By closer analysis in section 16.2 we will find out, that indeed (8.104) = (8.88) and (8.105) = (8.89). Thus it's merely a matter of taste, whether $\mathcal{L}$ = (8.24) or $\mathcal{L}'$ = (8.99) is used.

We conclude this section with an remark on General Relativity Theory (GRT). The field-equation of GRT describes, how the geometry of four-dimensional space-time is deformed by it's content of energy and momentum:

$$R_{\mu\nu}(x) - \frac{R(x)}{2}\,g_{\mu\nu}(x) + \Lambda\, g_{\mu\nu}(x) = -\frac{8\pi G}{c^4}\,\mathcal{T}_{\mu\nu}(x) \tag{8.106}$$

The tensor $R_{\mu\nu}$ and it's contraction R are describing the curvature of space-time at x. Λ is the cosmological constant, and G is Newton's constant of gravitation. $\mathcal{T}_{\mu\nu}$ is the ES-tensor of all fields at space-time point x, with exception of the metric field $g_{\mu\nu}$.

It has been shown in section 2.1, that the metric tensor of GRT is symmetric

$$g_{\,\mu\nu} \overset{(2.5)}{=} g_{\,\nu\mu} \tag{8.107}$$

and thus only 10 of it's components are independent. Consequently, according to (8.106) $\mathcal{T}_{\mu\nu}$ must be symmetric as well. But neither $\mathcal{T}$ = (8.86) nor $\mathcal{T}'$ = (8.102) is symmetric, i. e. for both forms we have $\mathcal{T}^{\rho\sigma} \neq \mathcal{T}^{\sigma\rho}$ and $\mathcal{T}'^{\rho\sigma} \neq \mathcal{T}'^{\sigma\rho}$.

At first sight, there seems to be a serious problem. But it is shown in appendix A.12 that symmetric ES-tensors can be constructed for arbitrary vector- and spinor-fields, which are resulting into the same equations of continuity as the asymmetric ES-tensors of the form (8.86). In case of the Dirac field, the symmetric ES-tensor is

$$\begin{aligned}\widetilde{\mathcal{T}}^{\rho\sigma} &\overset{(A.86)}{=} \frac{i\hbar c}{4}\Big(- (\mathrm{d}^\rho\overline{\psi})\gamma^\sigma\psi - (\mathrm{d}^\sigma\overline{\psi})\gamma^\rho\psi + \overline{\psi}\gamma^\rho\mathrm{d}^\sigma\psi + \\ &+ \overline{\psi}\gamma^\sigma\mathrm{d}^\rho\psi\Big) - g^{\rho\sigma}\,\underbrace{\overline{\psi}(i\hbar c\gamma^\nu\mathrm{d}_\nu - mc^2)\psi}_{\mathcal{L}}\ .\end{aligned} \tag{8.108}$$

To stay in line with most textbooks on quantum field theory, we keep in mind that the symmetric ES-tensor (8.108) exists (and must be used for computations in the framework of GRT), but we will use throughout this book the asymmetric form (8.86).

8.7 Standard Form of the Dirac-Matrices

In (8.15) we decided for the chiral form of the Dirac-matrices, and we will stay with this choice everywhere in this book. But as the "standard form" of the matrices is often encountered in the literature, the reader should become familiar with it as well.

We define the four-dimensional unitary spinor transformation

$$U \equiv \sqrt{\frac{1}{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \qquad U^+ = \sqrt{\frac{1}{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = U^{-1} \,. \tag{8.109}$$

$\mathbb{1}$ is the two-dimensional unit matrix. The four-dimensional unit matrix $\mathbb{1} = U^+U$ is inserted two times into the Dirac field's Lagrangian:

$$\begin{aligned} \mathcal{L} &\overset{(8.21)}{=} \overline{\psi}(x)\Big(i\hbar c\gamma^\nu\big(\mathrm{d}_\nu + \frac{i}{\hbar}qA_\nu(x)\big) - mc^2\Big)\psi(x) \\ &= \overline{\psi}(x)U^\dagger\Big(i\hbar cU\gamma^\nu U^+\big(\mathrm{d}_\nu + \frac{i}{\hbar}qA_\nu(x)\big) - mc^2\Big)U\psi(x) \end{aligned} \tag{8.110}$$

The transformed γ-matrices are

$$\begin{aligned} U\gamma^0U^\dagger &= \frac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ U\gamma^jU^\dagger &= \frac{1}{2}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \end{aligned} \tag{8.111}$$

This is the "standard form" of the Dirac-matrices. It differs from the chiral form only in γ^0. The three matrices γ^j of both forms are identical. The spinors in the standard form get the shape

$$U\psi = \sqrt{\frac{1}{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} L \\ R \end{pmatrix} = \begin{pmatrix} L+R \\ -L+R \end{pmatrix} . \tag{8.112}$$

The standard form is most suitable for the exploration of the non-relativistic limit. This is described e. g. in [33, chapter 1.4]. It's disadvantage is, that the close relation of Dirac-spinors to the direct sum of left- and right-handed Weyl-spinors is obscured. In the standard form it's not immediately evident, that the four-dimensional Dirac-representation of high-energy fields can be reduced – as we will discuss in section (8.9) – into two-dimensional Weyl-representations.

8.8 A Remark on the Electron's Spin

The spin $\hbar/2$ of the electron has been detected in the early twenties of the last century due to analysis of the optical absorption spectra of atoms. An unexpected doubling of almost all spectral lines (a phenomenon called anomalous Zeeman-effect) could be explained as the effect of an additional degree of freedom of the electron, namely the two possible orientations of it's "spin" along some axis of position space. Pauli expanded the Schrödinger-equation to a non-relativistic equation for two-component spinors (one component for each possible spin projection onto that axis), and thus could accommodate most spectroscopic observations with reasonable accuracy. Still spin seemed to be a surprising and "unnecessary" complication. Why couldn't the electron be a simple elementary point-particle which is completely characterized by it's mass and charge?

In the introduction of his publication [35] of equation (8.5), Dirac wrote: "The question remains as to why Nature should have chosen this particular model for the electron instead of being satisfied with the point-charge. One would like to find some incompleteness in the previous methods of applying quantum mechanics to the point-charge electron such that, when removed, the whole of the duplexity phenomena follow without arbitrary assumptions. In the present paper it is shown that this is the case, the incompleteness of the previous theories lying in their disagreement with relativity".

Indeed, Dirac only required (8.3) to be a correct, Lorentz-invariant equation, and linearized it in the form (8.4). As shown in the previous sections,

the Dirac-equation and the form of the γ-matrices are resulting from this ansatz without any further artfully arranged assumptions. And in [33, chapter 1.4] it is shown, that the Pauli-equation is nothing but the non-relativistic limit of the Dirac-equation (8.5) (with two spinor components being negligible in that limit). Thus the electron with spin $\hbar/2$ is really the simplest design Nature could choose to construct a point-charge in compliance with Special Relativity.

Finally a word of warning may be appropriate. The often encountered picture of the electron as a rotating sphere must not be taken literally, because that picture is inconsistent with observation in several respects: In that picture, the electron's speed at the equator must be much higher than the speed of light in vacuum, unless a much larger electron diameter is assumed than compatible with experimental evidence. Dirac in contrast assumes the electron to be a true mathematical point particle, with no diameter at all. Furthermore the orbital magnetic moment of a particle with charge q, mass m, and orbital angular momentum $\boldsymbol{L}$ is

$$\boldsymbol{\mu}_L = g_L \frac{q}{2m} \boldsymbol{L} \quad \text{with } g_L = 1 \ , \tag{8.113a}$$

while the electron's spin magnetic moment is in good approximation

$$\boldsymbol{\mu}_S = g_S \frac{(-e)}{2m} \boldsymbol{S} \quad \text{with } g_S = 2 \text{ and } |\boldsymbol{S}| = \hbar/2 \ . \tag{8.113b}$$

In Dirac's theory, the observed factor $g_S \approx 2$ results — as shown in [33, chapter 1.4] — quite simple and without any additional assumptions. (In chapter 26 we will compute quantum-field-theoretical corrections of that factor.) Surprisingly, the g-factors are no insurmountable problem for the rotating-sphere-model, because the factor $g_S = 2$ can be explained in that model as an effect of relativistic kinematics [36]. But Dirac emphasizes in his publication [35] that only the sum of the orbital angular momentum and of the spin are conserved, but not the electron's orbital angular momentum nor it's spin alone. This again is showing that the picture of a rotating sphere would be misleading, as in that case (e. g. in case of a rotating planet which is orbiting around the sun) both orbital angular momentum and spin

would be separately conserved.

Thus the spin must be interpreted as a quite abstract mathematical concept, which has no analogue in classical physics, and can not be squeezed into the pictorial model of a rotating sphere.

8.9 Weyl Equations

The Dirac equation $(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\psi \overset{(8.5)}{=} 0$ can be written by insertion of the γ-matrices (8.15) as

$$(i\hbar c \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \mathrm{d}_0 + i\hbar c \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \mathrm{d}_k - mc^2)\psi = \begin{pmatrix} -mc^2 & i\hbar c(\mathrm{d}_0 + \sigma^k \mathrm{d}_k) \\ i\hbar c(\mathrm{d}_0 - \sigma^k \mathrm{d}_k) & -mc^2 \end{pmatrix} \psi = 0 \ . \tag{8.114}$$

If $m = 0$, or if the field's energy is so high that it's rest energy mc^2 is negligible, then the components of the four-dimensional Dirac equation (8.114) split into the two-dimensional *Weyl equations*:

$$i\hbar c(\mathrm{d}_0 - \sigma^k \mathrm{d}_k) \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = 0 \tag{8.115a}$$

$$i\hbar c(\mathrm{d}_0 + \sigma^k \mathrm{d}_k) \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = 0 \tag{8.115b}$$

Here we resumed the notation of section 6.2.1, in which two-dimensional spinors were used as bases of left- and right-handed representations of the Lorentz group's covering group $\{\mathcal{B}\}$. In the next section we will discuss solutions of the Weyl equations (8.115), and we will clarify the notion of their helicity (sometimes also called chirality or handedness).

8.10 The free Weyl Field

As the four-component Dirac equation decomposes in case of negligible rest mass (and the more so in case of rest mass zero) into a two-component left-

handed and a two-component right-handed Weyl equation, the solutions of the Weyl equations can be derived from the solutions of the Dirac equation due to the limit $m \to 0$.

We again constrain our considerations to those solutions $\psi(x)$, in which only one single Fourier coefficient in (8.83) is 1, while all other Fourier coefficients are 0. We start with the spinors (8.75), and apply such a strong boost in direction of the *negative* x^3-axis, that $E \approx c^2 p^2 \gg mc^2$. Now we have $p^3 > 0$, $p_3 < 0$ and $E + cp_3 \approx 0$, $E - cp_3 \approx 2E$. Thus instead of (8.75) we get these spinors:

$$
{}^1u^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} \mathcal{O}(mc^2) \\ 0 \\ 2E \\ 0 \end{pmatrix} \qquad {}^2u^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} 0 \\ 2E \\ 0 \\ \mathcal{O}(mc^2) \end{pmatrix}
$$

$$
{}^1v^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} \mathcal{O}(mc^2) \\ 0 \\ -2E \\ 0 \end{pmatrix} \qquad {}^2v^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} 0 \\ 2E \\ 0 \\ \mathcal{O}(mc^2) \end{pmatrix} \tag{8.116}
$$

In the limit $m \to 0$ from the two top components of these spinors, left-handed Weyl spinors can be derived, and right-handed Weyl spinors from the two bottom components. We mark left-handed spinors by the index L, right-handed spinors by the index R. The additional index $>$, which codes $p^3 = \hbar k^3 > 0$, will soon be skipped:

$$
{}^1u^{\boldsymbol{k}R>} = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad {}^2u^{\boldsymbol{k}L>} = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

$$
{}^1v^{\boldsymbol{k}R>} = \sqrt{2E} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad {}^2v^{\boldsymbol{k}L>} = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{8.117}
$$

Spinors $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, which are trivial solutions of the Weyl equations, have been ignored from the outset.

A very strong passive boost in direction of the *positive* x^3-axis results in $p^3 < 0$, $p_3 > 0$ and $E + cp_3 \approx 2E$, $E - cp_3 \approx 0$. Thus one gets instead of (8.75) the spinors

$$
{}^1u^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} 2E \\ 0 \\ \mathcal{O}(mc^2) \\ 0 \end{pmatrix} \qquad {}^2u^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} 0 \\ \mathcal{O}(mc^2) \\ 0 \\ 2E \end{pmatrix}
$$

$$
{}^1v^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} 2E \\ 0 \\ \mathcal{O}(mc^2) \\ 0 \end{pmatrix} \qquad {}^2v^{\boldsymbol{k}} = \sqrt{\frac{1}{2E}} \begin{pmatrix} 0 \\ \mathcal{O}(mc^2) \\ 0 \\ -2E \end{pmatrix} . \tag{8.118}
$$

From these Dirac spinors one can in the limit $m \to 0$ extract these non-trivial Weyl spinors:

$$
{}^1u^{\boldsymbol{k}L<} = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad {}^2u^{\boldsymbol{k}R<} = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

$$
{}^1v^{\boldsymbol{k}L<} = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad {}^2v^{\boldsymbol{k}R<} = \sqrt{2E} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{8.119}
$$

Dirac spinors of type v are linearly independent of spinors of type u of the Dirac field in (8.73a) resp. (8.74). This isn't true any more for the Weyl spinors. In both (8.117) and (8.119), the v spinors depend linearly from (or are even identical to) the u spinors. Therefore we only keep the spinors of type u.

The helicity of a two-component spinor is by definition the eigenvalue of the

$$
\text{helicity operator} \equiv Z \equiv -\frac{1}{2} \frac{p_j}{p} \sigma^j \ . \tag{8.120}
$$

If the field is moving in direction of the positive x^3-axis, then $p^3 > 0$, and there is a factor $p_3/p = -1$ within the helicity operator. In this case the spinors have the helicity

$$
Z {}^1u^{\boldsymbol{k}R>} = +\frac{1}{2}\sigma^3 \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2} {}^1u^{\boldsymbol{k}R>} \tag{8.121a}
$$

$$Z^2{}^1u^{\boldsymbol{k}L>} = +\frac{1}{2}\sigma^3\ \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}{}^2u^{\boldsymbol{k}L>}\ . \tag{8.121b}$$

If the field is moving in direction of the negative x^3-axis, then there is the factor $p_3/p = +1$ within the helicity operator because of $p^3 < 0$. In this case, the spinors have the helicity

$$Z^1{}^1u^{\boldsymbol{k}L<} = -\frac{1}{2}\sigma^3\ \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix} = -\frac{1}{2}{}^1u^{\boldsymbol{k}L<} \tag{8.121c}$$

$$Z^2{}^2u^{\boldsymbol{k}R<} = -\frac{1}{2}\sigma^3\ \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} = +\frac{1}{2}{}^2u^{\boldsymbol{k}R<}\ . \tag{8.121d}$$

All four spinors are eigenvectors of the helicity operator. The R-spinors have positive helicity, and the L-spinors have negative helicity. This justifies by hindsight the classification of the spinors into right- and left-handed ones, which we had established already in (6.78) and (8.115).

We have derived the solutions of the free Weyl-equations only for boosts along the x^3-axis. But similar solutions hold for boosts in arbitrary directions. In the general form (8.73) of the Dirac spinors there are all three products $p_j\sigma^j$, and in the helicity operator (8.120) there are the same products. Weyl-spinors, which are built from the two top components of boosted Dirac spinors, always have helicity $-\frac{1}{2}$, independent of the boost's direction. Weyl-spinors, which are built from the two bottom components of boosted Dirac spinors, always have helicity $+\frac{1}{2}$.

From now on, we skip the indices $>$ and $<$, because these informations are already supplied by the index $\boldsymbol{k}$. But we continue to distinguish left-handed from right-handed spinors, because they belong — even though they are looking formally identical — to solutions of the different equations (8.115a) and (8.115b).

$${}^1u^{\boldsymbol{k}L} = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix} \qquad {}^2u^{\boldsymbol{k}L} = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \tag{8.122a}$$

$${}^1u^{\boldsymbol{k}R} = \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix} \qquad {}^2u^{\boldsymbol{k}R} = \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \tag{8.122b}$$

The products

$$({}^{r}u^{\boldsymbol{k}L})^{\dagger}\,{}^{s}u^{\boldsymbol{k}L} = ({}^{r}u^{\boldsymbol{k}R})^{\dagger}\,{}^{s}u^{\boldsymbol{k}R} = 2E\,\delta_{rs} \tag{8.123}$$

are not lorentz-invariant, in contrast to the corresponding products (8.72) of Dirac-spinors. Same as for the Dirac field in (8.80), we choose

$$N \equiv 2E \tag{8.124}$$

as normalization factor of the complete solution, and get as complete Weyl-spinors

$$\psi(x) = \frac{1}{\sqrt{2E\Omega}} \begin{cases} {}^{1}u^{\boldsymbol{k}L}\exp\{-ikx\} \\ {}^{2}u^{\boldsymbol{k}L}\exp\{-ikx\} \\ {}^{1}u^{\boldsymbol{k}L}\exp\{+ikx\} \\ {}^{2}u^{\boldsymbol{k}L}\exp\{+ikx\} \\ {}^{1}u^{\boldsymbol{k}R}\exp\{-ikx\} \\ {}^{2}u^{\boldsymbol{k}R}\exp\{-ikx\} \\ {}^{1}u^{\boldsymbol{k}R}\exp\{+ikx\} \\ {}^{2}u^{\boldsymbol{k}R}\exp\{+ikx\} \end{cases} \tag{8.125}$$

with $E = c\hbar k^0 = \hbar\omega_{\boldsymbol{k}} > 0$ and ${}^{r}u^{X} =$ (8.122).

Weyl fields have mass $m = 0$, or negligibly small mass, spin $s = 1/2$, and helicity $Z = +1/2$ or $Z = -1/2$. Left-handed Weyl fields (helicity $= -1/2$) emerge in the limit $m \to 0$ from the two top components of Dirac spinors. Right-handed Weyl fields (helicity $= +1/2$) emerge in the limit $m \to 0$ from the two bottom components of Dirac spinors. (8.126)

The two last sentences are only true with our definitions (8.15) of the Dirac-matrices. With other definitions of the γ-matrices, the relations inbetween Weyl spinors and Dirac spinors are less transparent.

8.11 Weyl field: Probability Density

We found out in section 8.4, that the Dirac field is normalizable only if the product $\bar{\psi}\,\psi$, but not the product $\psi^\dagger\,\psi$, is defined as probability density. No complication of that type does exist for the two-component Weyl-spinors, even though they are identical to the upper two or to the lower two components of Dirac-spinors. Weyl-spinors are normalizable with the probability density $\psi^\dagger\psi$.

The transformation of a Weyl-spinor $\psi(x)$ under a rotation of the space-time coordinates is

$$\psi' \overset{(6.78)}{=} \underbrace{\exp\left\{\frac{i}{\hbar}\Big(\Theta_k\frac{\hbar\sigma^k}{2} \pm \eta_k i\frac{\hbar\sigma^k}{2}\Big)\right\}}_{D}\psi\,. \tag{8.127}$$

Using $\sigma^j \overset{(8.14)}{=} \sigma^{j\dagger}$ we get the infinitesimal transformation (in which terms $\mathcal{O}(\Theta^2), \mathcal{O}(\eta^2), \mathcal{O}(\Theta\eta)$ may be neglected):

$$\begin{aligned}
\psi'^\dagger\,\psi' &= (D_{\text{INF}}\psi)^\dagger\,(D_{\text{INF}}\psi) = \psi^\dagger\,D_{\text{INF}}^\dagger\,D_{\text{INF}}\,\psi \;= \\
&\overset{(8.127)}{=} \psi^\dagger\left(1 - \frac{i}{\hbar}\Big(\Theta_k\frac{\hbar\sigma^{k\dagger}}{2} - \eta_k i\frac{\hbar\sigma^{k\dagger}}{2}\Big)\right)\cdot \\
&\qquad\cdot\left(1 + \frac{i}{\hbar}\Big(\Theta_k\frac{\hbar\sigma^{k}}{2} + \eta_k i\frac{\hbar\sigma^{k}}{2}\Big)\right)\psi \;= \\
&= \psi^\dagger\left(1 + \frac{i}{2}\Theta_k(\sigma^k - \sigma^{k\dagger}) - \frac{1}{2}\eta_k(\sigma^k + \sigma^{k\dagger})\right)\psi \;= \\
&= \psi^\dagger\Big(1 - \eta_k\sigma^k\Big)\psi \neq \psi^\dagger\psi
\end{aligned} \tag{8.128}$$

While the Weyl field's probability density $\psi^\dagger\,\psi$ is invariant under a pure space-like rotation of coordinates ($\boldsymbol{\Theta} \neq 0, \boldsymbol{\eta} = 0$), it is not invariant under a Lorentz boost ($\boldsymbol{\eta} \neq 0$).

Recall, however, that it is not the probability density but the three-dimensional space integral over the probability density, which must be

Lorentz-invariant. The Weyl field does meet this criterion:

$$\int_\Omega \mathrm{d}^3x\, \psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x}) \overset{(8.123),(8.125)}{=} \frac{1}{\Omega}\int_\Omega \mathrm{d}^3x \;= 1 \tag{8.129}$$

9 The classical Electromagnetic Field

We have seen in section 4.5 that the electromagnetic field is the gauge field of fields with an electric charge (for example the Dirac field), and that it's Lagrangian is almost completely fixed by the postulate, that the charged field shall be invariant under local gauge transformations.

9.1 Lagrangian and Field Equation

This is the Lagrangian of the free electromagnetic field, i. e. the electromagnetic field which is not interacting with any other field:

$$\boxed{\mathcal{L} \overset{(4.120)}{=} -\frac{1}{4\mu_0} F_{\sigma\tau} F^{\sigma\tau} \overset{(4.114)}{=} -\frac{1}{4\mu_0} (\mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma)(\mathrm{d}^\sigma A^\tau - \mathrm{d}^\tau A^\sigma)} \tag{9.1}$$

The second term in the field equation

$$\mathrm{d}_\nu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu A_\mu)} - \underbrace{\frac{\partial \mathcal{L}}{\partial A_\mu}}_{-j^\mu} \overset{(3.37)}{=} 0 \quad \text{with } \mu = 0, 1, 2, 3 \tag{9.2}$$

is the current density $j^\mu = (4.113)$. In case of the free electromagnetic field, this term is zero. With

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu A_\mu)} &= -\frac{2}{4\mu_0} (g_\nu{}^\sigma g_\mu{}^\tau - g_\nu{}^\tau g_\mu{}^\sigma)(\mathrm{d}^\sigma A^\tau - \mathrm{d}^\tau A^\sigma) = \\ &= -\frac{1}{2\mu_0} (\mathrm{d}^\nu A^\mu - \mathrm{d}^\mu A^\nu - \mathrm{d}^\mu A^\nu + \mathrm{d}^\nu A^\mu) = \\ &= -\frac{1}{\mu_0} (\mathrm{d}^\nu A^\mu - \mathrm{d}^\mu A^\nu) \overset{(4.114)}{=} -\frac{1}{\mu_0} F^{\nu\mu} \end{aligned} \tag{9.3}$$

we get this equation for the free electromagnetic field:

$$\boxed{\mathrm{d}_\nu F^{\nu\mu} = \mathrm{d}_\nu(\mathrm{d}^\nu A^\mu - \mathrm{d}^\mu A^\nu) = 0 \quad \text{with } \mu = 0,1,2,3} \tag{9.4}$$

It has been shown at the end of section 4.5, that this four-dimensional equation is equivalent to the well-known four three-dimensional Maxwell equations.

9.2 Conserved Quantities

For the free electromagnetic field's ES-tensor

$$\begin{aligned}\boxed{\mathcal{T}^{\sigma\tau} \overset{(4.32)}{=} \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\sigma A_\rho)}\,\mathrm{d}^\tau A_\rho - g^{\sigma\tau}\mathcal{L}} \\ \overset{(9.3),(9.1)}{=} -\frac{1}{\mu_0}F^{\sigma\rho}\,\mathrm{d}^\tau A_\rho + g^{\sigma\tau}\frac{1}{4\mu_0}F_{\nu\rho}F^{\nu\rho}\end{aligned} \tag{9.5}$$

these four equations of continuity hold:

$$\mathrm{d}_\sigma \mathcal{T}^{\sigma\tau} \overset{(4.33)}{=} 0 \quad \text{with } \sigma\tau = 0,1,2,3 \tag{9.6}$$

Remember that $\mathcal{H} \overset{(4.34)}{=} \mathcal{T}^{00}$ is the field's energy density, and that the j-component of it's momentum density is $\mathcal{P}^j \overset{(4.34)}{=} \mathcal{T}^{0j}/c$.

In section 5.7 we found that the six equations of continuity

$$\mathrm{d}_\rho \mathcal{M}^{\rho\sigma\tau} \overset{(5.102)}{=} 0 \quad \text{with } \sigma\tau = 10,20,30,23,31,12 \tag{9.7}$$

hold for the angular-momentum-density tensor

$$\mathcal{M}^{\rho\sigma\tau} \overset{(5.100)}{=} x^\sigma \frac{\mathcal{T}^{\rho\tau}}{c} - x^\tau \frac{\mathcal{T}^{\rho\sigma}}{c} + \mathcal{S}^{\rho\sigma\tau}\ , \tag{9.8}$$

in which

$$\mathcal{S}^{\rho\sigma\tau} \overset{(5.99)}{=} \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho A_\sigma)}A^\tau - \frac{1}{c}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho A_\tau)}A^\sigma \tag{9.9}$$

is the electromagnetic field's spin density. We discussed already at equation (4.74) that only the purely space-like angular momenta

$$M^{jl} \overset{(5.104)}{=} \underbrace{\int_\Omega \mathrm{d}^3x \left(x^j \mathcal{P}^l - x^l \mathcal{P}^j\right)}_{\text{orbital angular momentum}} + \underbrace{\int_\Omega \mathrm{d}^3x\, \mathcal{S}^{0jl}}_{\text{spin}} \qquad (9.10)$$

$$\text{with } jl = 23, 31, 12$$

are useful for practical applications. Note that conservation laws only hold for the total angular momenta M^{jl}, but not for orbital angular momenta or spins separately.

The ES-tensor (9.5) is obviously not symmetric, i. e. $\mathcal{T}^{\sigma\tau} \neq \mathcal{T}^{\tau\sigma}$. As explained around (8.106), this will result into problems if computations are done in the framework of General Relativity Theory, because in that theory ES-tensors *must* be symmetric. It is shown in appendix A.12, however, that symmetric ES-tensors can be constructed for arbitrary vector- and spinor-fields, which are resulting into the same equations of continuity as the asymmetric ES-tensors of the form (9.5). In case of the electromagnetic field, the symmetric ES-tensor is

$$\widetilde{\mathcal{T}}^{\sigma\tau} \overset{(A.83)}{=} -\frac{1}{\mu_0} F^{\sigma\nu} F^{\tau}{}_{\nu} + g^{\sigma\tau} \frac{1}{4\mu_0} F_{\nu\rho} F^{\nu\rho} \,. \qquad (9.11)$$

To stay in line with most textbooks on quantum field theory, we keep in mind that the symmetric ES-tensor (9.11) exists (and must be used for computations in the framework of GRT), but we will use throughout this book the asymmetric form (9.5).

10 The classical Klein-Gordon Field

10.1 Field Equation

In section 8.1 we iterated the Ansatz for the Dirac equation, and thereby found an equation for the squared energy:

$$(i\hbar c)^2\Big(\gamma^0\gamma^0 \mathrm{d}_0\mathrm{d}_0 + (\gamma^0\gamma^k + \gamma^k\gamma^0)\mathrm{d}_0\mathrm{d}_k + \\ + \gamma^k\gamma^j \mathrm{d}_k\mathrm{d}_j\Big)\phi \overset{(8.7)}{=} (mc^2)^2\phi \tag{10.1}$$

From this we derived the Dirac equation, which is linear in the energy. Alternatively we may stick to equation (10.1). Inserting the γ-matrices from (8.15), and regarding

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \overset{(8.9)}{=} 2g^{\mu\nu}\mathbb{1}\ ,$$

we find

$$\Big((i\hbar c)^2(\mathbb{1}\mathrm{d}_0\mathrm{d}_0 - \mathbb{1}\mathrm{d}_k\mathrm{d}_k) - (mc^2)^2\Big)\phi = 0\ .$$

All four components of this equation are identical. Therefore this actually is only one equation:

$$\boxed{(\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu + m^2c^4)\phi = 0} \tag{10.2}$$

Thus the field $\phi(x)$ is a spinor with one component only, i. e. a scalar in spinor-space (and a scalar in space-time). Schrödinger found this equation even previous to his non-relativistic equation

$$\Big(i\hbar\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\hbar^2}{2m}\nabla^2\Big)\psi=0\,. \tag{10.3}$$

But he discarded it when he noticed, that no equation of continuity can be derived from it. Searching for a continuity equation, he multiplied (10.2) by the complex-conjugate function ϕ^*, multiplied the complex-conjugate equation by ϕ, and subtracted one equation from the other:

$$\begin{aligned}0&=\phi^*(\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu+m^2c^4)\phi-\phi(\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu+m^2c^4)\phi^*\\&=\hbar^2c^2(\phi^*\mathrm{d}^\mu\mathrm{d}_\mu\phi-\phi\mathrm{d}^\mu\mathrm{d}_\mu\phi^*)=\hbar^2c^2\mathrm{d}^\mu(\phi^*\mathrm{d}_\mu\phi-\phi\mathrm{d}_\mu\phi^*)\\\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi^*\frac{\mathrm{d}\phi}{\mathrm{d}t}-\phi\frac{\mathrm{d}\phi^*}{\mathrm{d}t}\Big)&=-c^2\mathrm{d}^k\Big(\phi^*\mathrm{d}_k\phi-\phi\mathrm{d}_k\phi^*\Big)\end{aligned} \tag{10.4}$$

The term $\phi^*\dot\phi-\phi\dot\phi^*$ can become negative. If instead (10.3) is multiplied by ψ^*, the complex-conjugate equation is multiplied by ψ, and one equation is subtracted from the other, then the result is

$$\begin{aligned}i\hbar\Big(\psi^*\frac{\mathrm{d}\psi}{\mathrm{d}t}+\psi\frac{\mathrm{d}\psi^*}{\mathrm{d}t}\Big)&=-\frac{\hbar^2}{2m}\Big(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\Big)\\i\hbar\frac{\mathrm{d}(\psi^*\psi)}{\mathrm{d}t}&=-\frac{\hbar^2}{2m}\nabla\Big(\psi^*\nabla\psi-\psi\nabla\psi^*\Big)\,.\end{aligned} \tag{10.5}$$

$\psi^*\psi$ obviously is positive definite, and thus – in contrast to $(\phi^*\dot\phi-\phi\dot\phi^*)$ – is a reasonable expression for the electron field's probability density (or rather matter density according to Schrödinger's understanding by the end of 1925). For this reason, Schrödinger dismissed his ansatz (10.2), and focused for the time being on the non-relativistic equation (10.3). Motivated by the need of different names for different equations, for Schrödinger's obviously relativistically invariant equation (10.2) "unjustly" the name *Klein-Gordon equation* became conventional.

While (10.2) because of (10.4) is not reasonable as an equation for one particle, the Klein-Gordon equation turned out to be a sensible and useful equation in relativistic quantum field theory.

10.2 The free Field

$$(\hbar^2 c^2 \mathrm{d}^\mu \mathrm{d}_\mu + m^2 c^4)\phi(x) \overset{(10.2)}{=} 0 \tag{10.6}$$

is the equation of a free, not with any other field interacting, Klein-Gordon field. To check that this equation is solved by plane waves

$$\phi(x) = \frac{1}{\sqrt{N}} \exp\{\mp ikx\} = \frac{1}{\sqrt{N}} \exp\{\mp i(\omega_{\boldsymbol{k}} t - \boldsymbol{kx})\} \ , \tag{10.7}$$

we insert this Ansatz into the field equation:

$$(-\hbar^2\omega_{\boldsymbol{k}}^2 + \hbar^2 c^2 \boldsymbol{k}^2 + m^2 c^4) \frac{\exp\{\mp ikx\}}{\sqrt{N}} \overset{(7.17)}{=} 0 \cdot \frac{\exp\{\mp ikx\}}{\sqrt{N}}$$

Any real or complex solution of the Klein-Gordon equation can — according to the general considerations of chapter 7 — be written in the form

$$\phi(x) \overset{(7.19)}{=} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} + b_{\boldsymbol{k}}^* \exp\{+ikx\} \Big) \tag{10.8}$$

as a linear combination of plane waves. Interpreted as state functions of point-particle quantum mechanics, the energy of these solutions would be

$$\begin{aligned} i\hbar \frac{\mathrm{d}\phi}{\mathrm{d}t} &= E\phi \ = \\ &= \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}} \Big(\hbar\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} \exp\{-ikx\} - \hbar\omega_{\boldsymbol{k}} b_{\boldsymbol{k}}^* \exp\{+ikx\} \Big) \ . \end{aligned}$$

Because we fixed in (7.18) that $k^0 = \omega/c > 0$ always is positive, the solutions $b_{\boldsymbol{k}}^* \exp\{+ikx\}$ would have negative energy $-\hbar\omega_{\boldsymbol{k}} < 0$. We will see, that energy is computed by a different method in quantum field theory, and that the energy of the quantized Klein-Gordon field always is positive.

Real and complex solutions of the Klein-Gordon equation both are important. In section 4.4 we found out, that the global phase transformation

$$\phi(x) \xrightarrow{\Gamma} \phi'(x) \overset{(4.77)}{=} e^{\frac{i}{\hbar}Kq}\phi(x) \quad \text{with } K, q \in \mathbb{R}$$

correlates with conservation of the charge q. This transformation makes sense only for a complex field.

Real solutions of the Klein-Gordon equations are suited for the description of fields with spin $s = 0$ and mass $m \neq 0$ or $m = 0$, which have no conserved charge. They have the form (10.8) with $b_{\boldsymbol{k}} = a_{\boldsymbol{k}}$ for arbitrary wavenumbers $\boldsymbol{k}$.
Complex solutions of the Klein-Gordon equation are suited for the description of fields with spin $s = 0$ and mass $m \neq 0$ or $m = 0$, which have (minimum) one conserved charge. They have the form (10.8) with $a_{\boldsymbol{k}} \neq b_{\boldsymbol{k}}$ for at least one wavenumber $\boldsymbol{k}$. (10.9)

10.3 Lagrangian

The equation (10.2) of the Klein-Gordon field can be derived from the Lagrangian

$$\boxed{\mathcal{L} = c^2\hbar^2(\mathrm{d}_\mu\phi^*)\mathrm{d}^\mu\phi - m^2c^4\phi^*\phi}\,. \tag{10.10}$$

In the lines after (3.37c) we explained, why ϕ and ϕ^* must be treated as independent variables of $\mathcal{L}$ for the derivation of the field equations. Because of

$$\begin{aligned}(\mathrm{d}_\mu\phi^*)\mathrm{d}^\mu\phi &= g_{\mu\sigma}g^{\mu\tau}(\mathrm{d}^\sigma\phi^*)\mathrm{d}_\tau\phi = \delta_{\sigma\tau}(\mathrm{d}^\sigma\phi^*)\mathrm{d}_\tau\phi \\ &= (\mathrm{d}^\sigma\phi^*)\mathrm{d}_\sigma\phi = (\mathrm{d}^\mu\phi^*)\mathrm{d}_\mu\phi\end{aligned}$$

the Lagrangian is completely symmetric in ϕ and ϕ^*. Therefore it's variation to ϕ^* and ϕ results into the identical field equations

$$\begin{aligned}\mathrm{d}_\mu\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\mu\phi_r)} - \frac{\partial\mathcal{L}}{\partial\phi_r} &\overset{(3.37b)}{=} 0 \\ c^2\hbar^2\mathrm{d}_\mu\mathrm{d}^\mu\phi + m^2c^4\phi &= 0\end{aligned} \tag{10.11a}$$

$$c^2\hbar^2 \mathrm{d}_\mu \mathrm{d}^\mu \phi^* + m^2 c^4 \phi^* = 0 \,, \tag{10.11b}$$

namely the Klein-Gordon equation (10.2).

At closer inspection one notices, that a re-definition of the units of the field $\phi(x)$ is necessary. We derived the Klein-Gordon equation (10.2) as an iterated Dirac equation. The Dirac field's dimension is volume$^{-1/2}$. If we would stick to this definition for the Klein-Gordon field, then it's Lagrangian (10.10) would get the exotic dimension energy^2/volume. That's not acceptable. To avoid that the canonical formalism goes to pieces, the dimension of Lagrangian and Hamiltonian compulsory must be energy/volume. Therefore we define for the Klein-Gordon field the dimension

$$[\text{Klein-Gordon field}] = [\phi(x)] = \sqrt{\frac{1}{\text{energy} \cdot \text{volume}}} \,. \tag{10.12}$$

Consequently, as the Fourier coefficients are dimension-less numbers, the normalization factor N in

$$\phi(x) \overset{(10.8)}{=} \sum_{\boldsymbol{f}} \frac{1}{\sqrt{N\Omega}} \Big(a_{\boldsymbol{f}} \exp\{-ifx\} + b^*_{\boldsymbol{f}} \exp\{+ifx\} \Big) \tag{10.13a}$$

$$\phi^*(x) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}} \Big(a^*_{\boldsymbol{k}} \exp\{+ikx\} + b_{\boldsymbol{k}} \exp\{-ikx\} \Big) \tag{10.13b}$$

must have the dimension $[N] = \text{energy}$.

Using $\dot\phi = c\mathrm{d}_0\phi = c\mathrm{d}^0\phi$, the canonical conjugate momentum density π of the field ϕ is

$$\pi(x) \overset{(3.57)}{=} \frac{\partial \mathcal{L}}{\partial \dot\phi(x)} = c\hbar^2 \mathrm{d}^0 \phi^* = \hbar^2 \dot\phi^* \,, \tag{10.14a}$$

and the conjugate momentum density of the field ϕ^* is

$$\frac{\partial \mathcal{L}}{\partial \dot\phi^*(x)} = c\hbar^2 \mathrm{d}^0 \phi = \hbar^2 \dot\phi = \pi^*(x) \,. \tag{10.14b}$$

The conjugate momentum densities' dimension is energy$^{\frac{1}{2}} \cdot$action$\cdot$volume$^{-\frac{1}{2}}$.

Thus the dimension of the product of field amplitude $\phi(x)$ and it's conjugate momentum density $\pi(x)$ is action $\cdot$ volume^{-1}, and the dimension of the product of the field's amplitude and it's conjugate momentum is action. The dimension of a product of canonically conjugate quantities *always* must be action. Thus our choice of dimensions seems reasonable.

10.4 Conserved Quantities

The components of the classical Klein-Gordon field's ES-tensor are

$$\mathcal{T}^{\rho\sigma} \overset{(4.32)}{=} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi)}\, \mathrm{d}^\sigma \phi + (\mathrm{d}^\sigma \phi^*)\, \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi^*)} - g^{\rho\sigma} \mathcal{L} \,. \tag{10.15}$$

Note that we placed here, and in the Lagrangian (10.10), the complex conjugate factors ϕ^* and $\mathrm{d}^\sigma \phi^*$ always left of the factors ϕ and $\mathrm{d}^\sigma \phi$. As the Klein-Gordon field is a scalar, the general rule (8.87) is silent about the arrangement of these factors, and indeed the order of these factors is completely arbitrary. That will change, however, when the field will be quantized in chapter 15. Then the amplitudes will become operators with non-commutative algebra, and their sequence will have significant consequences. As we derived both the Dirac equation and the Klein-Gordon equation from (10.1), these fields are closely related. For that reason we arrange the factors in Klein-Gordon theory like we did in Dirac theory, with the complex conjugate factors ϕ^* and $\mathrm{d}^\sigma \phi^*$ left of ϕ and $\mathrm{d}^\sigma \phi$.

Inserting

$$\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi)} = c^2 \hbar^2 \mathrm{d}^\rho \phi^* \qquad\qquad \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi^*)} = c^2 \hbar^2 \mathrm{d}^\rho \phi$$

into (10.15), we get these components of the ES-tensor:

$$\begin{aligned} \mathcal{T}^{\rho\sigma} = c^2 \hbar^2 \Big(& (\mathrm{d}^\rho \phi^*)\, \mathrm{d}^\sigma \phi + (\mathrm{d}^\sigma \phi^*)\, \mathrm{d}^\rho \phi - \\ & - g^{\rho\sigma} (\mathrm{d}_\mu \phi^*)\, \mathrm{d}^\mu \phi + g^{\rho\sigma} \frac{m^2 c^2}{\hbar^2}\, \phi^* \phi \Big) \end{aligned} \tag{10.16a}$$

In particular, the Hamiltonian is

$$\mathcal{H} \overset{(4.35)}{\equiv} \mathcal{T}^{00} = 2c^2\hbar^2(\mathrm{d}_0\phi^*)\mathrm{d}^0\phi - c^2\hbar^2(\mathrm{d}_\mu\phi^*)\mathrm{d}^\mu\phi + m^2c^4\phi^*\phi$$
$$= c^2\hbar^2(\mathrm{d}^\mu\phi^*)\mathrm{d}^\mu\phi + m^2c^4\phi^*\phi \ , \tag{10.16b}$$

and the physical momentum density is

$$\mathcal{P}^j \overset{(4.35)}{\equiv} \frac{1}{c}\mathcal{T}^{0j} = c\hbar^2\Big((\mathrm{d}^0\phi^*)\,\mathrm{d}^j\phi + (\mathrm{d}^j\phi^*)\,\mathrm{d}^0\phi\Big) \ . \tag{10.16c}$$

By insertion of the fields (10.13), the ES-tensor (10.16a) can be evaluated a little bit further:

$$\begin{aligned}
\mathcal{T}^{\rho\sigma} = \sum_{\boldsymbol{k},\boldsymbol{f}} \frac{c^2\hbar^2}{2\Omega\hbar\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}} \Big[& i^2 k^\rho f^\sigma \Big(a^*_{\boldsymbol{k}}\exp\{+ikx\} - b_{\boldsymbol{k}}\exp\{-ikx\}\Big)\cdot \\
& \cdot\Big(-a_{\boldsymbol{f}}\exp\{-ifx\} + b^*_{\boldsymbol{f}}\exp\{+ifx\}\Big) + \\
& + i^2 f^\rho k^\sigma \Big(a^*_{\boldsymbol{k}}\exp\{+ikx\} - b_{\boldsymbol{k}}\exp\{-ikx\}\Big)\cdot \\
& \cdot\Big(-a_{\boldsymbol{f}}\exp\{-ifx\} + b^*_{\boldsymbol{f}}\exp\{+ifx\}\Big) - \\
& - g^{\rho\sigma} i^2 k_\mu f^\mu \Big(a^*_{\boldsymbol{f}}\exp\{+ifx\} - b_{\boldsymbol{f}}\exp\{-ifx\}\Big)\cdot \\
& \cdot\Big(-a_{\boldsymbol{k}}\exp\{-ikx\} + b^*_{\boldsymbol{k}}\exp\{+ikx\}\Big) + \\
& + g^{\rho\sigma}\frac{m^2c^2}{\hbar^2}\Big(a^*_{\boldsymbol{k}}\exp\{+ikx\} + b_{\boldsymbol{k}}\exp\{-ikx\}\Big)\cdot \\
& \cdot\Big(a_{\boldsymbol{f}}\exp\{-ifx\} + b^*_{\boldsymbol{f}}\exp\{+ifx\}\Big)\Big] = \\
= \sum_{\boldsymbol{k},\boldsymbol{f}} \frac{c^2\hbar^2}{2\Omega\hbar\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}} \Big[& i^2(k^\rho f^\sigma + f^\rho k^\sigma)\Big(\\
& - a^*_{\boldsymbol{k}}a_{\boldsymbol{f}}\exp\{+i(k-f)x\} + a^*_{\boldsymbol{k}}b^*_{\boldsymbol{f}}\exp\{+i(k+f)x\} + \\
& + b_{\boldsymbol{k}}a_{\boldsymbol{f}}\exp\{-i(k+f)x\} - b_{\boldsymbol{k}}b^*_{\boldsymbol{f}}\exp\{-i(k-f)x\}\Big) -
\end{aligned}$$

$$
\begin{aligned}
&- g^{\rho\sigma} i^2 k_\mu f^\mu \Big(\\
&\quad - a^*_{\boldsymbol{f}} a_{\boldsymbol{k}} \exp\{+i(f-k)x\} + a^*_{\boldsymbol{f}} b^*_{\boldsymbol{k}} \exp\{+i(f+k)x\} + \\
&\quad + b_{\boldsymbol{f}} a_{\boldsymbol{k}} \exp\{-i(f+k)x\} - b_{\boldsymbol{f}} b^*_{\boldsymbol{k}} \exp\{-i(f-k)x\} \Big) + \\
&+ g^{\rho\sigma} \frac{m^2 c^2}{\hbar^2} \Big(\\
&\quad a^*_{\boldsymbol{k}} a_{\boldsymbol{f}} \exp\{+i(k-f)x\} + a^*_{\boldsymbol{k}} b^*_{\boldsymbol{f}} \exp\{+i(k+f)x\} + \\
&\quad + b_{\boldsymbol{k}} a_{\boldsymbol{f}} \exp\{-i(k+f)x\} + b_{\boldsymbol{k}} b^*_{\boldsymbol{f}} \exp\{-i(k-f)x\} \Big) \Big] \qquad (10.17)
\end{aligned}
$$

Now we will integrate this result in three-dimensional position space over the normalization volume Ω. We want to make use of the Kronecker symbol

$$
\frac{1}{\Omega} \int_\Omega \mathrm{d}^3 x \, \exp\{\pm i(\boldsymbol{k} - \boldsymbol{f})\boldsymbol{x}\} \overset{(7.12)}{=} \delta_{\boldsymbol{k}\boldsymbol{f}} \qquad (10.18)
$$

in all terms. As the sums over $\boldsymbol{k}$ and $\boldsymbol{f}$ are running symmetrically over all positive and negative wavenumbers, we can e. g. write

$$
\begin{aligned}
&\sum_{\boldsymbol{k},\boldsymbol{f}} a^*_{\boldsymbol{f}} b^*_{\boldsymbol{k}} \frac{1}{\Omega} \int_\Omega \mathrm{d}^3 x \, \exp\{+i(f+k)x\} = \sum_{\boldsymbol{k},\boldsymbol{f}} a^*_{-\boldsymbol{f}} b^*_{\boldsymbol{k}} \cdot \\
&\cdot \exp\{+i(f^0 + k^0)x_0\} \frac{1}{\Omega} \int_\Omega \mathrm{d}^3 x \, \exp\{+i(-f^j + k^j)x_j\} \overset{(10.18)}{=} \\
&= \sum_{\boldsymbol{k}} a^*_{-\boldsymbol{k}} b^*_{\boldsymbol{k}} \exp\{+i2k^0 x_0\} \, . \qquad (10.19)
\end{aligned}
$$

Here we applied our convention (7.18), that the null-component of the wavenumbers of free fields is chosen always ≥ 0. With (10.19) we get

$$
T^{\rho\sigma} \equiv \int_\Omega \mathrm{d}^3 x \, \mathcal{T}^{\rho\sigma} = \sum_{\boldsymbol{k}} \frac{c^2 \hbar^2}{2\hbar\omega_{\boldsymbol{k}}} \Big[
$$

$$
\begin{aligned}
&- 2k^\rho k^\sigma \Big(- a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + a^*_{\boldsymbol{k}} b^*_{-\boldsymbol{k}} \exp\{+i2k^0 x_0\} + \\
&\quad + b_{\boldsymbol{k}} a_{-\boldsymbol{k}} \exp\{-i2k^0 x_0\} - b_{\boldsymbol{k}} b^*_{\boldsymbol{k}} \Big) - \\
&+ g^{\rho\sigma} k_\mu k^\mu \Big(- a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + a^*_{-\boldsymbol{k}} b^*_{\boldsymbol{k}} \exp\{+i2k^0 x_0\} + \\
&\quad + b_{-\boldsymbol{k}} a_{\boldsymbol{k}} \exp\{-i2k^0 x_0\} - b_{\boldsymbol{k}} b^*_{\boldsymbol{k}} \Big) + \\
&+ g^{\rho\sigma} \frac{m^2 c^2}{\hbar^2} \Big(+ a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + a^*_{\boldsymbol{k}} b^*_{-\boldsymbol{k}} \exp\{+i2k^0 x_0\} + \\
&\quad + b_{\boldsymbol{k}} a_{-\boldsymbol{k}} \exp\{-i2k^0 x_0\} + b_{\boldsymbol{k}} b^*_{\boldsymbol{k}} \Big) \Big] \, . \qquad (10.20)
\end{aligned}
$$

The terms with the exponential functions are describing rapid oscillations. Their frequency does depend on the mass of the field. Even for relatively light particles with rest energy 1 MeV, the frequency of these oscillations is as high as

$$2k^0 c > \frac{2\,\mathrm{MeV}}{\hbar} \approx 3 \cdot 10^{21}\mathrm{Hz} \, . \qquad (10.21)$$

Schrödinger [34], who detected them (theoretically, these oscillations have never been observed experimentally), named them "Zitterbewegung" (trembling motion). The mean value of these terms is zero, and due to the high frequency, we certainly may neglect them.

Using that for free fields $k_\mu k^\mu = m^2 c^2/\hbar^2$, we thus arrive at

$$T^{\rho\sigma} = \sum_{\boldsymbol{k}} \frac{c^2 \hbar^2}{\hbar \omega_{\boldsymbol{k}}} k^\rho k^\sigma \Big(a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + b_{\boldsymbol{k}} b^*_{\boldsymbol{k}} \Big) \, . \qquad (10.22)$$

Note that we — in anticipation of the quantization to be done in chapter 15 — never in the derivation of this result interchanged the sequence of Fourier-coefficients in any product. While these coefficients are commutative numbers in the classical formula (10.22), they will become non-commutative operators upon quantization.

The Klein-Gordon field has a conserved charge q, if it is invariant under

a global phase transformation

$$\phi(x) \xrightarrow{\Gamma} \phi'(x) \overset{(4.77)}{=} e^{\frac{i}{\hbar}Kq}\phi(x) \quad \text{with } K, q \in \mathbb{R} \,. \tag{10.23}$$

This transformation is reasonable only for a complex field. A real Klein-Gordon field has no conserved charge.

To find out whether a $\mathcal{G}$ exists, with which the sufficient symmetry condition (4.11)

$$\exists\, \mathcal{G}: \quad \mathcal{L} \xrightarrow{I+\frac{i}{\hbar}w\gamma} \mathcal{L}\,' = \mathcal{L} + \frac{i}{\hbar}w\gamma\mathcal{L} = \mathcal{L} + \mathrm{d}_\rho \mathcal{G}^\rho \tag{10.24}$$

can be fulfilled, we investigate the infinitesimal transformation. Because of

$$e^{\frac{i}{\hbar}Kq} = \lim_{n\to\infty}\left(1 + \frac{i}{\hbar}\underbrace{\frac{K}{n}}_{\equiv k} q\right)^n \quad \text{with } n \in \mathbb{N} \tag{10.25}$$

it's form is

$$\phi(x) \xrightarrow{I+\frac{i}{\hbar}w\gamma} \phi'(x) = \phi(x) + \frac{i}{\hbar}kq\phi(x) \,. \tag{10.26}$$

The transformed complex-conjugated field is

$$\phi^{*\prime}(x) = \phi^*(x) - \frac{i}{\hbar}kq\phi^*(x) \,. \tag{10.27}$$

Note the minus sign. Now we can compute the impact of the infinitesimal transformation on $\mathcal{L}$ (terms $\mathcal{O}(k^2)$ may be neglected):

$$\begin{aligned}\mathcal{L}' = c^2\hbar^2(\mathrm{d}_\mu\phi^* - \frac{i}{\hbar}kq\mathrm{d}_\mu\phi^*)(\mathrm{d}^\mu\phi + \frac{i}{\hbar}kq\mathrm{d}^\mu\phi)\\ - m^2c^4(\phi^* - \frac{i}{\hbar}kq\phi^*)(\phi + \frac{i}{\hbar}kq\phi)\end{aligned}$$

$$= \mathcal{L} + c^2\hbar^2(\mathrm{d}_\mu\phi^*)\frac{i}{\hbar}kq\mathrm{d}^\mu\phi - c^2\hbar^2\frac{i}{\hbar}kq(\mathrm{d}_\mu\phi^*)\mathrm{d}^\mu\phi$$
$$- m^2c^4\phi^*\frac{i}{\hbar}kq\phi + m^2c^4\frac{i}{\hbar}kq\phi^*\phi + \mathcal{O}(k^2) = \mathcal{L}$$

The sufficient symmetry condition (10.24) is fulfilled by

$$\mathrm{d}_\rho\mathcal{G}^\rho = 0 \qquad \Longleftrightarrow \qquad \mathcal{G} = \text{constant} \ . \tag{10.28}$$

We can avoid unnecessary writing efforts by choosing $\mathcal{G} \equiv 0$. The conserved current density

$$j^\rho \overset{(4.16)}{=} C\Big(\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}w\gamma\phi_r + i\hbar\mathcal{G}^\rho\Big)$$

with $C \equiv 1/(i\hbar k)$ becomes

$$\begin{aligned} j^\rho &= \frac{1}{i\hbar k}\left(\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi)}kq\phi - kq\phi^*\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi^*)}\right) \\ &= \frac{1}{i\hbar k}\Big(c^2\hbar^2(\mathrm{d}^\rho\phi^*)kq\phi - c^2\hbar^2kq\phi^*\mathrm{d}^\rho\phi\Big) \\ &= -iqc^2\hbar\Big((\mathrm{d}^\rho\phi^*)\phi - \phi^*\mathrm{d}^\rho\phi\Big) \ . \end{aligned} \tag{10.29}$$

j fulfills the equation of continuity

$$\mathrm{d}_\rho j^\rho = 0 \tag{10.30}$$
$$-ic^2\hbar q\mathrm{d}_0\Big((\mathrm{d}^0\phi^*)\phi - \phi^*\mathrm{d}^0\phi\Big) = +ic^2\hbar q\mathrm{d}_j\Big((\mathrm{d}^j\phi^*)\phi - \phi^*\mathrm{d}^j\phi\Big) \ .$$

Integration over the volume V and application of Gauß' theorem results into

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V}\mathrm{d}^3x\, i\hbar q\Big(\frac{\mathrm{d}\phi^*}{\mathrm{d}t}\phi-\phi^*\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big)}_{Q(V)}=$$

$$=-\int\limits_{O(V)}\mathrm{d}f_j\ \underbrace{iqc^2\hbar q\Big((\mathrm{d}^j\phi^*)\phi-\phi^*\mathrm{d}^j\phi\Big)}_{j^j}\ . \tag{10.31}$$

In the integral Q we recognize – apart from constant factors – the term (10.4). As probability density, this term is useless, but as charge density it is acceptable, because a charge very well may be negative. Thus a complex Klein-Gordon field may have a positive or a negative conserved charge.

If the volume integral is extended over the complete normalization volume Ω, then the right side of (10.31) is zero, because ϕ and ϕ^* are zero on the normalization volume's surface. If in addition

$$\pi(x)\overset{(10.14a)}{=}\hbar^2\dot{\phi}^*(x)\qquad\qquad \pi^*(x)\overset{(10.14b)}{=}\hbar^2\dot{\phi}(x)$$

is inserted, then the total conserved charge of the Klein-Gordon field can be written in the form

$$\boxed{Q=\int\limits_{\Omega}\mathrm{d}^3x\,\frac{iq}{\hbar}\Big(\pi\phi-\phi^*\pi^*\Big)}\ . \tag{10.32}$$

10.5 The Real Klein-Gordon-Field

Compared to the vector field of electromagnetic interaction, or to the spinor field of Dirac theory, the charged Klein-Gordon field already is easy to handle. Even simpler is the formal treatment of the real Klein-Gordon field. It is defined by it's Lagrangian

$$\boxed{\mathcal{L}=\frac{c^2\hbar^2}{2}(\mathrm{d}_\mu\phi)\mathrm{d}^\mu\phi-\frac{1}{2}m^2c^4\phi^2}\ . \tag{10.33}$$

Like the charged KG-field, the uncharged KG-field by definition has the dimension (10.12) = (energy · volume)$^{-1/2}$. Due to this definition it's ensured, that the Lagrangian and the Hamiltonian will have the dimension energy/volume.

Variation of $\mathcal{L}$ with respect to ϕ leads to the field equation

$$\begin{aligned} \mathrm{d}_\mu \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} &\overset{(3.37b)}{=} 0 \\ c^2\hbar^2 \mathrm{d}_\mu \mathrm{d}^\mu \phi + m^2 c^4 \phi &= 0 \ , \end{aligned} \tag{10.34}$$

which is identical to (10.11a).

The canonically conjugate momentum density π of the real KG-field is

$$\pi(x) \overset{(3.57)}{=} \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = c\hbar^2 \mathrm{d}^0 \phi \ , \tag{10.35}$$

and the components of it's energy-density-stress tensor are

$$\begin{aligned} \mathcal{T}^{\rho\sigma} &\overset{(4.32)}{=} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho \phi)} \mathrm{d}^\sigma \phi - g^{\rho\sigma} \mathcal{L} = \\ &= c^2\hbar^2 (\mathrm{d}^\rho \phi)\mathrm{d}^\sigma \phi - g^{\rho\sigma} \frac{c^2\hbar^2}{2} (\mathrm{d}_\mu \phi)\mathrm{d}^\mu \phi + g^{\rho\sigma} \frac{1}{2} m^2 c^4 \phi^2 \ . \end{aligned} \tag{10.36}$$

In particular, it's energy density (Hamiltonian) is

$$\begin{aligned} \mathcal{H} &\overset{(4.34)}{=} \mathcal{T}^{00} = \frac{c^2\hbar^2}{2} (\mathrm{d}^0 \phi)\mathrm{d}^0 \phi - \frac{c^2\hbar^2}{2} (\mathrm{d}_j \phi)\mathrm{d}^j \phi + \frac{1}{2} m^2 c^4 \phi^2 \\ &= \frac{c^2\hbar^2}{2} (\mathrm{d}^\mu \phi)\mathrm{d}^\mu \phi + \frac{1}{2} m^2 c^4 \phi^2 \ , \end{aligned} \tag{10.37}$$

it's physical momentum density is

$$\mathcal{P}^j \overset{(4.34)}{=} \frac{1}{c} \mathcal{T}^{0j} = c\hbar^2 (\mathrm{d}^0 \phi)\mathrm{d}^j \phi \overset{(10.35)}{=} \pi(x)\mathrm{d}^j \phi \ , \tag{10.38}$$

and it's mean pressure P is

$$
\begin{aligned}
\mathrm{P} &\overset{(4.45a)}{=} \frac{1}{3}\Big(\mathcal{S}^{P11} + \mathcal{S}^{P22} + \mathcal{S}^{P33}\Big) = \frac{1}{3}\,\mathcal{T}^{jj} \\
&= \frac{c^2\hbar^2}{3}(\mathrm{d}^j\phi)\mathrm{d}^j\phi - \frac{g^{jj}}{3}\frac{c^2\hbar^2}{2}(\mathrm{d}_\mu\phi)\mathrm{d}^\mu\phi + \frac{g^{jj}}{6}\,m^2c^4\phi^2 \\
&= +\frac{c^2\hbar^2}{2}(\mathrm{d}_0\phi)\mathrm{d}_0\phi - \frac{c^2\hbar^2}{6}(\mathrm{d}_j\phi)\mathrm{d}_j\phi - \frac{1}{2}\,m^2c^4\phi^2\ . \qquad (10.39)
\end{aligned}
$$

11 Discrete Symmetries

11.1 The Extended Lorentz Group

The Lorentz group $\{\ell\}$ is a continuous group. Each if it's elements can be expanded in a Taylor series around the group's unit element. This means that any of it's elements can be represented by the product of infinitely many infinitely small elements.

In addition, there are two discrete changes of the space-time coordinates, which are as well leaving invariant the product $A_\nu B^\nu$: The inversion P of the three space coordinates x^1, x^2, x^3, and the inversion T of the time coordinate x^0. The coordinate inversion P is called parity transformation. The set union $\{\{\ell\}, T, P\}$ contains the totality of space-time coordinate transformations, which are leaving invariant the product $A_\nu B^\nu$, because the inversion of a single coordinate axis or of arbitrary combinations of coordinate axes can be constituted by appropriate products of P, T, and the elements of $\{\ell\}$.

The Lorentz group $\{\ell\}$ is more specifically called *proper orthochrone* Lorentz group $\{\ell_+^\uparrow\}$. The arrow $^\uparrow$ and the word orthochron are indicating, that the Lorentz group's elements don't rotate the time axis out of the light cone. The plus sign $_+$ and the word proper are meaning, that all elements of all matrix representations of the Lorentz group have determinant $+1$. We will keep on calling the proper orthochrone Lorentz group $\{\ell_+^\uparrow\}$ simply Lorentz group $\{\ell\}$.

The *orthochrone improper* Lorentz set $\{\ell_-^\uparrow\}$ is the set $\{P\{\ell\}\}$ of all products $P\ell = \ell P$ of the parity transformation and of elements of the Lorentz group. $\{\ell_-^\uparrow\}$ is no group, because the product of two elements of $\{\ell_-^\uparrow\}$ is not an element of $\{\ell_-^\uparrow\}$, but an element of $\{\ell\}$. The set union $\{\{\ell\}, P\} = \{\{\ell\}, \{\ell_-^\uparrow\}\}$ is a group.

The *non-orthochrone improper* Lorentz set $\{\ell_-^\downarrow\}$ is the set $\{T\{\ell\}\}$ of all

products $T\ell = \ell T$ of time inversion and the elements of the Lorentz group. $\{\ell^{\downarrow}_{-}\}$ is not a group, because the product of two elements of $\{\ell^{\downarrow}_{-}\}$ is not an element of $\{\ell^{\downarrow}_{-}\}$, but an element of $\{\,\ell\,\}$. The set union $\{\{\,\ell\,\}, T\} = \{\{\,\ell\,\}, \{\ell^{\downarrow}_{-}\}\}$ is a group.

The *non-orthochrone proper* Lorentz set $\{\ell^{\downarrow}_{+}\}$ is the set $\{TP\{\,\ell\,\}\}$ of all products $TP\ell$ of time inversion and parity transformation and the elements of the Lorentz group. As T and P commute mutually and with all elements of the Lorentz group, the sequence of the factors does not matter. $\{\ell^{\downarrow}_{+}\}$ is not a group, because the product of two elements of $\{\ell^{\downarrow}_{+}\}$ is not an element of $\{\ell^{\downarrow}_{+}\}$, but an element of $\{\,\ell\,\}$. The set union $\{\{\,\ell\,\}, TP\} = \{\{\,\ell\,\}, \{\ell^{\downarrow}_{+}\}\}$ is a group.

The set union $\{\{\,\ell\,\}, T, P\} = \{\{\,\ell\,\}, \{\ell^{\uparrow}_{-}\}, \{\ell^{\downarrow}_{-}\}, \{\ell^{\downarrow}_{+}\}\}$ is called *extended Lorentz group*. Due to it's elements T and P, the extended Lorentz group is no continuous group. It's parameter manifold is not connected, but split into the four parameter manifolds of it's components $\{\,\ell\,\}$, $\{\ell^{\uparrow}_{-}\}$, $\{\ell^{\downarrow}_{-}\}$, and $\{\ell^{\downarrow}_{+}\}$. No point A in one of these four components can be connected with a point B in one of the three other components by a continuous path, which is nowhere leaving the parameter manifold. Each one of these four components of the manifold is twofold connected, as each one contains the parameter manifold of $\{\,\ell\,\}$, which is twofold connected.

11.2 Discrete Symmetries of the Dirac Field

Using the definition of the γ-matrices

$$\gamma^0 \overset{(8.15a)}{=} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^k \overset{(8.15)}{=} \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} , \tag{11.1}$$

the generators

$$S^{\sigma\tau} \overset{(8.43)}{=} \frac{i\hbar}{4}[\gamma^\sigma, \gamma^\tau] , \tag{11.2}$$

which were defined in theorem (8.43), can be written in 2×2 block format:

$$S^{k0} = \frac{i\hbar}{4}\left(\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\right)$$
$$= \frac{i\hbar}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{11.3a}$$

$$S^{kl} = \frac{i\hbar}{4}\left(\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^l \\ -\sigma^l & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}\right)$$
$$= -\frac{i\hbar}{4}\begin{pmatrix} [\sigma^k, \sigma^l] & 0 \\ 0 & [\sigma^k, \sigma^l] \end{pmatrix} \overset{(6.24)}{=} \epsilon^{klm}\frac{\hbar}{2}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \tag{11.3b}$$

Using furthermore the single-indexed notation

$$\Theta_k S_R^k + \eta_k S_B^k \equiv \frac{1}{2}\Omega_{\sigma\tau} S^{\sigma\tau} \tag{11.4}$$

with rotation generators S_R and boost generators S_B, the transformations (8.30) of the Dirac field under rotations of the space-time coordinates become the form

$$\psi'(x') = D\psi(\Lambda^{-1}x)$$
$$= \exp\left\{\frac{i}{\hbar}\left(\Theta_k\frac{\hbar}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} + \eta_k\frac{i\hbar}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}\right)\right\}\psi(\Lambda^{-1}x)\ . \tag{11.5}$$

$\{D\}$ is nothing other than the direct sum of the left- and right-handed two-dimensional representations (6.78) of the group $\{\mathcal{B}\}$. Consequently the four-dimensional Dirac spinors

$$\psi(x) = \begin{pmatrix} L(x) \\ R(x) \end{pmatrix} \tag{11.6}$$

are nothing other than the direct sum of the two-dimensional Weyl spinors L and R, which are the bases of the left- and right-handed representations (6.78).

Why then do we bother at all with the seemingly reducible representations (11.5), and don't decompose them into the two irreducible representations

(6.78)? We stick to (11.5) because the Dirac field is not only invariant under the continuous transformations of the Lorentz group, but also under the discrete symmetries P and T, and in addition under a further discrete transformation, namely the charge conjugation C. As a representation of the group $\{\mathcal{B}\}$, the group $\{D = (11.5)\}$ would be reducible. But it is *irreducible* as a representation of $\{\{\mathcal{B}\}, P, T, C\}$ – the covering group of the extended Lorentz group plus C. We will see in sections 11.2.1 and 11.2.3, that the right- and left-handed spinor components are coupled by the parity transformation P and the charge conjugation C.

When we constructed the Dirac equation, we never demanded this equation to be not only lorentz-invariant but also invariant under parity transformation and under charge conjugation. But we demanded in section 8.1 that the representation must be compatible with the energy-momentum equation $\sqrt{E^2 - (c\boldsymbol{p})^2} \overset{(8.2)}{=} mc^2$ of Special Relativity theory. Due to $E^2 = (-E)^2$ and $\boldsymbol{p}^2 = (-\boldsymbol{p})^2$ this equation is invariant under time inversion and under parity transformation, and guided us to a theory which is not only lorentz-invariant, but also invariant under these discrete transformations.

11.2.1 Parity

We are looking for a representation of the parity transformation on the basis of four-vectors in time-position space, and we are looking for a representation of the parity transformation on the basis of four-component spinors. The four-dimensional representation

$$\Lambda_P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{11.7}$$

on the basis of four-vectors in time-position space is forced by definition. The representation D_P on the basis of spinors follows by computation, using the equation

$$D_P^{-1} \gamma^\nu D_P \overset{(8.32)}{=} \Lambda^\nu_{P\mu} \gamma^\mu \,. \tag{11.8}$$

From $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \overset{(8.9)}{=} 2g^{\mu\nu}\mathbb{1}$ follows $\gamma^0\gamma^0\gamma^0 = +\gamma^0$ and $\gamma^0\gamma^k\gamma^0 = -\gamma^k$. Therefore

$$D_P \equiv e^{i\varphi}\gamma^0 \overset{(8.15a)}{=} e^{i\varphi}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{11.9}$$

with arbitrary phase factor φ is a representation of the parity transformation on the basis of Dirac spinors, which is a solution of equation (11.8).

We apply the elements of this representation of P onto the Dirac field $\psi(x)$. With $\Lambda_P^{-1} = \Lambda_P$ follows

$$\begin{aligned} P\Big(\psi(x)\Big) = \psi'(x') = D_P\psi(\Lambda_P^{-1}x) \; = \\ = \begin{pmatrix} L_1'(x') \\ L_2'(x') \\ R_1'(x') \\ R_2'(x') \end{pmatrix} = e^{i\varphi}\begin{pmatrix} R_1(x^0,-\boldsymbol{x}) \\ R_2(x^0,-\boldsymbol{x}) \\ L_1(x^0,-\boldsymbol{x}) \\ L_2(x^0,-\boldsymbol{x}) \end{pmatrix} . \end{aligned} \tag{11.10}$$

D_P permutes the spinor components R and the spinor components L. Therefore the representation $\{D, D_P, D_T, D_C\}$ of the group $\{\{\mathcal{B}\}, P, T, C\}$ can not be reduced into two two-dimensional representations.

11.2.2 Time Inversion

We are looking for a representation of time inversion on the basis of four-vectors in time-position space, and we are looking for a representation of time inversion on the basis of four-component spinors. Regarding the vectors in time-position space, there is a complication: While the matrix

$$\Lambda_T \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.11}$$

is transforming x and d_ν correctly, a different time-inversion transformation is needed for the gauge field A:

$$T(A) = A' = \begin{pmatrix} \phi'/c \\ \boldsymbol{A}' \end{pmatrix} = \begin{pmatrix} +\phi/c \\ -\boldsymbol{A} \end{pmatrix} \tag{11.12}$$

This can be seen e.g. with the help of Maxwell's equation

$$\boldsymbol{E} \overset{(4.128a)}{=} -\boldsymbol{\nabla}\Phi - \mathrm{d}_t \boldsymbol{A} \; . \tag{11.13}$$

$P(\boldsymbol{E}) = -\boldsymbol{E}$ holds under space inversion P. From $P(\boldsymbol{\nabla}) = -\boldsymbol{\nabla}$ and $P(\mathrm{d}_t) = +\mathrm{d}_t$ follows $P(\phi) = +\phi$ and $P(\boldsymbol{A}) = -\boldsymbol{A}$. But $T(\boldsymbol{E}) = +\boldsymbol{E}$ under time inversion T. And from $T(\boldsymbol{\nabla}) = +\boldsymbol{\nabla}$ and $T(\mathrm{d}_t) = -\mathrm{d}_t$ follows $T(\phi) = +\phi$ and $T(\boldsymbol{A}) = -\boldsymbol{A}$. Thus parity transformation and time inversion are acting differently onto x^ν and d_ν, but identically onto the gauge field A! Due to the opposite sign changes of d_ν and A_ν under time inversion, there exists – different from parity transformation – no matrix operator, which can represent this transformation.

For this reason time inversion T is defined such, that a solution ψ of the Dirac equation

$$\Big(i\hbar c\gamma^\nu(\mathrm{d}_\nu + \frac{i}{\hbar}qA_\nu) - mc^2\Big)\psi \overset{(8.26a)}{=} 0 \tag{11.14a}$$

is transformed into a solution ψ^* of its complex-conjugated equation

$$\Big(- i\hbar c\gamma^{\nu *}(\mathrm{d}_\nu - \frac{i}{\hbar}qA_\nu) - mc^2\Big)\psi^* = 0 \; . \tag{11.14b}$$

This is achieved by the following definitions:

$$T\Big(\gamma^\nu\Big) \equiv -D_T^{-1}\gamma^{\nu *}D_T \tag{11.15a}$$

$$T\Big(V\Big) \equiv \Lambda_T V \quad \text{mit } \Lambda_T = (11.11) \tag{11.15b}$$

$$T\Big(q\Big) \equiv -q \tag{11.15c}$$

$$T\Big(\psi(x)\Big) \equiv D_T\psi^*(\Lambda_T^{-1}x) \tag{11.15d}$$

V is an arbitrary Lorentz vector. By this definition of T, the sign changes

of d_ν and A_ν in (11.14) are opposite – exactly as needed. Like charge conjugation C, which will be treated in the next section, also time inversion T changes the sign of the charge parameter q. But besides that, T and C are different transformations. D_T is a representation of time inversion on the basis of spinors. It can be found by means of (8.32):

$$-D_T^{-1}\gamma^{\nu *}D_T \overset{(8.32)}{=} \Lambda^\nu_{T\mu}\gamma^\mu \overset{(11.11)}{=} -g_{\nu\mu}\gamma^\mu \tag{11.16}$$

Note that $-\gamma^{\nu *}$ from (11.14b) shows up on the equation's left side, while on the right side γ^ν from (11.14a) is used. The left side of (8.32) is describing the transformation of γ-matrices in spinor space, and therefore (11.15a) must be inserted. But the right side of (8.32) is describing the transformation of the four γ-matrices, if they are formally considered as the components of a Lorentz vector, and therefore (11.15b) must be inserted. This equation is solved by

$$\begin{aligned} D_T &= e^{i\chi}\gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\ &= e^{i\chi}\begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix} = ie^{i\chi}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \end{aligned} \tag{11.17}$$

with an arbitrary phase factor χ. This can be proved by direct insertion, using the definitions of γ^ν in (8.15), and

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \overset{(8.9)}{=} 2g^{\mu\nu}\mathbb{1} \implies (\gamma^\alpha)^{-1} = \gamma^\alpha \ . \tag{11.18}$$

Due to the structure of the matrix (11.17), the L- and R-components of the Dirac field $\psi(x)$ are *not* coupled by time inversion:

$$\begin{aligned} &T\Big(\psi(x)\Big) = \psi'(x') = D_T\psi^*(\Lambda_T^{-1}x) \ = \\ &= ie^{i\chi}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} L_1^*(\Lambda_T^{-1}x) \\ L_2^*(\Lambda_T^{-1}x) \\ R_1^*(\Lambda_T^{-1}x) \\ R_2^*(\Lambda_T^{-1}x) \end{pmatrix} = -e^{i\chi}\begin{pmatrix} -L_2^*(-x^0,\boldsymbol{x}) \\ +L_1^*(-x^0,\boldsymbol{x}) \\ -R_2^*(-x^0,\boldsymbol{x}) \\ +R_1^*(-x^0,\boldsymbol{x}) \end{pmatrix} \end{aligned} \tag{11.19}$$

The procedure of time inversion seems rather artificial and cumbersome. Again we see that the notion "time" does not fit well into quantum theory.

11.2.3 Charge Conjugation

The name of this discontinuous transformation does only quite incompletely reflect it's properties. Charge conjugation by definition is a transformation, which changes solutions of the Dirac equation

$$(8.26a) = i\hbar c\gamma^\nu(\mathrm{d}_\nu + \frac{i}{\hbar}qA_\nu)\psi - mc^2\psi = 0 \tag{11.20a}$$

into solutions of it's adjoint transposed equation

$$(8.26b)^\sim = i\hbar c(-\gamma^{\nu\sim})(\mathrm{d}_\nu - \frac{i}{\hbar}qA_\nu)\,\overline{\psi}^\sim - mc^2\,\overline{\psi}^\sim = 0\ . \tag{11.20b}$$

In other contexts, "adjoint transposed" would have the same meaning as "complex conjugate", because normally "adjoint" is the same as "complex conjugate and transposed". But in Dirac theory, the adjoint spinor is not only complex conjugate and transposed, but in addition multiplied (from right) by the matrix γ^0. Therefore $\overline{\psi}^\sim \neq \psi^*$.

With reference to (11.20), charge conjugation C is defined as follows:

$$C\Big(\gamma^\nu\Big) \equiv -D_C^{-1}\gamma^{\nu\sim}D_C \tag{11.21a}$$

$$C\Big(q\Big) \equiv -q \tag{11.21b}$$

$$C\Big(V\Big) \equiv \Lambda_C V(\Lambda_C^{-1}x) \overset{(11.22)}{=} V \tag{11.21c}$$

$$C\Big(\psi(x)\Big) \equiv D_C\overline{\psi}^\sim(\Lambda_C^{-1}x)\ . \tag{11.21d}$$

V is an arbitrary Lorentz vector. Λ_C is a representation of C on the basis of Lorentz vectors, which will be defined immediately. D_C is a representation of C on the basis of spinors.

Charge conjugation C is a symmetry of the Dirac field, if $D_C\overline{\psi}^\sim$ is describing an observable state, provided that ψ is describing an observable

state. $D_C\overline{\psi}^{\sim}$ is describing the antiparticles of those particles, which are described by ψ. (Here – as everywhere in this book – we are using the notions "field", "wave", and "particle" synonymously.) Therefore in the quantized theory, charge conjugation is often called particle-antiparticle conjugation.

As representation of charge conjugation C on the basis of Lorentz vectors we choose the trivial representation

$$\Lambda_C V \equiv \mathbb{1} V = V \ . \tag{11.22}$$

Then in particular for the four-dimensional time-position vector

$$\Lambda_C x = x \ .$$

Due to it's definition, charge conjugation results into modifications in spinor space, and it changes the charge parameter q. Different from P and T, C does not affect the space-time coordinates of the fields.

Now we determine D_C by means of (8.32):

$$-D_C^{-1}\gamma^{\nu\sim}D_C \overset{(8.32)}{=} \Lambda^{\nu}_{C\mu}\gamma^{\mu} \overset{(11.22)}{=} \gamma^{\nu} \tag{11.23}$$

Note that $-\gamma^{\nu\sim}$ from (11.20b) shows up on the equation's left side, while on the right side γ^{ν} from (11.20a) is used. The left side of (8.32) is describing the transformation of γ-matrices in spinor space, and therefore (11.21a) must be inserted. But the right side of (8.32) is describing the transformation of the four γ-matrices, if they are formally considered as the components of a Lorentz vector, and therefore (11.21c) must be inserted. This equation is solved by

$$\begin{aligned} D_C &\equiv ie^{i\zeta}\gamma^0\gamma^2 = \\ &= ie^{i\zeta}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = ie^{i\chi}\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \ . \end{aligned} \tag{11.24}$$

This is proved by direct computation of the four products, using the definition of γ^{ν} in (8.15), and equation (11.18). ζ is an arbitrary phase factor. Compare

the results (11.24) and (11.17)! D_C differs from D_T only by one sign (resp. 2 signs, if the two components of σ^2 are counted separately). But because under charge conjugation in addition the transition $\psi \to \overline{\psi}^{\sim}$ is made, charge conjugation does – different from time inversion – interchange the L- and R-components of spinors:

$$C\Big(\psi(x)\Big) = \psi'(x') = D_C \overline{\psi}^{\sim}(\Lambda_C^{-1} x) =$$
$$= ie^{i\zeta}\gamma^0\gamma^2 \begin{pmatrix} \psi_3^*(x) \\ \psi_4^*(x) \\ \psi_1^*(x) \\ \psi_2^*(x) \end{pmatrix} = e^{i\zeta} \begin{pmatrix} -\psi_4^*(x) \\ +\psi_3^*(x) \\ +\psi_2^*(x) \\ -\psi_1^*(x) \end{pmatrix} = e^{i\zeta} \begin{pmatrix} -R_2^*(x) \\ +R_1^*(x) \\ +L_2^*(x) \\ -L_1^*(x) \end{pmatrix}$$

Charge conjugation interchanges right- and left-handed spinor components, interchanges in addition the both right-handed components mutually, interchanges the both left-handed components mutually, changes one sign each of the right- and left-handed components, and takes the complex-conjugated.

Besides the symmetry under parity transformation, symmetry under charge conjugation is the second (and last) cause, why the four-dimensional representation $\{D, D_P, D_T, D_C\}$ of the group $\{\{\mathcal{B}\}, P, T, C\}$ can not be reduced into two two-dimensional representations.

11.3 Discrete Symmetries of the Weyl Field

The four-dimensional Dirac-representation of the group $\{\mathcal{B}\}$ is the direct sum of the left- and right-handed two-dimensional Weyl-representation (6.78). To phrase it conversely: The four-dimensional Dirac-representation of the group $\{\mathcal{B}\}$ can be reduced into the two two-dimensional Weyl-representations, *if* it does not in addition contain the discrete symmetries of parity and/or charge conjugation. As these both symmetries interchange the right- and left-handed components of Dirac spinors, they make the four-dimensional representation irreducible, and thus are no symmetries of the Weyl-representations. Only the representation of time inversion on the basis of four-dimensional Dirac spinors

$$T\Big(\psi(x)\Big) = D_T\psi^*(\Lambda_T^{-1}x) \overset{(11.19)}{=} ie^{i\chi}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\psi^*(\Lambda_T^{-1}x) \tag{11.25}$$

is reducible, and thus also a discrete symmetry of the left- and right-handed Weyl fields:

$$D_T = ie^{i\chi}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix} = ie^{i\chi}(\sigma^2 \oplus \sigma^2) \tag{11.26}$$

11.4 Discrete Symmetries of the Klein-Gordon Field

The Klein-Gordon field's Lagrangian

$$\mathcal{L} \overset{(10.10)}{=} c^2\hbar^2(\mathrm{d}_\mu\psi^*)\mathrm{d}^\mu\psi - m^2c^4\psi^*\psi \tag{11.27}$$

is due to it's quadratic structure and it's invariance under the change to the complex-conjugated equation obviously invariant under time inversion, under parity transformation, and under charge conjugation. As we derived the Klein-Gordon equation from the Dirac equation, it's no surprise that both are not only lorentz-invariant, but also feature the same discrete symmetries.

12 Inhomogeneous Fields

12.1 Greensfunction of the Klein-Gordon Field

The homogeneous Klein-Gordon equation

$$(\hbar^2 c^2 \mathrm{d}^\mu \mathrm{d}_\mu + m^2 c^4)\psi^{(0)}(x) \overset{(10.2)}{=} 0 \tag{12.1}$$

describes a free, not interacting field. If there are interactions, then this equation changes to an inhomogeneous equation, in which the interactions show up as source-terms $j(x)$:

$$(\hbar^2 c^2 \mathrm{d}^\mu \mathrm{d}_\mu + m^2 c^4)\psi(x) = j(x) \tag{12.2}$$

For the solutions of this inhomogeneous equation we make the ansatz

$$\psi(x) = \psi^{(0)}(x) + \int\limits_{\Omega} \mathrm{d}^3 y \int\limits_{-\infty}^{+\infty} \mathrm{d}y^0 \, \frac{i}{\hbar c} \, G(x,y) j(y) \ . \tag{12.3}$$

$\psi^{(0)}(x)$ is a solution of the homogeneous equation (12.1). The source term j creates at the space-time point y an additional wave, which spreads – as described by the function $\frac{i}{\hbar c} G(x,y)$ – from y to x. According to Huygens'principle, $\frac{i}{\hbar c} G(x,y) j(y)$ can be imagined as a wave, which emanates from the source point y. At space-time point x, it combines with the free field and with the waves, which emanate from all other source points, to the inhomogeneous field $\psi(x)$.

The sum in (12.3) is running not only over the space-time points y in the backwards lightcone of x, but over all position-space points in the normalization volume Ω, and over all time points in the past and the future. There are two reasons for this surprising procedure: Firstly we will interpret

waves, which are spreading backwards in time, as the waves of anti-fields, which are spreading forward in time. Secondly we are already anticipating the intended quantization. We will see, that space-like interactions of fields and antifields outside the forward- and backwards-lightcone are mutually compensating exactly to zero. While antifields can be defined very well as classical fields, the compensation of space-like virtual fields is comprehensible only in the framework of quantum theory.

$G(x,y)$ is called Greens-function[1]. In field theory, the name propagator[2] is conventional. Note: Instead of the factor $i/\hbar c$, some authors factor out $-i/\hbar c$, and some authors factor out nothing at all of the Greens-function.

Inserting the Ansatz (12.3) into the inhomogeneous equation (12.2) one finds

$$\begin{aligned} 0 + \int_\Omega \mathrm{d}^3y \int_{-\infty}^{+\infty} \mathrm{d}y^0 \, (\hbar^2c^2 \mathrm{d}^\mu \mathrm{d}_\mu + m^2c^4) \, \frac{i}{\hbar c} \, G(x,y) j(y) \; = \\ = j(x) = \int_\Omega \mathrm{d}^3y \int_{-\infty}^{+\infty} \mathrm{d}y^0 \, \delta^{(4)}(x-y) j(y) \\ \Longrightarrow \quad (\hbar^2c^2 \mathrm{d}^\mu \mathrm{d}_\mu + m^2c^4) \, \frac{i}{\hbar c} \, G(x,y) = \delta^{(4)}(x-y) \; . \end{aligned} \tag{12.4}$$

$\frac{i}{\hbar c} G(x,y)$ is a solution of the inhomogeneous differential equation, if the inhomogeneous source $j(x)$ is condensed into an infinitely short, infinitely strong "test-stroke" in form of a delta function.

If the investigated system is invariant under translations, then $G(x,y)$ only depends on the difference $(x-y)$, and the Fourier-transformation

$$G(x-y) \overset{(7.15)}{=} \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \, \widetilde{G}(k) \exp\{-ik(x-y)\} \tag{12.5}$$

can be performed. Inserting (12.5) and

[1] named in honor of it's explorer George Green (1793 - 1841)
[2] latin: propagare = to spread, to propagate

$$\delta^{(4)}(x-y) \overset{(7.16a)}{=} \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \exp\{-ik(x-y)\} \tag{12.6}$$

into (12.4), one gets

$$(-\hbar^2c^2k^2 + m^2c^4)\,\frac{i}{\hbar c}\,\widetilde{G}(k) = 1$$

$$\boxed{\widetilde{G}(k) = \frac{+i}{\hbar c\Big(k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)}} \tag{12.7a}$$

$$\widetilde{G}(k) = \frac{+i}{\hbar c\Big(k^0 + \frac{\omega_{\boldsymbol{k}}}{c} - i\epsilon\Big)\Big(k^0 - \frac{\omega_{\boldsymbol{k}}}{c} + i\epsilon\Big)}\,. \tag{12.7b}$$

Here we used

$$k^2 - \frac{m^2c^2}{\hbar^2} = (k^0)^2 - \Big(\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}\Big) = \Big(k^0 + \frac{\omega_{\boldsymbol{k}}}{c}\Big)\Big(k^0 - \frac{\omega_{\boldsymbol{k}}}{c}\Big)$$

$$\frac{\omega_{\boldsymbol{k}}}{c} \overset{(7.18)}{=} +\sqrt{\boldsymbol{k}^2 + \frac{m^2c^2}{\hbar^2}}\,. \tag{12.8}$$

Only "on mass shell" k^0 is the null-component of a four-vector with the relativistically invariant square $k^2 = (k^0)^2 - \boldsymbol{k}^2 = m^2c^2/\hbar^2$. But under the integral (12.5), the integration variable k^0 is a parameter, which is independent of $\boldsymbol{k}$, and assumes all values in the continuum inbetween $-\infty$ and $+\infty$. $\widetilde{G}(k)$ would have poles of first order at $k = \mp mc/\hbar$ resp. $k^0 = \mp\omega_{\boldsymbol{k}}/c$, if not the infinitesimal small imaginary term $-i\epsilon$ had been added to $\omega_{\boldsymbol{k}}/c$:

$$\omega_{\boldsymbol{k}}/c \;\to\; \omega_{\boldsymbol{k}}/c - i\epsilon \quad \text{with} \quad \begin{cases} \epsilon \in \mathbb{R}\,,\ \epsilon > 0 \\ \dfrac{\epsilon}{\omega_{\boldsymbol{k}}/c} \le \dfrac{\epsilon}{mc/\hbar} \ll 1 \end{cases} \tag{12.9a}$$

This shifts the poles, which are indicated in fig. 12.1 as red points, from the real axis into the complex plane. There are overall four different ways to shift the two poles into the upper or lower complex k^0-plane. The advantage

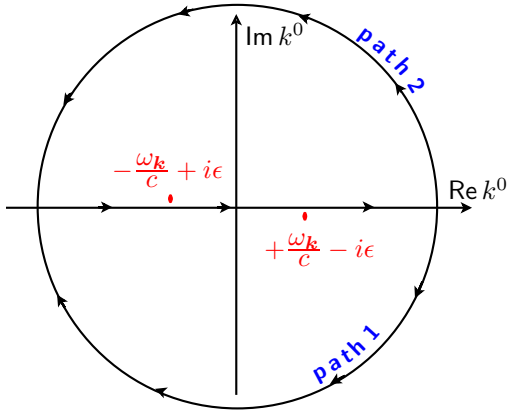


Fig. 12.1 : Poles and integration paths

of (12.9), which leads to the Feynman-propagator, will be explained once the integration is done. The infinitesimal parameter ϵ' is defined by

$$\begin{aligned}&\Big(k^0+(\omega_{\boldsymbol{k}}/c-i\epsilon)\Big)\Big(k^0-(\omega_{\boldsymbol{k}}/c-i\epsilon)\Big)\;=\\&=k^2-m^2\frac{c^2}{\hbar^2}+i\epsilon\,\frac{2\omega_{\boldsymbol{k}}}{c}+\mathcal{O}(\epsilon^2)\equiv k^2-m^2\frac{c^2}{\hbar^2}+i\epsilon'\\&\quad\text{with }\epsilon'\in\mathbb{R}\;,\;0<\epsilon'\ll m^2\frac{c^2}{\hbar^2}\;.\end{aligned}\tag{12.9b}$$

In case $x^0 > y^0$, the integration path of (12.5) can be closed in the lower complex half-plane, without changing the integral's value (path 1 in fig. 12.1), and the integral can be computed by means of the residue theorem[3]. The integral over the bottom half circle is zero due to the term $(k^0)^2$ in the denominator for large real part of k^0, and due to the exponential function in the nominator for large negative imaginary part of k^0. The pole at $k^0 = +\omega_{\boldsymbol{k}}/c - i\epsilon$ is enclosed by the integration path. Thus one finds, neglecting terms $\mathcal{O}(\epsilon^2)$, as solution of (12.5) the Greens function

[3] A comprehensive introduction, which is tailored to the need of physicists, to this important mathematical tool can be found in [37].

$$
\begin{aligned}
G(x-y) &\overset{x^0\geq y^0}{=} \frac{i}{\Omega 2\pi}\sum_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot \\
&\quad \cdot \oint_{\circlearrowleft} \mathrm{d}k^0 \, \frac{\exp\{-ik^0(x^0-y^0)\}}{\hbar c\Big(k^0+\frac{\omega_{\boldsymbol{k}}}{c}-i\epsilon\Big)\Big(k^0-\frac{\omega_{\boldsymbol{k}}}{c}+i\epsilon\Big)} \\
&= \frac{+1}{\Omega}\sum_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot \\
&\quad \cdot \frac{\exp\{-i(\omega_{\boldsymbol{k}}/c-i\epsilon)(x^0-y^0)\}}{2\hbar\omega_{\boldsymbol{k}}-2i\epsilon} \ . \qquad (12.10)
\end{aligned}
$$

There was a change of sign because of the clockwise integration path. Now ϵ is negligible versus $\omega_{\boldsymbol{k}}/c$, and one gets the Greens function

$$
G(x-y) \overset{x^0\geq y^0}{=} \sum_{\boldsymbol{k}} \frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \equiv G^a(x-y) \ . \qquad (12.11a)
$$

In case $x^0 < y^0$ the integration path is closed in the upper complex halfplane (path 2 in fig. 12.1):

$$
\begin{aligned}
G(x-y) &\overset{x^0\leq y^0}{=} \frac{i}{\Omega 2\pi}\sum_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot \\
&\quad \cdot \oint_{\circlearrowleft} \mathrm{d}k^0 \, \frac{\exp\{-ik^0(x^0-y^0)\}}{\hbar c\Big(k^0+\frac{\omega_{\boldsymbol{k}}}{c}-i\epsilon\Big)\Big(k^0-\frac{\omega_{\boldsymbol{k}}}{c}+i\epsilon\Big)} \\
&= \frac{-1}{\Omega}\sum_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot \\
&\quad \cdot \frac{\exp\{-i(-\omega_{\boldsymbol{k}}/c+i\epsilon)(x^0-y^0)\}}{-2\hbar\omega_{\boldsymbol{k}}+2i\epsilon}
\end{aligned}
$$

As the summation is done symmetrically over all positive and negative wave numbers $\boldsymbol{k}$, and because of $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, we may change $\boldsymbol{k}$ versus $-\boldsymbol{k}$. Neglecting ϵ we get

$$G(x-y) \overset{x^0 \leq y^0}{=} \sum_{\boldsymbol{k}} \frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \equiv G^b(y-x) \ . \tag{12.11b}$$

The sequence of x and y in the arguments of $G^a(x-y)$ and $G^b(y-x)$ is always arranged to the form $G^x(\text{later}-\text{earlier})$. There is no such rule for $G(x-y)$. Only as integration variable, k^0 is assuming all values $-\infty$ to $+\infty$ independent of $\boldsymbol{k}$. In (12.11), $k^0 = \omega_{\boldsymbol{k}}/c = +\sqrt{\boldsymbol{k}^2 + m^2c^2/\hbar^2} > 0$, and k^0 is fixed by $\boldsymbol{k}$.

In both propagators (12.11) the null-component of the exponential has the form

$$\exp\{-i \cdot (\underbrace{\text{frequency}}_{\geq 0} \cdot \underbrace{\text{time-interval}}_{> 0})\} \tag{12.12}$$

with positive frequency-timeinterval product. This is a result of the choice we made for the shift of the poles, which is sketched in figure 12.1. Any other of the three possible shifts of the poles into the complex plane would lead to minimum one residue with negative frequency-timeinterval product. Only solution with positive frequency-timeinterval product can be interpreted as fields or antifields with positive energy, which are moving forward in time.

Being sums of plane waves, $G^a(x-y)$ and $G^b(y-x)$ are solutions of the homogeneous Klein-Gordon equation (12.1), but no solutions of the inhomogeneous equation (12.4). But we derived (12.11) from (12.4)! The explanation is hidden in the boundary conditions $x^0 > y^0$ resp. $x^0 < y^0$, which are indicated in (12.11) above the $=$. These conditions must be integrated formally into the solutions, to make them solutions of the inhomogeneous equation (12.4). This can be achieved by means of the step function

$$\theta(x^0-y^0) \equiv \begin{cases} 1 & \text{if } x^0 > y^0 \\ 1/2 & \text{if } x^0 = y^0 \\ 0 & \text{if } x^0 < y^0 \end{cases} \tag{12.13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x^0}\theta(x^0-y^0) = -\frac{\mathrm{d}}{\mathrm{d}x^0}\theta(y^0-x^0) = \delta(x^0-y^0) \ . \tag{12.13b}$$

The derivative of the step function at $x^0 = y^0$ is the delta function. Using the step function, the two equations (12.11) can be combined to

$$G(x-y) \overset{x^0 \neq y^0}{=} \theta(x^0-y^0) \sum_{\boldsymbol{k}} \frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} + \\ + \theta(y^0-x^0) \sum_{\boldsymbol{k}} \frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \,. \tag{12.14}$$

To check whether this now is a solution of the inhomogeneous equation (12.4), we first insert the solution $(i/\hbar c)\theta(x^0-y^0)G^a(x-y)$ into (12.4). d_μ are derivatives to x^μ, under which y^μ must be regarded as a constant. Furthermore in (12.17) we will use

$$\mathrm{d}_\mu \theta(x^0-y^0) = \delta_{0\mu}\, \delta(x^0-y^0) \tag{12.15}$$

and the property

$$f(\tau)\,\frac{\mathrm{d}}{\mathrm{d}\tau}\delta(\tau) = -\delta(\tau)\,\frac{\mathrm{d}}{\mathrm{d}\tau}f(\tau) \tag{12.16}$$

of the delta function.

$$\begin{aligned}
&(\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu + m^2c^4)\frac{i}{\hbar c}\theta(x^0-y^0)G^a(x-y) = \\
&= i\hbar c\Big(\mathrm{d}^\mu\mathrm{d}_\mu\theta(x^0-y^0)\Big)G^a(x-y) \\
&\quad + 2i\hbar c\Big(\mathrm{d}^\mu\theta(x^0-y^0)\Big)\mathrm{d}_\mu G^a(x-y) \\
&\quad + \theta(x^0-y^0)\underbrace{(\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu + m^2c^4)\frac{i}{\hbar c}G^a(x-y)}_{0} = \\
&= -i\hbar c\delta(x^0-y^0)\,\mathrm{d}_0 G^a(x-y) + 2i\hbar c\delta(x^0-y^0)\,\mathrm{d}_0 G^a(x-y) \\
&= i\delta(x^0-y^0)\sum_{\boldsymbol{k}}\frac{(-i\hbar ck^0)\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}}
\end{aligned}$$

$$= +\frac{1}{2}\underbrace{\sum_{\boldsymbol{k}}\frac{\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}}{\Omega}}_{\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}\underbrace{\delta(x^0-y^0)\exp\{-ik^0(x_0-y_0)\}}_{\delta(x^0-y^0)}$$

$$= \frac{1}{2}\delta^{(4)}(x-y) \tag{12.17}$$

If $\theta(y^0-x^0)G^b(y-x)$ is inserted into (12.4) instead of $\theta(x^0-y^0)G^a(x-y)$, there is a change of sign due to the different signs of x in the exponential function, and a second change of sign due to (12.13b). Thus in total the result is identical. The sum of both solutions is identical to (12.4).

12.2 Greens function of the Dirac field

In the definition and computation of the Dirac field's Greens function we will proceed in close analogy to the Klein-Gordon field. The free, not interacting Dirac field $\psi^{(0)}(x)$ is a solution of the equation

$$(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\psi^{(0)}(x) \overset{(8.5)}{=} 0\ . \tag{12.18}$$

If interaction terms $j(x)$ show up on the right side of the equation

$$(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\psi(x) = j(x)\ , \tag{12.19}$$

then the field $\psi(x)$ can be described as the overlay of waves, which propagate from source terms $j(y)$ to the space-time point x, and combine with the free field $\psi^{(0)}(x)$ to the inhomogeneous field

$$\psi(x) = \psi^{(0)}(x) + \int_\Omega \mathrm{d}^3y \int_{-\infty}^{+\infty}\mathrm{d}y^0\,\frac{(-i)}{\hbar c}\,S(x,y)j(y)\ . \tag{12.20}$$

$S(x,y)$ is the Greens function of the Dirac field. As in case of the Klein-Gordon field, the summation in this equation is running not only over the points y in the backwards-lightcone of x, but over all position points in the normalization volume and over all time points in the past and the future.

The Greens function is a solution of the inhomogeneous Dirac equation, if the source $j(x)$ is condensed to a delta function:

$$(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\,\frac{(-i)}{\hbar c}\,S(x,y) = \delta^{(4)}(x-y) \tag{12.21}$$

If the investigated system is invariant under translations, $S(x,y)$ is depending only of the difference $(x-y)$, and the Fourier transformation

$$S(x-y) \overset{(7.15)}{=} \frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\widetilde{S}(k)\exp\{-ik(x-y)\} \tag{12.22}$$

is feasible. This equation, and the form (12.6) of the delta function, are inserted into the inhomogeneous equation (12.21):

$$(+\hbar c\gamma^\mu k_\mu - mc^2)\,\frac{(-i)}{\hbar c}\,\widetilde{S}(k) = \mathbb{1} \tag{12.23}$$

Because of

$$\gamma^\mu k_\mu \gamma^\nu k_\nu = \tfrac{1}{2}\gamma^\mu\gamma^\nu k_\mu k_\nu + \tfrac{1}{2}\gamma^\nu\gamma^\mu k_\nu k_\mu = \underbrace{\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}}_{\overset{(8.9)}{=}\,g^{\mu\nu}} k_\mu k_\nu = k^2$$

we have

$$\gamma^\mu k_\mu - m\tfrac{c}{\hbar} = \left(\gamma^\mu k_\mu - m\tfrac{c}{\hbar}\right)\frac{\left(\gamma^\nu k_\nu + m\frac{c}{\hbar}\right)}{\left(\gamma^\nu k_\nu + m\frac{c}{\hbar}\right)} = \frac{k^2 - m^2\frac{c^2}{\hbar^2}}{\left(\gamma^\nu k_\nu + m\frac{c}{\hbar}\right)}\,.$$

Using this, we get

$$\boxed{\widetilde{S}(k) = \frac{i(\gamma^\nu k_\nu + m\frac{c}{\hbar})}{k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'}}\,. \tag{12.24}$$

$\gamma^\nu k_\nu$ can be a diagonal spinor matrix only in case $k=0$, see (8.15). Thus the factor $\gamma^\nu k_\nu + \mathbb{1}m\frac{c}{\hbar}$ in the numerator is never zero. If we had not converted

the fraction, but had left the factor $\gamma^\mu k_\mu - \mathbb{1} m\frac{c}{\hbar}$ in the denominator, then it would not be easy to see, that it has zeros. But it has zeros, as $\widetilde{S}(k)$ *has* poles. Evidently the properties of terms with gamma-matrices in the denominator are hard to identify. Therefore it's a good idea, always to transfer all spinors and spinor-matrices into the numerator.

$\widetilde{S}(k)$ has two poles of first order at $k = \mp mc/\hbar$ resp. $k^0 = \mp\omega_{\boldsymbol{k}}/c$ because of

$$k^2 - m^2\tfrac{c^2}{\hbar^2} \overset{(12.8)}{=} \Big(k^0 + \frac{\omega_{\boldsymbol{k}}}{c}\Big)\Big(k^0 - \frac{\omega_{\boldsymbol{k}}}{c}\Big) \,.$$

To avoid the poles in the integration (12.22), the infinitesimal small imaginary terms $-i\epsilon$ resp. $+i\epsilon'$ have bee added to $\omega_{\boldsymbol{k}}/c$ resp. to k^2, compare (12.9). As shown in fig. 12.1 this shifts the pole at $k^0 = -\omega_{\boldsymbol{k}}/c$ from the real axis into the second quadrant, and the pole at $k^0 = +\omega_{\boldsymbol{k}}/c$ into the fourth quadrant of the complex k^0-plane. These shifts lead to the Feynman-propagator, whose frequency-timeinterval product is always positive, see (12.12).

In case $x^0 > y^0$, the integration path can be closed in the lower complex half-plane (path 1 in fig. 12.1) without changing the integral's value. Thus one finds the Greens function

$$\begin{aligned} S(x-y) &\overset{x^0>y^0}{=} \frac{i}{\Omega 2\pi}\sum_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot \\ &\qquad \cdot \oint_{\cup} \mathrm{d}k^0\, \frac{(\gamma^\nu k_\nu + m\frac{c}{\hbar})\exp\{-ik^0(x^0-y^0)\}}{(k^0 + \frac{\omega_{\boldsymbol{k}}}{c} - i\epsilon)(k^0 - \frac{\omega_{\boldsymbol{k}}}{c} + i\epsilon)} \\ &= \frac{+1}{\Omega}\sum_{\boldsymbol{k}} \exp\{-i(\omega_{\boldsymbol{k}}/c - i\epsilon)(x^0-y^0) + i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\cdot \\ &\qquad \cdot \frac{(\gamma^0(\omega_{\boldsymbol{k}}/c - i\epsilon) + \gamma^j k_j + m\frac{c}{\hbar})}{2\omega_{\boldsymbol{k}}/c - 2i\epsilon} \,. \end{aligned} \tag{12.25}$$

The integration variable assumes all values $-\infty \le k^0 \le +\infty$. After the integration, $k^0 = +\omega_{\boldsymbol{k}}/c$ is again the k^0-value on mass shell. Furthermore ϵ now can be neglected versus $\omega_{\boldsymbol{k}}/c$, and one gets the Greens function

$$S(x-y) \overset{x^0 \geq y^0}{=} \sum_{\boldsymbol{k}} \frac{(\gamma^\mu k_\mu + m\frac{c}{\hbar}) \exp\{-ik(x-y)\}}{\Omega 2\omega_{\boldsymbol{k}}/c} \,. \tag{12.26a}$$

In case $x^0 < y^0$, the integration path is closed in the upper complex half-plane (path 2 in fig. 12.1), resulting in the Greens function

$$S(x-y) \overset{x^0 \leq y^0}{=} \frac{-1}{\Omega} \sum_{\boldsymbol{k}} \exp\{+i(\omega_{\boldsymbol{k}}/c - i\epsilon)(x^0 - y^0) + i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \cdot \\ \cdot \frac{(-\gamma^0(\omega_{\boldsymbol{k}}/c - i\epsilon) + \gamma^j k_j + m\frac{c}{\hbar})}{-2\omega_{\boldsymbol{k}}/c + 2i\epsilon} \,.$$

As the sum is running symmetrically over all positive and negative wavenumbers $\boldsymbol{k}$, and because of $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, we may change $\boldsymbol{k}$ versus $-\boldsymbol{k}$. Neglecting ϵ one finds

$$S(x-y) \overset{x^0 \leq y^0}{=} \sum_{\boldsymbol{k}} \frac{(-\gamma^\mu k_\mu + m\frac{c}{\hbar}) \exp\{+ik(x-y)\}}{\Omega 2\omega_{\boldsymbol{k}}/c} \,. \tag{12.26b}$$

We keep $\hbar k_\mu = p_\mu$ in mind, multiply the numerators and denominators of the two equations (12.26) by $\hbar c$, and combine them by means of the step function (12.13) to the Feynman-propagator:

$$\begin{aligned} &S(x-y) \overset{x^0 \neq y^0}{=} \\ &= \theta(x^0-y^0) \sum_{\boldsymbol{k}} \frac{(+\gamma^\mu c p_\mu + mc^2) \exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} + \\ &+ \theta(y^0-x^0) \sum_{\boldsymbol{k}} \frac{(-\gamma^\mu c p_\mu + mc^2) \exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \end{aligned} \tag{12.27}$$

Note the different signs of $\pm\gamma$. The propagator is a 4×4 spinor matrix, and the term mc^2 is to be interpreted as $\mathbb{1}mc^2$. The same holds for the Fourier-transformed propagator (12.24).

12.3 Greens function of the gauge field $A(x)$

The gauge field's equation

$$\mathrm{d}_\nu F^{\nu\rho} \overset{(4.125)}{=} \mu_0 j^\rho \quad \text{with } \rho = 0,1,2,3 \ ,$$

which was derived in section 4.5, becomes with (4.114) for the source-free field ($j^\rho = 0$)

$$\mathrm{d}_\nu \mathrm{d}^\nu A^\rho - \mathrm{d}_\nu \mathrm{d}^\rho A^\nu = 0 \quad \text{with } \rho = 0,1,2,3 \ .$$

$A(x)$ is an analytical function. Therefore $\mathrm{d}_\nu \mathrm{d}^\rho A^\nu = \mathrm{d}^\rho \mathrm{d}_\nu A^\nu$. Because of the field's gauge invariance, the four-divergence $\mathrm{d}_\nu A^\nu$ may be chosen arbitrarily. (In section 17.1 we will explain this aspect of gauge invariance in detail.) In this book, we always choose the

$$\text{Lorentz gauge:} \quad \mathrm{d}_\nu A^\nu \equiv 0 \ , \tag{12.28}$$

by which the source-free field equation assumes the simple form

$$\mathrm{d}_\nu \mathrm{d}^\nu A^\rho = 0 \quad \text{with } \rho = 0,1,2,3 \ . \tag{12.29}$$

If interactions show up as source terms $\mu_0 j(x)$ on the right side of the equation

$$\mathrm{d}_\nu \mathrm{d}^\nu A^\rho(x) = \mu_0 j^\rho(x) \quad \text{with } \rho = 0,1,2,3 \ , \tag{12.30}$$

then the field $A(x)$ can be described as the overlay of waves, which propagate from the source terms $\mu_0 j(y)$ to the space-time point x, and add-up there with the free field $A^{(0)}(x)$ to the inhomogeneous field with the components

$$A^\rho(x) = A^{\rho(0)}(x) + \int_\Omega \mathrm{d}^3 y \int_{-\infty}^{+\infty} \mathrm{d}y^0 \, \frac{(-i)}{\mu_0 \hbar c} \, D^{\rho\sigma}(x,y) \, g_{\sigma\tau} \, \mu_0 j^\tau(y) \ . \tag{12.31}$$

The matrix $D(x,y)$, which has 4×4 space-time components, is the Greens function of the gauge field $A(x)$. We will explain in section 12.4, why $-i/\mu_0\hbar c$ was factored out of the Greens function.

Inserting the Ansatz (12.31) into the inhomogeneous equation (12.30), one finds

$$\begin{aligned} 0 + \int_\Omega \mathrm{d}^3y \int_{-\infty}^{+\infty} \mathrm{d}y^0\, \mathrm{d}_\nu \mathrm{d}^\nu \frac{(-i)}{\hbar c} D^{\rho\sigma}(x,y) g_{\sigma\tau} j^\tau(y) \;=\; \\ = \mu_0 j^\rho(x) = \int_\Omega \mathrm{d}^3y \int_{-\infty}^{+\infty} \mathrm{d}y^0\, \delta^{(4)}(x-y) \underbrace{g^\rho{}_\tau}_{g^{\rho\sigma}g_{\sigma\tau}} \mu_0\, j^\tau(y) \\ \Longrightarrow \quad \mathrm{d}_\nu \mathrm{d}^\nu \frac{(-i)}{\hbar c\mu_0} D^{\rho\sigma}(x,y) = \delta^{(4)}(x-y) g^{\rho\sigma}\ . \end{aligned} \tag{12.32}$$

If the system under investigation is invariant under translations, $D^{\rho\sigma}(x,y)$ depends only of the difference $(x-y)$, and the Fourier transformation

$$D^{\rho\sigma}(x-y) \overset{(7.15)}{=} \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \widetilde{D}^{\rho\sigma}(k) \exp\{-ik(x-y)\}$$

can be performed. Inserting this equation and (12.6) into (12.32), one gets

$$-k^2 \frac{(-i)}{\hbar c} \widetilde{D}^{\rho\sigma}(k) = g^{\rho\sigma}\mu_0$$

$$\boxed{\widetilde{D}^{\rho\sigma}(k) = \frac{-ig^{\rho\sigma}\mu_0\hbar c}{k^2 + i\epsilon'}} \tag{12.33a}$$

$$\widetilde{D}^{\rho\sigma}(k) = \frac{-ig^{\rho\sigma}\mu_0\hbar c}{\left(k^0 + \frac{\omega_{\boldsymbol{k}}}{c} - i\epsilon\right)\left(k^0 - \frac{\omega_{\boldsymbol{k}}}{c} + i\epsilon\right)}\ . \tag{12.33b}$$

For the mass-less gauge field $A(x)$

$$\frac{\omega_{\boldsymbol{k}}}{c} \overset{(7.18)}{=} + \sqrt{\boldsymbol{k}^2} = |\boldsymbol{k}| \;> 0\ .$$

As usual, a small imaginary term was inserted, to avoid the divergence at the poles $k^0 = \pm\omega_{\boldsymbol{k}}/c$. From comparison with the propagator of the Klein-Gordon field

$$\begin{aligned}\widetilde{G}(k) &\overset{(12.7)}{=} \frac{+i}{\hbar c\Big(k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)}\\ G(x-y) &\overset{(12.14)}{=} \theta(x^0-y^0)\sum_{\boldsymbol{k}}\frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} + \\ &\quad + \theta(y^0-x^0)\sum_{\boldsymbol{k}}\frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}}\ ,\end{aligned}$$

the propagator of the gauge field $A(x)$ in time-position-space can immediately be deduced:

$$\begin{aligned}D^{\rho\sigma}(x-y) &\overset{x^0\neq y^0}{=} \theta(x^0-y^0)(-g^{\rho\sigma}\mu_0\hbar^2c^2)\sum_{\boldsymbol{k}}\frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} + \\ &\quad + \theta(y^0-x^0)(-g^{\rho\sigma}\mu_0\hbar^2c^2)\sum_{\boldsymbol{k}}\frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}}\end{aligned} \tag{12.34}$$

$D(x-y)$ and $\widetilde{D}(k)$ are 4×4-matrices in time-position-space. All their off-diagonal elements are zero.

12.4 Units

The units of the Klein-Gordon field, the Dirac field, and it's gauge field, are according to our definitions

$$\begin{aligned}\Big[\psi_{\text{KG}}(x)\Big] &\overset{(10.12)}{=} \sqrt{\frac{1}{\text{energy}\cdot\text{volume}}}\\ \Big[\psi_{\text{Dirac}}(x)\Big] &= \sqrt{\frac{1}{\text{volume}}}\\ \Big[A(x)\Big] &\overset{(4.106)}{=} \frac{\text{momentum}}{\text{charge}}\ .\end{aligned}$$

The propagators were defined by means of the inhomogeneous field equations:

$$\begin{aligned}(\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu + m^2c^4)\psi_{\mathrm{KG}}(x) &\overset{(12.2)}{=} j(x)\\ (\hbar^2c^2\mathrm{d}^\mu\mathrm{d}_\mu + m^2c^4)\,\frac{i}{\hbar c}\,G(x,y) &\overset{(12.4)}{=} \delta^{(4)}(x-y)\end{aligned} \tag{12.35a}$$

$$\begin{aligned}(i\hbar c\gamma^\mu\mathrm{d}_\mu - mc^2)\psi_{\mathrm{Dirac}}(x) &\overset{(12.19)}{=} j(x)\\ (i\hbar c\gamma^\mu\mathrm{d}_\mu - mc^2)\,\frac{(-i)}{\hbar c}\,S(x,y) &\overset{(12.21)}{=} \delta^{(4)}(x-y)\end{aligned} \tag{12.35b}$$

$$\begin{aligned}\mathrm{d}_\nu\mathrm{d}^\nu A^\rho(x) &\overset{(12.30)}{=} \mu_0 j^\rho(x)\\ \mathrm{d}_\nu\mathrm{d}^\nu\frac{(-i)}{\hbar c\mu_0}D^{\rho\sigma}(x,y) &\overset{(12.32)}{=} \delta^{(4)}(x-y)g^{\rho\sigma}\end{aligned} \tag{12.35c}$$

Only the $\pm i$ is arbitrary in the out-factored terms. For the factors $\hbar c$ resp. $\hbar c\mu_0$ there is a cogent reason, which becomes clear from the Greens functions' units. Because of

$$\Big[\delta^{(4)}(x-y)\Big] = \frac{1}{\mathrm{length}^4}$$

their units are

$$\begin{aligned}\Big[G(x,y)\Big] &\overset{(12.35a)}{=} \frac{1}{\mathrm{energy}\cdot\mathrm{volume}} = \Big[\psi_{\mathrm{KG}}(x)\,\psi_{\mathrm{KG}}(y)\Big]\\ \Big[S(x,y)\Big] &\overset{(12.35b)}{=} \frac{1}{\mathrm{volume}} = \Big[\psi_{\mathrm{Dirac}}(x)\,\psi_{\mathrm{Dirac}}(y)\Big]\\ \Big[D(x,y)\Big] &\overset{(12.35c)}{=} \frac{\mathrm{momentum}^2}{\mathrm{charge}^2} = \Big[A(x)\,A(y)\Big]\ .\end{aligned}$$

By quantization, the fields' amplitudes will become field-operators, and in the second part of this book we will construct the propagators of the quantum field theories as bi-linear forms of their field operators. Thanks to the definitions (12.35) they have from start on the appropriate units.

Part 2: Field Quantization

13 Why Quantum Field Theory?

This question can be understood on two different levels:

1. Why are quantum field theories needed instead of the classical field theories?
2. Why is quantum field theory needed instead of point-particle quantum mechanics?

In this chapter we will try to answer both of these questions.

The first question immediately leads back into the year 1900, when quantum theory was born. This theory emerged from the analysis of two theories, which where firmly established and had performed quite well in many experimental tests: Classical thermodynamics and Maxwell's electrodynamics. The problem, with which Planck was struggling, was not that one of these theories had failed when used separately. The problem was, that it seemed impossible to combine them without running into inconsistencies.

Planck tried at that time to compute – by means of Maxwell's electrodynamics and classical thermodynamics – the energy density of "blackbody radiation" as a function of temperature. A blackbody is a furnace, whose walls are in thermodynamic equilibrium with the electromagnetic radiation field within the furnace. The number of degrees of freedom of the electromagnetic field in a narrow frequency interval $\Delta\nu$ around the frequency ν equals the number of radiation modes (i.e. standing waves) with this frequency, which fit into the furnace. The result [38] computed by Maxwell's electrodynamics for a furnace with volume V is

$$\text{number of degrees of freedom} = V \cdot \frac{16\pi\nu^2}{c^3}\Delta\nu \; . \tag{13.1}$$

On the other hand, there is the "equipartition theorem" of classical thermodynamics. According to that theorem, each degree of freedom of a dynamical system in thermal equilibrium at temperature T has the energy

$$E = \frac{1}{2}kT \ , \tag{13.2}$$

with the Boltzmann-constant $k = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$. Combining these equations, the electromagnetic energy within a furnace of volume V in the frequency range $\Delta\nu$ around the frequency ν at temperature T should equal

$$E = V\frac{16\pi\nu^2}{c^3}\Delta\nu \cdot \frac{1}{2}kT \ , \tag{13.3}$$

and the energy density in the frequency range $\Delta\nu$ should equal

$$\rho = \frac{8\pi\nu^2}{c^3} \cdot kT \ . \tag{13.4}$$

This radiation law follows stringently from the combination of classical electrodynamics and classical thermodynamics. If it is wrong, then there must be basic failures at least in one of these classical theories. Actually this radiation law *is* wrong in two aspects:

Firstly according to (13.4) the electromagnetic field's energy density at a given temperature T is increasing quadratically with the frequency ν. For low frequencies this fits very well to the observations, but for high frequencies the equation diverges. This obviously absurd result was dubbed "ultraviolet-catastrophe".

Secondly, (13.4) is depending linearly upon temperature T. Thus according to this equation, the energy density of all frequencies should increase uniformly at increasing temperature. But actually, when a furnace is slowly heated up, we see at low temperature a dark red color, which is changing with raising temperature to orange, then to yellow, and eventually to white. Obviously the energy is not at all distributed equally into all radiation modes. Instead the energy share of high-frequency modes is relatively low at low temperature, but raises significantly with increasing temperature.

Eventually Planck found out how a radiation law could be constructed,

which matched reality. He had to make a simple, but quite unexpected assumption, which was absolutely incompatible with the principles of classical physics: The exchange of energy between the oscillating charges in the walls of the furnace and the electromagnetic radiation field is not a continuous process, but happens in energy quanta of size $h\nu$, with the constant $h = 6.63 \cdot 10^{-34}\mathrm{J}$, which later was named in honor of Planck. Based on this assumption, Planck derived the radiation law

$$\rho = \frac{8\pi h\nu^3/c^3}{e^{\frac{h\nu}{kT}} - 1} \ , \tag{13.5}$$

which excellently matches all observations.

It soon turned out, that the quantized energy is not – as the cautious Planck had assumed – a property of the oscillating charges in the walls of the furnace, but a property of the electromagnetic field itself, and as well a property of any other field. The "ultraviolet-catastrophe" according to (13.4) is unavoidable in any field, which firstly can oscillate in infinitely many modes (i. e. with infinitely many different wave numbers), and whose oscillation modes secondly can assume a continuum of energy values. Conversely stated: The specific heat of a field with infinitely many oscillation modes and a continuous energy spectrum must diverge according to (13.4), because an infinite amount of energy is needed, to assign the finite amount of energy $\frac{1}{2}kT$ to each of infinitely many degrees of freedom.

In contrast, quantum field theory states that each oscillation mode can assume only discrete energy values, which are proportional to it's wave number. Therefore, if a finite amount of energy is distributed to the field's oscillation modes, the modes with highest frequencies are not excited at all. The energy is distributed only to the finite number of low-frequency oscillation modes. Thus the specific heat is finite.

The retrospection to the birth of quantum theory has led us directly to the theme of the second part of this book: We see the imperative to formulate the theory of *all* fields such, that the energy in thermodynamic equilibrium is *not* distributed equally to all degrees of freedom. How this can be done remained a mystery for a quarter of a century after Planck's seminal discovery.

A physical theory describes by means of mathematical equations the course of events in nature, and the interactions and correlations between these events. This is not to say that the objects of physical research are identical with the elements of the mathematical language. But there exists a "correspondence" between them: The elements of the mathematical language are assigned as representatives to the physical objects. The relations and mutual dependencies between the mathematical elements are representing analogous relations and dependencies between the objects of physical observation. Thereby a *structural similarity* exists between the theory, which is formulated in mathematical language, and the described physical phenomena.

It was the conviction of physicists since the days of Galilei, that numbers – possibly multiplied by units like meter or second or volt times coulomb – are suitable and appropriate representatives of physical objects of whatever type. Therefore it came as a big surprise, when Heisenberg in June 1925 puzzled out a strange system – he named it "number schemes" – , by which the discrete energy values, which a quantum mechanical oscillator can assume, could be computed correctly. It turned out, that these "number schemes" actually were matrices, and that their success rested on the fact, that the algebra of matrices is not commutative, in contrast to the algebra of numbers.

Schrödinger was upset so much by Heisenberg's "horrible" matrix mechanics, that he immediately started to work out a more "beautiful" quantum theory. Already some few months later he could publish his wave-mechanics, which indeed goes without matrices, but does not lead back to the commutative algebra of numbers. In Schrödinger's wave-mechanics measurable physical quantities instead are represented by operators, and these operators have exactly the same non-commutative algebra as Heisenberg's matrices. Within soon, Schrödinger himself proved, that Heisenberg's matrix mechanics and his own wave-mechanics are indeed mathematically completely equivalent, and nothing but different formulations of the selfsame mathematical structure.

Whether one prefers matrices, or operators, or mathematical elements of whatever type – the essential point is, that the representatives of measurable quantities of quantum phenomena must follow a non-commutative algebra.

The commutator or anti-commutator of canonically conjugated observables, for example of a particle's position and momentum, equals in this algebra Planck's reduced quantum of action $\hbar$, multiplied by $i = \sqrt{-1}$.

Thereby already the second difference of quantum theory's mathematics versus the mathematics of classical theory is addressed: While all classical theories can be formulated with real numbers, and only for convenience sometimes imaginary numbers are used, quantum theory can not do without imaginary numbers.

Now we turn to the second of the two questions posed at the beginning of this chapter: Why is quantum field theory needed, although there is a quantum mechanics of point-particles available? There are three main reasons for this, which are closely connected mutually:

* All attempts to construct a quantum mechanics of point-particles, which is consistent with special relativity theory, inevitably lead to solutions with negative energies. For fermions, Dirac found a workaround by means of his holes-theory, while no remedy is known for bosons. Quantum field theory solves this problem completely. The energy of quantized fields always is positive.
* It is known from observations, that particles can decompose into other particles, or can be created in collisions of other particles. Quantum field theory describes such events qualitatively and quantitatively in excellent compliance with experimental observations, while in relativistic point-particles quantum mechanics the number of particles is invariable.
* If two measurements of one quantum system are performed, and the two measurements are space-like (i.e. one measurement is not within the lightcone of the other), then these two measurements must not influence one another, if the theory is to comply with relativity theory. Nobody succeeded to fulfill this requirement with a relativistic quantum mechanics of point-particles. Quantum field theory meets this objective with elegance and perfection due to exchange of virtual space-like particles and antiparticles.

Thus there are convincing arguments for the development of a quantum theory of fields. In the following chapters we will clarify, how this can be done successfully.

14 Canonical Quantization

The canonical quantization of fields starts with the construction of their canonically conjugate momentum densities. Then a non-commutative algebra is postulated for the fields and their canonically conjugate momenta, according to which their commutator or anti-commutator must equal $i\hbar$. We will see that there is no freedom to choose arbitrarily either the commutator or the anticommutator in the algebra's definition. Instead the commutator is mandatory in the definition of boson field algebra, and the anticommutator is mandatory in the definition of fermion field algebra, because only then the quantized fields will (a) be compatible with the principles of relativity theory, and (b) never have negative energy.

14.1 Quantum Mechanics

We presume that the reader is familiar with non-relativistic quantum mechanics. In this section we will very shortly recapitulate five topics of quantum mechanics, which will prove most useful when we subsequently turn to the quantization of fields:

* The canonical quantization of point-particle mechanics
* Representations of quantum mechanics
* The time-evolution operator
* Schrödinger-picture and Heisenberg-picture
* The harmonic oscillator

14.1.1 Quantization of Point-Particle Mechanics

Let q_j and p_j with $j = 1, 2, \ldots, 3N$ be the generalized coordinates and the canonically conjugate momenta of a classically described system of N point

particles. The Poisson[1]-brackets of arbitrary quantities A and B are defined by

$$\{A, B\}_{\mathrm{PB}} \equiv \sum_j \left(\frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} - \frac{\partial B}{\partial p_j} \frac{\partial A}{\partial q_j} \right) . \tag{14.1}$$

The total derivative of a function $A(q_j(t), p_j(t), t)$ with respect to time is

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\mathrm{d}q_j}{\mathrm{d}t} + \frac{\partial A}{\partial p_j} \frac{\mathrm{d}p_j}{\mathrm{d}t} \right) + \frac{\partial A}{\partial t} \\ &\overset{(3.22)}{=} \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial H}{\partial q_j} \right) + \frac{\partial A}{\partial t} \\ &= \{H, A\}_{\mathrm{PB}} + \frac{\partial A}{\partial t} . \end{aligned} \tag{14.2}$$

H is the system's Hamiltonian. Thus in the canonical formalism, the time dependence of a quantity A is decomposed into two parts: The *canonical time dependence* (some authors call it dynamical time dependence), which is determined by the Poisson-bracket $\{H, A\}_{\mathrm{PB}}$, and the *explicit time dependence* (some authors call it parametric time dependence), which is expressed by the partial derivative $\partial A/\partial t$.

The system of N point particles has $6N + 1$ variables, namely the $3N$ coordinates $q_j(t)$ (which again are depending on time), the $3N$ momenta $p_j(t)$ (which again are depending on time), and the time t. Any function $A(q_j(t), p_j(t), t)$ is a function of some or all of these $6N + 1$ variables. This also holds for the functions $q_k(q_j(t), p_j(t), t)$ and $p_k(q_j(t), p_j(t), t)$, for which the canonical formalism is making a somewhat subtle difference between functions and variables. The functions

$$q_k(q_j(t), p_j(t), t) \equiv q_k(q_k(t)) \neq q_k(t) \tag{14.3a}$$

$$\underbrace{p_k(q_j(t), p_j(t), t) \equiv p_k(p_k(t))}_{\text{functions}} \neq \underbrace{p_k(t)}_{\text{variables}} \tag{14.3b}$$

[1] named in honor of Siméon Denis Poisson, 1781 - 1840

are considered to be functions of the variables $q_k(t)$ resp. $p_k(t)$, but not to be functions of the variable t. Thus for these functions

$$\frac{\mathrm{d}\,q_k}{\mathrm{d}\,t} \neq 0 = \frac{\partial q_k}{\partial t} \qquad\qquad \frac{\mathrm{d}\,p_k}{\mathrm{d}\,t} \neq 0 = \frac{\partial p_k}{\partial t}\ . \tag{14.4}$$

That the definitions (14.3) are reasonable becomes clear, when we compute the total derivatives of the functions q_k and p_k with respect to time:

$$\frac{\mathrm{d}\,q_k}{\mathrm{d}\,t} = \sum_j \left(\underbrace{\frac{\partial q_k}{\partial q_j}}_{\delta_{kj}} \underbrace{\frac{\mathrm{d}\,q_j}{\mathrm{d}\,t}}_{\frac{\partial H}{\partial p_j}} + \underbrace{\frac{\partial q_k}{\partial p_j}}_{0} \underbrace{\frac{\mathrm{d}\,p_j}{\mathrm{d}\,t}}_{-\frac{\partial H}{\partial q_j}} \right) + \underbrace{\frac{\partial q_k}{\partial t}}_{0} = \{H, q_k\}_{\mathrm{PB}} \tag{14.5a}$$

$$\frac{\mathrm{d}\,p_k}{\mathrm{d}\,t} = \sum_j \left(\underbrace{\frac{\partial p_k}{\partial q_j}}_{0} \underbrace{\frac{\mathrm{d}\,q_j}{\mathrm{d}\,t}}_{\frac{\partial H}{\partial p_j}} + \underbrace{\frac{\partial p_k}{\partial p_j}}_{\delta_{kj}} \underbrace{\frac{\mathrm{d}\,p_j}{\mathrm{d}\,t}}_{-\frac{\partial H}{\partial q_j}} \right) + \underbrace{\frac{\partial p_k}{\partial t}}_{0} = \{H, p_k\}_{\mathrm{PB}} \tag{14.5b}$$

Our result would not be consistent with (3.22), if we would consider q_k and p_k to be functions of t. Therefore definition (14.3) is necessary. The variables $q_k(t)$ and $p_k(t)$ depend explicitly on time, while the functions q_k and p_k depend on time only canonically, but not explicitly.

Canonical quantization of point-particle mechanics is the replacement of the Poisson-brackets by commutators, which are multiplied by $i/\hbar$:

$$\boxed{\begin{aligned} \{A, B\}_{\mathrm{PB}} \equiv \sum_j \left(\frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j} - \frac{\partial B}{\partial p_j}\frac{\partial A}{\partial q_j} \right) \quad \longrightarrow \\ \longrightarrow \quad \frac{i}{\hbar}[A\,, B] \equiv \frac{i}{\hbar}(AB - BA) \end{aligned}} \tag{14.6}$$

Thus equations (14.5) become

$$\frac{\mathrm{d}\,q_k}{\mathrm{d}\,t} = \frac{i}{\hbar}[H, q_k] \equiv \frac{i}{\hbar}(Hq_k - q_kH) \overset{(14.5a)}{=} \frac{\partial H}{\partial p_k} \tag{14.7a}$$

$$\frac{\mathrm{d}\,p_k}{\mathrm{d}\,t} = \frac{i}{\hbar}[H, p_k] \equiv \frac{i}{\hbar}(Hp_k - p_kH) \overset{(14.5b)}{=} -\frac{\partial H}{\partial q_k}\ , \tag{14.7b}$$

and (14.2) becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t} \ . \tag{14.8}$$

Only the canonical time dependence of the functions is affected by the quantization. The explicit time dependence stays unchanged.

If H does explicitly depend on p_k resp. on q_k, then the right sides of (14.7) are different from zero, and thus the commutators of H with p_k resp. q_k must be different from zero as well. Therefore H, p_k, and q_k can't any more be numbers (which possibly are multiplied by units), but become non-commuting quantities due to quantization. The commutator of q_k and p_l can be computed directly by insertion of q_k for A and p_l for B into (14.6):

$$\{q_k, p_l\}_{\mathrm{PB}} \equiv \sum_j \left(\underbrace{\frac{\partial q_k}{\partial p_j}}_{0} \underbrace{\frac{\partial p_l}{\partial q_j}}_{0} - \underbrace{\frac{\partial p_l}{\partial p_j}}_{\delta_{lj}} \underbrace{\frac{\partial q_k}{\partial q_j}}_{\delta_{kj}} \right) = -\delta_{kl} \quad \longrightarrow$$

$$\longrightarrow \quad \boxed{\frac{i}{\hbar}[q_k, p_l] \equiv \frac{i}{\hbar}(q_k p_l - p_l q_k) = -\delta_{kl}} \tag{14.9}$$

In Heisenberg's matrix mechanics, the non-commutative algebra (14.9) is realized by representing the measurable quantities q_k and p_l by matrices. In Schrödinger's wave-mechanics, the non-commutative algebra is realized by representing the measurable quantities q_k and p_l by operators, which are acting onto wave functions. Note, that the quantities, which are inserted into the commutator, must be measured at the same time t. (14.9) does not give any statement on the commutator's value, if the measurements of q_k and p_l are performed at different times.

If Planck's quantum of action is negligible compared to the action S of the mass point, then the commutator is negligible:

$$[q_k, p_l] = q_k p_l - p_l q_k \overset{(14.9)}{=} i\hbar\delta_{kl} \approx 0 \quad \text{if } \hbar/S \ll 1 \tag{14.10}$$

Exactly this limiting case defines the range of applicability of classical physics. Only in this case, the commutative algebra of numbers holds for the coordinates and their canonically conjugate momenta. Else the non-commutative algebra (14.9) must be applied.

We consider the non-commutative algebra (14.6) resp. (14.9) as a law of nature, which can not be derived. Heisenberg found it by ingenious guessing. It can be vindicated alone by the fact, that all conclusions drawn from it match the experimental observations in the applicability range of non-relativistic point-particle quantum mechanics. As soon as we require compliance with special relativity theory, the extension to (14.74) becomes mandatory.

14.1.2 Representations of Quantum Mechanics

In the years after 1925, quantum mechanics was formulated in different, seemingly independent representations. Dirac [39] suggested a method, by which the systematic relations between the different representations become transparent, and transformations between the representations can easily be performed:

All (possibly time-dependent) elements of Hilbert space are firstly considered as "basis-independent" vectors $|\psi(t)\rangle$. Then different "coordinate systems" are introduced in the Hilbert space due to the definition of bases of orthogonal vectors. Let

$$|F_1\rangle, |F_2\rangle, |F_3\rangle, \ldots, |F_f\rangle, \ldots \in \mathcal{H} \tag{14.11}$$

be a complete system of orthonormal vectors, which span the Hilbert space:

$$\langle F_f|F_g\rangle = \delta_{fg} \qquad , \qquad \sum_f |F_f\rangle\langle F_f| = 1 \tag{14.12}$$

In this notation, $|F_g\rangle$ is a "ket-vector", $\langle F_g|$ is it's dual "bra-vector", and the combination of the ket $|F_g\rangle$ and the bra $\langle F_f|$ to the bracket $\langle F_f|F_g\rangle = \langle F_g|F_f\rangle^*$ defines the scalar product of the vectors $|F_g\rangle$ and $|F_f\rangle$.

The projection operator $|F_f\rangle\langle F_f|$ creates, if it is applied to an arbitrary ket $|\phi\rangle$, the projection

$$|F_f\rangle\langle F_f|\phi\rangle \equiv |F_f\rangle\phi_f \tag{14.13}$$

of the vector $|\phi\rangle$ onto the vector $|F_f\rangle$. If the sum (14.12) over the projection

operators equals the identity operator 1, then the orthonormal system is called "complete". If a system of orthonormal basis vectors is complete, then any element $|\psi(t)\rangle \in \mathcal{H}$ can be expanded with respect to this basis:

$$|\psi(t)\rangle = \sum_f |F_f\rangle\langle F_f|\psi(t)\rangle \equiv \sum_f |F_f\rangle\psi_f(t) \tag{14.14}$$

The numbers $\psi_f(t) \in \mathbb{C}$ are the components of the vector $|\psi(t)\rangle$ in the F-representation. They may be considered as the "coordinates" of $|\psi(t)\rangle$ in the "coordinate system" $|F_1\rangle$, $|F_2\rangle$, $|F_3\rangle$, ... Complete knowledge of the functions $\psi_f(t)$ for all f is equivalent to complete knowledge of $|\psi(t)\rangle$. All results of the theory, which can be computed by means of $|\psi(t)\rangle$, can as well be computed by means of the complete set of functions $\psi_f(t)$.

If the basis-free state vector is normalized to 1

$$\langle\psi(t)|\psi(t)\rangle = 1 \ , \tag{14.15}$$

then the expectation value $\langle S\rangle$ of an arbitrary operator S in the F-representation is

$$\begin{aligned}\langle S\rangle &= \langle\psi(t)|S|\psi(t)\rangle = \sum_f \sum_g \langle\psi(t)|F_f\rangle\langle F_f|S|F_g\rangle\langle F_g|\psi(t)\rangle \\ &= \sum_f \sum_g \psi_f^*(t) \underbrace{\langle F_f|S|F_g\rangle}_{S_{fg}\in\mathbb{C}} \psi_g(t) \ .\end{aligned} \tag{14.16}$$

$S_{fg} \in \mathbb{C}$ is the fg-component of the operator S in the F-Representation. If the vectors $|F_g\rangle$ are eigenvectors of S, then only the diagonal elements of the matrix S_{fg} are different from zero because of

$$S_{fg} = \langle F_f|S|F_g\rangle = \underbrace{\langle F_f|F_g\rangle}_{\delta_{fg}} S_g = \begin{cases} S_f \text{ if } f = g \\ 0 \text{ if } f \neq g \ . \end{cases} \tag{14.17}$$

Does there also exist a basis, in which Schrödinger's wave function $\psi_{\boldsymbol{x}}(t) \equiv \psi(t, \boldsymbol{x})$ is the $\boldsymbol{x}$-component of the basis-free state vector $|\psi(t)\rangle$? It does, namely the eigenvectors $|\boldsymbol{y}\rangle$ of the position-operator $\boldsymbol{x}$ with eigenvalues $\boldsymbol{y}$:

$$\boldsymbol{x}|\boldsymbol{y}\rangle = \boldsymbol{y}|\boldsymbol{y}\rangle \tag{14.18a}$$

The index $\boldsymbol{x}$ of the basis $|\boldsymbol{x}\rangle$ is continuous, in contrast to the discrete index f of the basis $|F_f\rangle$. Therefore the scalar product

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle \equiv \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{14.18b}$$

of the basis vectors cannot be the Kronecker symbol as in equation (14.12). Dirac extended the Kronecker symbol to continuous indices by definition of the delta function:

$$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \equiv 0 \quad \text{if } \boldsymbol{x} \neq \boldsymbol{y} \tag{14.19a}$$

$$\int \mathrm{d}^3x\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{x}) \equiv \phi(\boldsymbol{y}) \tag{14.19b}$$

The integration volume must cover the point $\boldsymbol{y}$. The expansion of the basis-free state vector $|\psi(t)\rangle$ with respect to the vectors $|\boldsymbol{x}\rangle$ is done in complete analogy to (14.14). But because of the continuous index of the basis $|\boldsymbol{x}\rangle$, one gets an integral instead of a sum:

$$|\psi(t)\rangle = \int\limits_\Omega \mathrm{d}^3x\, |\boldsymbol{x}\rangle\langle \boldsymbol{x}|\psi(t)\rangle = \int\limits_\Omega \mathrm{d}^3x\, |\boldsymbol{x}\rangle\psi(t,\boldsymbol{x}) \tag{14.20}$$

The representation with the basis $|\boldsymbol{x}\rangle$ is called local representation or position-representation. It is identical to Schrödinger's wave mechanics, and the functions $\psi(t,\boldsymbol{x})$ are identical to Schrödinger's wave wave-functions. In the local representation, the scalar product of two vectors $\in \mathcal{H}$ has the form

$$\langle\phi|\psi\rangle = \int\limits_\Omega \mathrm{d}^3x\, \langle\phi|\boldsymbol{x}\rangle\langle\boldsymbol{x}|\psi\rangle = \int\limits_\Omega \mathrm{d}^3x\, \phi^*(\boldsymbol{x})\psi(\boldsymbol{x})\ . \tag{14.21}$$

Appreciably more effort is needed, to prove that the matrix elements of arbitrary operators in the local representation have the form

$$\langle\phi|S|\psi\rangle = \int\limits_{\Omega} \mathrm{d}^3x \int\limits_{\Omega} \mathrm{d}^3y \, \langle\phi|\boldsymbol{x}\rangle\langle\boldsymbol{x}|S|\boldsymbol{y}\rangle\langle\boldsymbol{y}|\psi\rangle = \int\limits_{\Omega} \mathrm{d}^3x \, \phi^*(\boldsymbol{x}) \, S \, \psi(\boldsymbol{x}) \; . \tag{14.22}$$

The proof can be found in appendix A.19.

14.1.3 The Time-Evolution Operator

If the Hamilton operator H does not depend explicitly on time, then the Schrödinger equation of wave mechanics

$$i\hbar \frac{\partial\psi(t,\boldsymbol{x})}{\partial t} = H\psi(t,\boldsymbol{x}) \tag{14.23}$$

can formally be integrated:

$$\psi(t,\boldsymbol{x}) = e^{-\frac{i}{\hbar}(t-t_0)H}\psi(t_0,\boldsymbol{x}) \tag{14.24}$$

The Hamilton operator H in the exponent is to be interpreted as a shortcut notation for the series

$$e^{-\frac{i}{\hbar}(t-t_0)H} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\Big(-\frac{i}{\hbar}(t-t_0)H\Big)^n \tag{14.25}$$
$$\text{with } H^n\psi \equiv \underbrace{H(\ldots H(H(H(H}_{n\times}\psi)))\ldots) \; .$$

Using (14.24), the state function can be computed for arbitrary time t, if it is known at time t_0.

We generalize (14.24) due to the definition of the time-evolution operator U, which is describing the evolution of the wave function in time:

$$\psi(t,\boldsymbol{x}) \equiv U(t,t_0)\psi(t_0,\boldsymbol{x}) \tag{14.26}$$

Insertion into (14.23) results into

$$i\hbar \frac{\partial U(t,t_0)}{\partial t}\,\psi(t_0,\boldsymbol{x}) = HU(t,t_0)\psi(t_0,\boldsymbol{x}) \; ,$$

respectively into the operator-equation

$$i\hbar \frac{\partial U(t,t_0)}{\partial t} = H U(t,t_0) \ . \tag{14.27}$$

Only if the Hamilton operator does not depend on time, this equation has the simple solution

$$U(t,t_0) \overset{(14.24)}{=} e^{-\frac{i}{\hbar}(t-t_0)H} \ . \tag{14.28a}$$

Without proof we state the solution of (14.27) with time-dependent Hamilton operator:

$$\begin{aligned} U(t,t_0) &= T \, \exp\Big\{ -\frac{i}{\hbar} \int_{t_0}^{t} \mathrm{d}\tau \, H(\tau) \Big\} \\ &= T \sum_{j=0}^{\infty} \frac{1}{j!} \Big(-\frac{i}{\hbar} \int_{t_0}^{t} \mathrm{d}\tau \, H(\tau) \Big)^j \end{aligned} \tag{14.28b}$$

T is the time-order operator. It changes the sequence of time-dependent operators – in this case $H(\tau_i)H(\tau_j)H(\tau_k)\ldots$ – such, that younger operators with larger time-argument are placed left of older operators with smaller time-argument. We list some properties of the time-evolution operator:

$$U^\dagger(t,t_0) = U^{-1}(t,t_0) \tag{14.29a}$$

$$U(t,t_0) = U(t,t_1)\, U(t_1,t_0) \tag{14.29b}$$

$$U(t,t_1)\, U(t_1,t_0) = U(t_1,t_0)\, U(t,t_1) \tag{14.29c}$$

$$U(t,t_0) = U^{-1}(t_0,t) \quad \Longrightarrow \quad U(t_0,t_0) = 1 \tag{14.29d}$$

$$[U,H] = 0 \tag{14.29e}$$

In the simple case (14.28a), these properties are evident. We state without proof that all relations (14.29) hold as well, if H is explicitly time-dependent.

Remember that the wave-function $\psi(t,\boldsymbol{x})$ is the scalar product of the basis-independent state vector $|\psi(t)\rangle$ and the vector $|\boldsymbol{x}\rangle$ of the local base:

$$\psi(t,\boldsymbol{x}) \overset{(14.20)}{=} \langle \boldsymbol{x}|\psi(t)\rangle \overset{(14.26)}{=} U(t,t_0)\psi(t_0,\boldsymbol{x}) = \\ \overset{(14.20)}{=} U(t,t_0)\langle \boldsymbol{x}|\psi(t_0)\rangle = \langle \boldsymbol{x}|U(t,t_0)|\psi(t_0)\rangle \tag{14.30}$$

The time-evolution operator could be pulled into the bracket, because $|\boldsymbol{x}\rangle$ does not depend on time. This result is suggesting a generalization of definition (14.26):

$$|\psi(t)\rangle \equiv U(t,t_0)|\psi(t_0)\rangle \tag{14.31}$$

Furthermore we define the time-dependent position base due to

$$|t,\boldsymbol{x}\rangle \equiv U^{-1}(t,0)\,|\boldsymbol{x}\rangle \ . \tag{14.32}$$

This definition is consistent with the previous equations because of

$$\psi(t,\boldsymbol{x}) \overset{(14.20)}{=} \langle \boldsymbol{x}|\psi(t)\rangle \overset{(14.31)}{=} \langle \boldsymbol{x}|\,U(t,0)\,|\psi(t{=}0)\rangle = \\ = \left(U^{-1}(t,0)\,|\boldsymbol{x}\rangle\right)^{\dagger}|\,\psi(t{=}0)\rangle = \langle t,\boldsymbol{x}|\,\psi(t{=}0)\rangle \ . \tag{14.33}$$

Note that the time evolution of the local basis is running inversely to the time evolution (14.31) resp. (14.26) of the state vector $|\psi(t)\rangle$ resp. the wave function $\psi(t,\boldsymbol{x})$. This is consistent with the fact, that the time parameter t in

$$\psi(t,\boldsymbol{x}) = \langle \boldsymbol{x}|\psi(t)\rangle = \langle t,\boldsymbol{x}|\psi\rangle \qquad \text{with } |\psi\rangle \equiv |\psi(t{=}0)\rangle \tag{14.34}$$

is located in the ket or in the bra, respectively. In $\langle \boldsymbol{x}|\psi(t)\rangle$ the time-dependent vector $|\psi(t)\rangle$ is projected onto the time-independent coordinates $|\boldsymbol{x}\rangle$ of the Hilbert space, i.e. this is an active translation in Hilbert space. In $\langle t,\boldsymbol{x}|\psi\rangle$ the time-independent vector $|\,\psi\rangle$ is projected onto the time-dependent coordinates $|t,\boldsymbol{x}\rangle$ of the Hilbert space, i.e. this is a passive translation in Hilbert space. Because of the quite peculiar type of time dependence, some authors deny that $|t,\boldsymbol{x}\rangle$ is a time-dependent vector at all. That isn't a consistent point of view. Both $|\psi(t)\rangle$ and $|t,\boldsymbol{x}\rangle$ clearly are time-

dependent vectors. The difference is due to the fact, that $|t, \boldsymbol{x}\rangle$ is a time-dependent basis vector, while $|\psi(t)\rangle$ is a time-dependent "normal" vector.

The vectors of the time-dependent local base are eigenvectors of the time-dependent position operator, provided that the time arguments t in the vectors and in the operator are identical:

$$\begin{aligned} \boldsymbol{y} &\overset{(14.18a)}{=} \langle \boldsymbol{y}|\boldsymbol{x}|\boldsymbol{y}\rangle = \\ &= \underbrace{\langle \boldsymbol{y}|\, U(t,0)}_{\langle t,\boldsymbol{y}|} \underbrace{U^{-1}(t,0)\,\boldsymbol{x}\,U(t,0)}_{\boldsymbol{x}(t)} \underbrace{U^{-1}(t,0)\,|\boldsymbol{y}\rangle}_{|t,\boldsymbol{y}\rangle} \end{aligned} \tag{14.35}$$

As the time-independent position vectors form an orthonormal basis of Hilbert space, the time-dependent position vectors are doing this as well, because (at same t in all vectors) they as well are orthonormal and complete:

$$\langle t, \boldsymbol{x}|t, \boldsymbol{y}\rangle = \langle \boldsymbol{x}|\, U(t,0)\, U^{-1}(t,0)\, |\boldsymbol{y}\rangle = \langle \boldsymbol{x}|\boldsymbol{y}\rangle \overset{(14.18b)}{=} \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{14.36a}$$

$$\begin{aligned} \int_\Omega \mathrm{d}^3x\, |t,\boldsymbol{x}\rangle\langle t,\boldsymbol{x}| &= \int_\Omega \mathrm{d}^3x\, |\boldsymbol{x}\rangle U^{-1}(t,0)\, U(t,0)\langle \boldsymbol{x}| = \\ &= \int_\Omega \mathrm{d}^3x\, |\boldsymbol{x}\rangle\langle \boldsymbol{x}| \overset{(14.20)}{=} 1 \end{aligned} \tag{14.36b}$$

For the method of path integrals, which will be derived in chapter 18, the probability amplitude

$$\begin{aligned} U(t_2,\boldsymbol{z},t_1,\boldsymbol{y}) &\equiv \langle \boldsymbol{z}|\, U(t_2,t_1)\, |\boldsymbol{y}\rangle = \langle \boldsymbol{z}|\, U(t_2,0)\, U(0,t_1)\, |\boldsymbol{y}\rangle = \\ &= \langle \boldsymbol{z}|\, U(t_2,0)\, U^{-1}(t_1,0)\, |\boldsymbol{y}\rangle = \langle t_2,\boldsymbol{z}|t_1,\boldsymbol{y}\rangle \end{aligned} \tag{14.37}$$

will be of central importance. This matrix element is interpreted as the amplitude of the probability, that a point-particle of quantum mechanics, which was observed at location $\boldsymbol{y}$ at time t_1, can be detected at the later time t_2 at position $\boldsymbol{z}$.

14.1.4 Schrödinger-Picture and Heisenberg-Picture

In section 14.1.1 we indicated the time-derivatives of the operators

$$\frac{\mathrm{d}\,q_{k\,\mathsf{H}}}{\mathrm{d}\,t} \overset{(14.7a)}{=} \frac{i}{\hbar}[H, q_{k\,\mathsf{H}}] \qquad \frac{\mathrm{d}\,p_{k\,\mathsf{H}}}{\mathrm{d}\,t} \overset{(14.7b)}{=} \frac{i}{\hbar}[H, p_{k\,\mathsf{H}}]$$

and

$$\frac{\mathrm{d}\,A_{\mathsf{H}}}{\mathrm{d}\,t} \overset{(14.8)}{=} \frac{i}{\hbar}[H, A_{\mathsf{H}}] + \frac{\partial A_{\mathsf{H}}}{\partial t} \ . \tag{14.38}$$

They are operators in the *Heisenberg-picture*. Here we marked them by an additional index H. Operators in the Heisenberg-picture are canonically time-dependent, and possibly in addition explicitly time-dependent. Their canonical time dependence is determined by their commutator with the Hamilton operator, multiplied by $i/\hbar$, while their partial derivatives with respect to time reflect their possible explicit time dependence.

It's conventional, though somewhat confusing, to change the notation of the variables upon quantization:

$$q_k\Big(q_k(t)\Big) \quad \longrightarrow \quad q_{k\,\mathsf{H}}(t) \tag{14.39a}$$

$$p_k\Big(p_k(t)\Big) \quad \longrightarrow \quad p_{k\,\mathsf{H}}(t) \tag{14.39b}$$

$$A\Big(q_j(t), p_j(t), t\Big) \quad \longrightarrow \quad A_{\mathsf{H}}(t) \tag{14.39c}$$

While for the classical functions we carefully discriminate between the variable t, which is indicating the explicit time dependence, and the canonical time dependence, which is mediated by the variables $q_j(t)$ and $p_j(t)$, both types of time dependence are combined in the variable t of the operators.

Using the time-evolution operator, the operators of the Heisenberg-picture can be transformed to operators of the *Schrödinger-picture*, which we mark by an index S:

$$A_{\mathsf{S}} \equiv U(t,0)\, A_{\mathsf{H}}(t)\, U^{-1}(t,0) \overset{\text{possibly}}{=} e^{-\frac{i}{\hbar}Ht} A_{\mathsf{H}}(t)\, e^{+\frac{i}{\hbar}Ht} \tag{14.40}$$

The equals sign is "possibly" valid. It does hold if and only if the Hamilton operator does not explicitly depend on time. In contrast, all equations of the time-evolution operator U in this chapter hold in any case, even if that condition is not fulfilled.

The Hamilton operator is identical in both pictures:

$$H_{\mathsf{S}} = U(t,0)\, H_{\mathsf{H}}\, U^{-1}(t,0) \overset{(14.29e)}{=} H_{\mathsf{H}} \overset{\text{possibly}}{=} e^{-\frac{i}{\hbar}Ht} H_{\mathsf{H}} e^{+\frac{i}{\hbar}Ht} \tag{14.41}$$

Therefore the indices H or S can be skipped in case of the Hamilton operator.

We compute the total derivative of the operator A_{S} in the Schrödinger-picture with respect to time:

$$\frac{\mathrm{d}\, A_{\mathsf{S}}}{\mathrm{d}\, t} = \frac{\mathrm{d}\, U}{\mathrm{d}\, t}\, A_{\mathsf{H}}\, U^{-1} + U\, \frac{\mathrm{d}\, A_{\mathsf{H}}}{\mathrm{d}\, t}\, U^{-1} + U\, A_{\mathsf{H}}\, \frac{\mathrm{d}\, U^{-1}}{\mathrm{d}\, t} \tag{14.42}$$

Because of $U^{\dagger}(t,t_0) \overset{(14.29a)}{=} U^{-1}(t,t_0)$, the equation which is adjoint to

$$\frac{\mathrm{d}}{\mathrm{d}\, t}\, U(t,0) = \frac{\partial}{\partial t}\, U(t,0) \overset{(14.27)}{=} -\frac{i}{\hbar} H\, U(t,0) \tag{14.43a}$$

is

$$\frac{\mathrm{d}}{\mathrm{d}\, t}\, U^{-1}(t,0) = +\frac{i}{\hbar} H\, U^{-1}(t,0)\ . \tag{14.43b}$$

Inserting this into (14.42) results into

$$\begin{aligned} \frac{\mathrm{d}\, A_{\mathsf{S}}}{\mathrm{d}\, t} &= -\frac{i}{\hbar} H \underbrace{U A_{\mathsf{H}} U^{-1}}_{A_{\mathsf{S}}} + U \underbrace{\frac{\mathrm{d}\, A_{\mathsf{H}}}{\mathrm{d}\, t}}_{\overset{(14.38)}{=} \frac{i}{\hbar}[H, A_{\mathsf{H}}] + \frac{\partial A_{\mathsf{H}}}{\partial t}} U^{-1} \overset{(14.29e)}{+} \frac{i}{\hbar} \underbrace{U A_{\mathsf{H}} U^{-1}}_{A_{\mathsf{S}}} H \\ &= -\frac{i}{\hbar}[H, A_{\mathsf{S}}] + \frac{i}{\hbar}[H, A_{\mathsf{S}}] + U\, \frac{\partial A_{\mathsf{H}}}{\partial t}\, U^{-1} = \frac{\partial A_{\mathsf{S}}}{\partial t}\ . \end{aligned} \tag{14.44}$$

The time dependence of A_{S} is caused solely by the explicit time dependence of A_{H}, while the canonical time-dependence, which is defined by $\frac{i}{\hbar}[H, A_{\mathsf{H}}]$, is exactly compensated due to the transformation (14.40) of the Operator A_{H} with the time-evolution operator. Note:

$$A_{\mathsf{H}}(t{=}0) = A_{\mathsf{S}} \tag{14.45}$$

Operators in the Heisenberg-picture depend canonically and possibly explicitly on time. Operators in the Schrödinger-picture do not depend canonically, but possibly explicitly, on time. The operators are transformed from one picture to the other by the time-evolution operator:

$$A_{\mathsf{S}} = U(t,0)\, A_{\mathsf{H}}(t)\, U^{-1}(t,0)$$
$$A_{\mathsf{H}}(t) = U^{-1}(t,0)\, A_{\mathsf{S}}\, U(t,0) \tag{14.46a}$$

Observables, which depend explicitly on time, are rarely encountered. Therefore many authors mark operators in the Heisenberg- resp. in the Schrödinger-picture not by indices H resp. S, but due to the indication resp. not-indication of the time dependence:

$$A(t) \equiv A_{\mathsf{H}}(t) \qquad\qquad A \equiv A_{\mathsf{S}}$$

We will in most cases adopt this simplified notation. Only in this section we keep the eye-catching indices H and S.

Definition: Eigenfunctions of operators in the Heisenberg-picture are considered vectors in the Heisenberg-picture. Eigenfunctions of operators in the Schrödinger-picture are considered vectors in the Schrödinger-picture. (14.46b)

In the time-dependent expectation value

$$\langle A\rangle(t) = \langle\psi_{\mathsf{S}}(t)|\, A_{\mathsf{S}}\, |\psi_{\mathsf{S}}(t)\rangle \tag{14.47a}$$
$$= \langle\psi_{\mathsf{S}}(t)|\, U(t,0)\, A_{\mathsf{H}}(t)\, U^{-1}(t,0)\, |\psi_{\mathsf{S}}(t)\rangle$$
$$= \langle\psi_{\mathsf{H}}|\, A_{\mathsf{H}}(t)\, |\psi_{\mathsf{H}}\rangle\ , \tag{14.47b}$$

the canonical time dependence is shifted back and forth between the operator and the state functions due to transformation with the time-evolution operator U. The transformation

$$|\psi_{\mathsf{S}}(t)\rangle = U(t,0)\, |\psi_{\mathsf{H}}\rangle = U(t,0)\, |\psi_{\mathsf{S}}(t{=}0)\rangle \tag{14.48}$$

transforms the time-independent vector $|\psi_{\mathsf{H}}\rangle$ of the Heisenberg-picture into

the time-dependent vector $|\psi_S(t)\rangle$ of the Schrödinger-picture. Inversely formulated, we may state: The vector $|\psi_H\rangle$ in the Heisenberg-picture is the vector $|\psi_S(t=0)\rangle$ of the Schrödinger-picture, which was frozen at time $t=0$. Sometimes it is useful, to freeze the vector of the Schrödinger-picture not at time $t=0$, but at some other time $t_0 \neq 0$:

$$|\psi_S(t-t_0)\rangle = U(t,t_0)\,|t_0,\psi_H\rangle = U(t,t_0)\,|\psi_S(t=t_0)\rangle \tag{14.49}$$

t_0 is not a running time-parameter, but an index which is documenting, at what time the time-independent vector $|t_0,\psi_H\rangle$ was frozen.

Comparing (14.47a) and (14.47b), one might guess that state vectors in the Schrödinger-picture always were time-dependent, while state vectors in the Heisenberg-picture were never time-dependent. But this is not generally true. There also exist time-dependent state vectors in the Heisenberg-picture, and time-independent state vectors in the Schrödinger-picture.

An important example are the vectors of the time-independent resp. the time-dependent position base, which we evaluated in section 14.1.2. They are eigenvectors of the time-independent position vector $\boldsymbol{x}_S$ in the Schrödinger-picture resp. eigenvectors of the time-dependent position vector $\boldsymbol{x}_H(t)$ in the Heisenberg-picture:

$$\langle \boldsymbol{x}_S\rangle = \underbrace{\langle \boldsymbol{y}_S|\overbrace{U(t,0)}^{1}}_{\langle t,\boldsymbol{y}_H|}\ \underbrace{U^{-1}(t,0)\,\boldsymbol{x}_S\overbrace{U(t,0)}^{1}\,U^{-1}(t,0)}_{\boldsymbol{x}_H(t)}\ \underbrace{|\boldsymbol{y}_S\rangle}_{|t,\boldsymbol{y}_H\rangle} = \langle \boldsymbol{x}_H(t)\rangle = \boldsymbol{y} \tag{14.50}$$

The state vectors

$$|\psi_H\rangle \overset{(14.48)}{=} U^{-1}(t,0)\,|\psi_S(t)\rangle \tag{14.51a}$$

$$|\boldsymbol{y}_S\rangle \overset{(14.50)}{=} U(t,0)\,|t,\boldsymbol{y}_H\rangle \tag{14.51b}$$

obviously are time-independent, while the vectors

$$|\psi_{\mathsf{S}}(t)\rangle \overset{(14.48)}{=} U(t,0)\,|\psi_{\mathsf{H}}\rangle \tag{14.51c}$$

$$|t, \boldsymbol{y}_{\mathsf{H}}\rangle \overset{(14.50)}{=} U^{-1}(t,0)\,|\boldsymbol{y}_{\mathsf{S}}\rangle \tag{14.51d}$$

obviously are time-dependent. Subsequent to (14.33), we already discussed and explained the reason of the anti-canonical time dependence of (14.51d) versus the canonical time dependence of (14.51c).

Because of $U\,U\,U^{-1} = U$, the time-evolution operator is trivially invariant under the time-evolution transformation. This operator does neither belong to the Schrödinger- nor to the Heisenberg-picture, but it is standing in-between the both pictures, and brings about the transformation between them.

14.1.5 The Harmonic Oscillator

The energy of a classical point particle of mass m with generalized coordinate q and generalized momentum p, which is oscillating harmonically with angular frequency ω, is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \quad . \tag{14.52}$$

To quantize this oscillator, one postulates for p and q the non-commutative algebra

$$[q,p] \equiv qp - pq \overset{(14.9)}{=} i\hbar \; . \tag{14.53}$$

The dimension-less ladder operator a is defined by

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}\, q + i \sqrt{\frac{1}{2m\omega\hbar}}\, p \quad . \tag{14.54a}$$

The operator adjoint to a is

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, q - i \sqrt{\frac{1}{2m\omega\hbar}}\, p \; . \tag{14.54b}$$

We don't need to write $q^\dagger$ and $p^\dagger$ in (14.54b), because the hermitean operators q and p are self-adjoint, i. e. $q^\dagger = q$ and $p^\dagger = p$. The commutator of the ladder operators is

$$\begin{aligned}[a, a^\dagger] &= \frac{m\omega}{2\hbar}\, q^2 - \frac{i}{2\hbar}[q,p] + \frac{1}{2m\omega\hbar}\, p^2 \\ &\quad - \frac{m\omega}{2\hbar}\, q^2 - \frac{i}{2\hbar}\,[q,p] - \frac{1}{2m\omega\hbar}\, p^2 \\ &= -\frac{i}{\hbar}[q,p] = -\frac{i}{\hbar} i\hbar = 1\ . \end{aligned} \tag{14.55}$$

Due to addition resp. subtraction of (14.54a) and (14.54b) one finds

$$q = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \tag{14.56a}$$

$$p = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)\ . \tag{14.56b}$$

Using these equations, the Hamilton operator (14.52) can be written in the form

$$\begin{aligned} H &= -\frac{1}{2m}\frac{m\omega\hbar}{2}(a^\dagger a^\dagger - a^\dagger a - a a^\dagger + a a) + \\ &\quad + \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}(a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a) \\ &= \frac{\hbar\omega}{2}(a^\dagger a + a a^\dagger) = \hbar\omega(a^\dagger a + \frac{1}{2}\underbrace{[a, a^\dagger]}_{1})\ . \end{aligned} \tag{14.57}$$

Let $|n\rangle$ be an eigenvector of H with energy E_n:

$$H|n\rangle = E_n|n\rangle \tag{14.58}$$

We compute the energy of the oscillator in the state $a^\dagger|n\rangle$:

$$
\begin{aligned}
Ha^\dagger|n\rangle &= \frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger)a^\dagger|n\rangle \\
&= \frac{\hbar\omega}{2}(a^\dagger aa^\dagger + aa^\dagger a^\dagger)|n\rangle \\
&= \frac{\hbar\omega}{2}\Big(a^\dagger(1+a^\dagger a) + (1+a^\dagger a)a^\dagger\Big)|n\rangle \\
&= \hbar\omega a^\dagger|n\rangle + a^\dagger \underbrace{\frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger)|n\rangle}_{=H|n\rangle = E_n|n\rangle} \\
&= (E_n + \hbar\omega)a^\dagger|n\rangle
\end{aligned}
\tag{14.59}
$$

Thus $a^\dagger|n\rangle$ again is an eigenvector of H. In the state $a^\dagger|n\rangle$, the oscillator's energy is by $\hbar\omega$ higher than in the state $|n\rangle$. In the state $a|n\rangle$, it's energy is

$$
\begin{aligned}
Ha|n\rangle &= \frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger)a|n\rangle \\
&= \frac{\hbar\omega}{2}(a^\dagger aa + aa^\dagger a)|n\rangle \\
&= \frac{\hbar\omega}{2}\Big((-1+aa^\dagger)a + a(-1+aa^\dagger)\Big)|n\rangle = \\
&= -\hbar\omega a|n\rangle + a\underbrace{\frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger)|n\rangle}_{=H|n\rangle = E_n|n\rangle} \\
&= (E_n - \hbar\omega)a|n\rangle \ .
\end{aligned}
\tag{14.60}
$$

Therefore, by applying the operator a j-times onto the state $|n\rangle$ with energy E_n, a state with energy $E_n - j\hbar\omega$ can be constructed. But this method cannot be repeated infinitely often. Being hermitean operators, all eigenvalues of p and q are real. As both show up in the Hamilton operator quadratically, the Hamilton operator cannot have negative eigenvalues. Infinite continuation of method (14.60) is avoided by the postulate, that there must be a lowest-energy state, called $|0\rangle$, for which

$$
a|0\rangle = 0 \ . \tag{14.61}
$$

Due to this postulate, the energy of the lowest state – usually called "vacuum" – is

$$H|0\rangle = \frac{\hbar\omega}{2}(a^\dagger \underbrace{a|0\rangle}_{=0} + aa^\dagger|0\rangle) = \frac{\hbar\omega}{2}(-a^\dagger \underbrace{a|0\rangle}_{=0} + aa^\dagger|0\rangle) =$$
$$= \frac{\hbar\omega}{2} \underbrace{[a, a^\dagger]}_{1} |0\rangle = \frac{\hbar\omega}{2}|0\rangle \ . \tag{14.62}$$

The energy of a quantized harmonic oscillator is at least $\frac{1}{2}\hbar\omega$. It's energy can't be zero.

Each application of the operator $a^\dagger$ raises the oscillator's energy by one energy quantum $\hbar\omega$ ("one energy quantum is created"). Each application of the operator a lowers the oscillator's energy by one energy quantum $\hbar\omega$ ("one energy quantum is annihilated"). Therefore $a^\dagger$ is called creation operator, a is called annihilation operator. Both also are called ladder operators, because by using them, one can step up and down on the energy ladder. We will see, that the ladder operators of quantum field theory emerge from the coefficients of the Fourier-series expansions, which were described in chapter 7.

Until now we didn't give any thought to the normalization of the state functions $|n\rangle$. The normalization is fixed by these two postulates:

$$\left.\begin{aligned} \langle n|n\rangle &= 1 \\ \langle n|a^\dagger a|n\rangle &= n \end{aligned}\right\} \quad \text{for } n = 0, 1, 2, 3, \ldots \tag{14.63}$$

$a^\dagger a$ is called particle-number operator. It's eigenvalue n is indicating, how many energy quanta $\hbar\omega$ are excited in the oscillator's state $|n\rangle$. We define two numbers r_n and s_n by

$$a|n\rangle = r_n|n-1\rangle \qquad\qquad a^\dagger|n\rangle = s_n|n+1\rangle \ . \tag{14.64}$$

The expectation value of the operator $a^\dagger a$ in state $|n\rangle$ is

$$\langle n|a^\dagger a|n\rangle = n = \langle an|an\rangle = |r_n|^2\langle n-1|n-1\rangle = |r_n|^2$$
$$\Longrightarrow \quad r_n = \sqrt{n} \tag{14.65}$$
$$\langle n|a^\dagger a|n\rangle = n = \sqrt{n}\langle n|a^\dagger|n-1\rangle = \sqrt{n}s_{n-1}\langle n|n\rangle$$
$$\Longrightarrow \quad s_{n-1} = \sqrt{n}\,, \tag{14.66}$$

from which

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{14.67}$$

follows. The normalized eigenfunction of the n-th state thus is

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n|0\rangle\,, \tag{14.68}$$

and it's energy is

$$H|n\rangle = \underbrace{(n+\tfrac{1}{2})\hbar\omega}_{E_n}\,|n\rangle\,. \tag{14.69}$$

14.2 The Quantization of Fields

We want to transfer the quantization method (14.9) of point-particle mechanics to field theory, i.e. we want to postulate for the field amplitude $\psi(t,\boldsymbol{x})$ and for it's canonically conjugate momentum density $\pi(t,\boldsymbol{x})$ the same non-commutative algebra, which we defined in section 14.1.1 when we quantized the coordinates q_j and the momenta p_j of point particles. Formally this means, that we consider $\psi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$ as operators, which are acting onto the elements $|s\rangle$ of a Hilbert space.

But the quantized q_j and p_j are time-independent operators, i.e. operators in the Schrödinger-picture, while the quantized $\psi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$ are time-dependent operators in the Heisenberg-picture. It's not immediately clear, how we should handle the time-coordinate when the field is being quantized, whether for example we should postulate

$$[\psi(t_1,\boldsymbol{x}),\pi(t_2,\boldsymbol{y})] \stackrel{?}{=} i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta(t_1-t_2) \quad \text{is wrong!}$$

Actually this approach is wrong. This becomes obvious, if the field operators are transformed into the Schrödinger-picture by means of the time-evolution operator $U(t)$:

$$\psi(\boldsymbol{x}) \overset{(14.46a)}{=} U(t_1,0)\,\psi(t_1,\boldsymbol{x})\,U^{-1}(t_1,0) \tag{14.70}$$

$$\pi(\boldsymbol{x}) \overset{(14.46a)}{=} U(t_2,0)\,\pi(t_2,\boldsymbol{x})\,U^{-1}(t_2,0) \tag{14.71}$$

In the Schrödinger-picture, the commutator must definitely be

$$[\psi(\boldsymbol{x}),\pi(\boldsymbol{y})] = i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{14.72}$$

because of the analogy with (14.9). We compare this equation with

$$\begin{aligned}[\psi(t_1,\boldsymbol{x}),\pi(t_2,\boldsymbol{y})] \;&= \\ &= U^{-1}(t_1,0)\psi(\boldsymbol{x})U(t_1,0)\,U^{-1}(t_2,0)\pi(\boldsymbol{y})U(t_2,0) \\ &\quad - U^{-1}(t_2,0)\pi(\boldsymbol{y})U(t_2,0)\,U^{-1}(t_1,0)\psi(\boldsymbol{x})U(t_1,0)\;.\end{aligned} \tag{14.73}$$

If $t_2 = t_1$, then

$$[\psi(t_1,\boldsymbol{x}),\pi(t_1,\boldsymbol{y})] \;= \\ = U^{-1}(t_1,0)\Big(\underbrace{\psi(\boldsymbol{x})\,\pi(\boldsymbol{y}) - \pi(\boldsymbol{y})\,\psi(\boldsymbol{x})}_{i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}\Big)U(t_1,0) = i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$$

follows, because the time-evolution operator and the number $i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$ commute.

But in case $t_2 \neq t_1$, the time-evolution operators in (14.73) do not compensate, and thus the commutator isn't a number $\in \mathbb{C}$, but an operator. We cannot derive from the analogy with (14.9), what might be the value of this operator. Therefore we don't give any statement for the case $t_2 \neq t_1$, but restrict in the following explicitly to the case $t_2 = t_1$.

With the indices a, b marking the components of vector- or spinor-fields, we postulate — in analogy to (14.9) — for boson fields the non-commutative algebra

$$
\begin{aligned}
&[\psi_a(t,\boldsymbol{x}),\pi_b(t,\boldsymbol{y})] \equiv \\
&\qquad \equiv \psi_a(t,\boldsymbol{x})\pi_b(t,\boldsymbol{y}) - \pi_b(t,\boldsymbol{y})\psi_a(t,\boldsymbol{x}) \\
&\qquad = i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{ab} \\
&[\psi_a(t,\boldsymbol{x}),\psi_b(t,\boldsymbol{y})] = [\pi_a(t,\boldsymbol{x}),\pi_b(t,\boldsymbol{y})] = 0
\end{aligned}
\tag{14.74a}
$$

for fields with integer spin $0, 1, 2, 3, \ldots$ and same time argument t in both operators.

For fermion fields we postulate — deviating from (14.9) — the non-commutative algebra

$$
\begin{aligned}
&\{\psi_a(t,\boldsymbol{x}),\pi_b(t,\boldsymbol{y})\} \equiv \\
&\qquad \equiv \psi_a(t,\boldsymbol{x})\pi_b(t,\boldsymbol{y}) + \pi_b(t,\boldsymbol{y})\psi_a(t,\boldsymbol{x}) \\
&\qquad = i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\delta_{ab} \\
&\{\psi_a(t,\boldsymbol{x}),\psi_b(t,\boldsymbol{y})\} = \{\pi_a(t,\boldsymbol{x}),\pi_b(t,\boldsymbol{y})\} = 0
\end{aligned}
\tag{14.74b}
$$

for fields with half-integer spin $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and same time argument t in both operators.

$[\psi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})]$ is called commutator, $\{\psi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})\}$ is called anti-commutator. Different from the quantum mechanics of point particles, here in addition an anticommutator shows up. The reason for this difference is, that we want to formulate the quantum theory of fields – different from the quantum mechanics of point particles – such, that it firstly is complying with Special Relativity theory, and that secondly the quantized field never will have negative energy. According to the "spin-statistics-theorem" [40] by Pauli[2] these both conditions can only be met in combination, if the quantization rule (14.74) for bosons and fermions is different. We will not prove the spin-statistic-theorem in general, but we will check it for the examples of those fields, which we will quantize in the following chapters.

The canonical quantization (14.74) is not lorentz-invariant, because the delta function is not lorentz-invariant. The delta function makes sense only under an integral

[2] Wolfgang Pauli (1900-1958) made numerous seminal contributions to quantum theory.

$$\int_V \mathrm{d}^3x\, \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) = 1 \qquad \text{if } \boldsymbol{y} \text{ in } V. \tag{14.75}$$

The number 1 is lorentz-invariant. The infinitesimal volume d^3x is not lorentz-invariant, but shrinks due to relativistic length contraction by a factor γ^{-1}, see the notes at equation (7.22). The delta function's dimension is volume^{-1}. Therefore it is stretched by the factor γ under Lorentz transformations. The objective of a quantum field theory, which complies with Special Relativity theory, seems to be challenged by (14.74). Still we try to stick as closely as possible to the quantization rule (14.9), which Heisenberg found by guessing in summer 1925. We even accept some deviations, which still eventually will guide us to a quantum field theory, which is complying with Special Relativity theory.

We can't derive any justification for the quantization condition (14.74) from classical physics. It is a law of nature, which was found by guessing, and it can be justified only by the fact, that the conclusions drawn from it are complying with all experimental observations.

14.3 No "second"Quantization

We mention the notion "second quantization", because it is still showing up (though with declining frequency) in many books in the context of field quantization. This notion probably arose from the fact, that Schrödinger formulated his wave equation, which clearly is a quantum theory, with a classical, not quantized field ψ. When it became visible, that this ψ-field (and the analogous fields of other quantum theories) needed quantization, the notion "second quantization" for this procedure seemed reasonable. But actually there exists only one quantization each in the quantum mechanics of point particles, and in the quantum theory of fields.

The variables of classical point-particle mechanics are numbers (multiplied by physical units), i. e. commuting quantities. To quantize point-particle mechanics, a non-commutative algebra is postulated for these variables. To realize this non-commutative algebra mathematically, the variables are represented by matrices, or they are promoted to operators, which are acting onto the elements of a Hilbert space. This is the first and only quantization

on the way from the classical mechanics of point-particles to the quantum mechanics of point-particles.

In the first part of this book, we described some classical fields and their canonically conjugate momentum densities. To quantize these fields, the non-commutative algebra (14.74) is postulated. Due to this quantization, indirectly also the conserved quantities (like e. g. energy, momentum, charge) of a field, which are functions of the field's amplitude and it's canonically conjugate momentum density, get a non-commutative algebra. To realize this non-commutative algebra mathematically, the field's amplitude and it's canonically conjugate momentum density (and thus indirectly the products formed from them) are promoted to operators, which are acting onto the elements of a Hilbert space. This is the first and only quantization on the way from classical field theory to quantum field theory.

Weizsäcker reports [41, chap. 11, section 1.f.γ], that Heisenberg forbade him to use the notion second quantization, "because that notion will inhibit any correct understanding of the physical meaning of that procedure". In the same book Weizsäcker explains, why he eventually defied that interdiction, and not only appended a second quantization to the first quantization, but also inserted a zeroth quantization before the first quantization. We thoroughly follow Heisenberg's view of quantum field theory. We define any field first as a classical field, and then quantize it exactly once. The misleading notion "second quantization" will not be used in this book.

15 The Free Klein-Gordon Field

15.1 Quantization

The free Klein-Gordon field, which is not interacting with any other field, is described by (10.8). First we consider the quantization in the Schrödinger picture, i.e. according to (14.45) the amplitudes $\phi(\boldsymbol{x}) \equiv \phi(t=0,\boldsymbol{x})$ and their canonically conjugate momentum densities $\pi(\boldsymbol{x}) \equiv \pi(t=0,\boldsymbol{x})$. For these operators we postulate the non-commutative algebra (14.74a), because the Klein-Gordon field's spin is $s = 0$, i. e. it is a boson field. The non-commutative algebra of ϕ and π is realized by the Fourier coefficients $a_{\boldsymbol{k}}, a^*_{\boldsymbol{k}}, b_{\boldsymbol{k}}, b^*_{\boldsymbol{k}}$, which become operators $a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}, b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}$ with a non-commutative algebra, while all other factors within ϕ and π stay commuting numbers:

$$\phi(\boldsymbol{x}) \overset{(10.8)}{=} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}} \Big(a_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}\boldsymbol{x}\} + b^\dagger_{\boldsymbol{k}} \exp\{-i\boldsymbol{k}\boldsymbol{x}\} \Big) \tag{15.1a}$$

$$\pi(\boldsymbol{x}) \overset{(10.14)}{=} \sum_{\boldsymbol{k}} \frac{i\hbar^2\omega_{\boldsymbol{k}}}{\sqrt{N\Omega}} \Big(a^\dagger_{\boldsymbol{k}} \exp\{-i\boldsymbol{k}\boldsymbol{x}\} - b_{\boldsymbol{k}} \exp\{+i\boldsymbol{k}\boldsymbol{x}\} \Big) \tag{15.1b}$$

The signs of the exponents are consistent with the signs in equation (10.8). Just the time components in the exponents have disappeared. We factor out the exponential functions, rename the wave numbers $\boldsymbol{k}$ of the conjugate momentum density to $\boldsymbol{f}$, and rename their space-time coordinates from $\boldsymbol{x}$ to $\boldsymbol{y}$:

$$\phi(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{N\Omega}} \Big(a_{\boldsymbol{k}} + b^\dagger_{-\boldsymbol{k}} \Big) \exp\{+i\boldsymbol{k}\boldsymbol{x}\} \tag{15.2a}$$

$$\pi(\boldsymbol{y}) = \sum_{\boldsymbol{f}} \frac{i\hbar^2\omega_{\boldsymbol{f}}}{\sqrt{N\Omega}} \Big(a^\dagger_{\boldsymbol{f}} - b_{-\boldsymbol{f}} \Big) \exp\{-i\boldsymbol{f}\boldsymbol{y}\} \tag{15.2b}$$

These operators are inserted into (14.74a):

$$[\phi(\boldsymbol{x}),\pi(\boldsymbol{y})] = \sum_{\boldsymbol{k},\boldsymbol{f}} \frac{i\hbar^2\omega_{\boldsymbol{f}}}{N\Omega}\Big([a_{\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] - [a_{\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] + \\ + [b^\dagger_{\text{-}\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] - [b^\dagger_{\text{-}\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}]\Big)\exp\{+i(\boldsymbol{k}\boldsymbol{x}-\boldsymbol{f}\boldsymbol{y})\} \tag{15.3}$$

By comparison with the general quantization rule

$$[\phi(\boldsymbol{x}),\pi(\boldsymbol{y})] \overset{(14.74a)}{=} i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \\ \overset{(7.9)}{=} \sum_{\boldsymbol{k}} \frac{i\hbar}{\Omega}\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}$$

we find this condition for the commutators:

$$[a_{\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] - [a_{\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] + [b^\dagger_{\text{-}\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] - [b^\dagger_{\text{-}\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] = \frac{N\delta_{\boldsymbol{k}\boldsymbol{f}}}{\hbar\omega_{\boldsymbol{k}}} \tag{15.4}$$

Due to quantization, the algebra of fields, of conjugate momenta, and of Fourier-coefficients is changed, but not their dimensions. In the lines after (10.8) we stated, that the Fourier-coefficients of the Klein-Gordon field are dimension-less, and that the dimension of the normalization factor N is energy. The definition

$$N \equiv 2\hbar\omega_{\boldsymbol{k}} \tag{15.5}$$

is consistent with those assumptions. The factor 2 is merely convention. The adjoint of a commutator is

$$[a,b]^\dagger = (ab)^\dagger - (ba)^\dagger = b^\dagger a^\dagger - a^\dagger b^\dagger = [b^\dagger,a^\dagger]\ .$$

We take the adjoint of (15.4), exchange $\boldsymbol{k} \leftrightarrow \boldsymbol{f}$, and add the thus achieved term to (15.4):

$$[a_{\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] + [a_{\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] - [a_{\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] - [b^\dagger_{\text{-}\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] + \\ + [b^\dagger_{\text{-}\boldsymbol{k}},a^\dagger_{\boldsymbol{f}}] + [a_{\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] - [b^\dagger_{\text{-}\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] - [b^\dagger_{\text{-}\boldsymbol{k}},b_{\text{-}\boldsymbol{f}}] = 4\delta_{\boldsymbol{k}\boldsymbol{f}}$$

$$\Longrightarrow \quad [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] + [b_{-\boldsymbol{f}}, b^\dagger_{-\boldsymbol{k}}] = 2\delta_{\boldsymbol{kf}}$$

This must hold even in case $a = b$. Therefore

$$[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] = [a_{-\boldsymbol{f}}, a^\dagger_{-\boldsymbol{k}}] = [b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}] = [b_{-\boldsymbol{f}}, b^\dagger_{-\boldsymbol{k}}] = \delta_{\boldsymbol{kf}}$$

must hold. Consequently, *for arbitrary* $\boldsymbol{f}$ and $\boldsymbol{k}$ the two commutators in the middle of (15.4) must be zero, even in case $a = b$:

$$[a_{\boldsymbol{k}}, b_{\boldsymbol{f}}] = [b^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] = [a_{\boldsymbol{k}}, a_{\boldsymbol{f}}] = [a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] = [b_{\boldsymbol{k}}, b_{\boldsymbol{f}}] = [b^\dagger_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}] = 0$$

In total we get these rules:

$$\begin{aligned} [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] = [b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}] &= \delta_{\boldsymbol{kf}} \\ [a_{\boldsymbol{k}}, a_{\boldsymbol{f}}] = [b_{\boldsymbol{k}}, b_{\boldsymbol{f}}] &= [a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] = \\ = [b^\dagger_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}] &= [a_{\boldsymbol{k}}, b_{\boldsymbol{f}}] = [a^\dagger_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}] = 0 \\ [a_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}] &= [a^\dagger_{\boldsymbol{k}}, b_{\boldsymbol{f}}] = 0 \text{ if } b_{\boldsymbol{k}} \neq a_{\boldsymbol{k}} \\ \text{In any case } a \neq b^\dagger &\text{ is presupposed!} \end{aligned} \tag{15.6}$$

Note the constraint in the last line! We allow for the choice $a = b$, namely for the description of real, uncharged Klein-Gordon fields, but never for the choice $a = b^\dagger$.

The quantization changed the Fourier-coefficients into Fourier-operators with non-commutative algebra. In addition, it has two remarkable side-effects: As long as $\phi(x)$ was a state-function of quantum mechanics, it's normalization factor N could be chosen arbitrarily, provided the constraints (7.2) were respected. Due to quantization, the normalization is uniquely fixed as

$$\frac{1}{N}\Big([a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}] + [b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{f}}]\Big) \overset{(15.5),(15.6)}{=} \frac{\delta_{\boldsymbol{kf}}}{\hbar\omega_{\boldsymbol{k}}} \,. \tag{15.7}$$

Secondly — different from the coefficients of a Fourier-series expansion in classical physics — *all* Fourier-operators $a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}, b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}$ for *all* wave numbers

$\boldsymbol{k}$ must be different from zero because of (15.6), and their commutator is equal for all $\boldsymbol{k}$:

$$\left.\begin{aligned} a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}, b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}} &\neq 0 \\ [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}}] = [b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}] &= 1 \end{aligned}\right\} \text{for arbitrary } \boldsymbol{k} \tag{15.8}$$

Thus the field operators (15.1) differ significantly from classical Fourier-expansions, and this difference is caused by the quantization condition (14.74).

15.2 Field Operators in the Heisenberg Picture

So far we used the time-independent field operators $\phi(\boldsymbol{x})$ and $\phi^\dagger(\boldsymbol{x})$ in the Schrödinger picture. For the investigation of time-dependent processes it is often advantageous to change to the Heisenberg picture. We presuppose in the following, that the Hamilton operator is not explicitly time-dependent. Thus the time-evolution operator has the simple form $U(t,0) = \exp\{-(i/\hbar)Ht\}$.

$$\begin{aligned} \phi(x) \equiv \phi(t,\boldsymbol{x}) &\overset{(14.46a)}{=} \exp\{+\frac{i}{\hbar}Ht\}\,\phi(\boldsymbol{x})\,\exp\{-\frac{i}{\hbar}Ht\} \overset{(15.1)}{=} \\ &= \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(\underbrace{\exp\{+\frac{i}{\hbar}Ht\}a_{\boldsymbol{k}}\exp\{-\frac{i}{\hbar}Ht\}}_{a_{\boldsymbol{k}}(t)} \exp\{+i\boldsymbol{k}\boldsymbol{x}\} + \\ &\quad + \underbrace{\exp\{+\frac{i}{\hbar}Ht\}b^\dagger_{\boldsymbol{k}}\exp\{-\frac{i}{\hbar}Ht\}}_{b^\dagger_{\boldsymbol{k}}(t)} \exp\{-i\boldsymbol{k}\boldsymbol{x}\}\Big)\,. \end{aligned} \tag{15.9}$$

To find the time-dependent Fourier operators $a_{\boldsymbol{k}}(t)$ and $b^\dagger_{\boldsymbol{k}}(t)$ in the Heisenberg picture, we first compute

$$\begin{aligned} [H, a_{\boldsymbol{k}}] &= \sum_{\boldsymbol{f}} \hbar\omega_{\boldsymbol{f}} \Big(a^\dagger_{\boldsymbol{f}} a_{\boldsymbol{f}} a_{\boldsymbol{k}} - a_{\boldsymbol{k}} a^\dagger_{\boldsymbol{f}} a_{\boldsymbol{f}} \Big) = \\ &= \sum_{\boldsymbol{f}} \hbar\omega_{\boldsymbol{f}} \Big(a^\dagger_{\boldsymbol{f}} a_{\boldsymbol{f}} a_{\boldsymbol{k}} - (\delta_{\boldsymbol{f}\boldsymbol{k}} a_{\boldsymbol{f}} + a^\dagger_{\boldsymbol{f}} a_{\boldsymbol{f}} a_{\boldsymbol{k}}) \Big) = -\hbar\omega_{\boldsymbol{k}} a_{\boldsymbol{k}} \end{aligned} \tag{15.10a}$$

$$[H, a_{\boldsymbol{k}}^\dagger] = \sum_{\boldsymbol{f}} \hbar\omega_{\boldsymbol{f}} \Big(a_{\boldsymbol{f}}^\dagger a_{\boldsymbol{f}} a_{\boldsymbol{k}}^\dagger - a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{f}}^\dagger a_{\boldsymbol{f}} \Big) =$$

$$= \sum_{\boldsymbol{f}} \hbar\omega_{\boldsymbol{f}} \Big((a_{\boldsymbol{f}}^\dagger \delta_{\boldsymbol{f}\boldsymbol{k}} + a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{f}}^\dagger a_{\boldsymbol{f}}) - a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{f}}^\dagger a_{\boldsymbol{f}} \Big) = +\hbar\omega_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger \ . \tag{15.10b}$$

This can be written in the form

$$H a_{\boldsymbol{k}} = a_{\boldsymbol{k}} (H - \hbar\omega_{\boldsymbol{k}}) \tag{15.11a}$$

$$H a_{\boldsymbol{k}}^\dagger = a_{\boldsymbol{k}}^\dagger (H + \hbar\omega_{\boldsymbol{k}}) \ . \tag{15.11b}$$

Iteration of this procedure results into

$$H^n a_{\boldsymbol{k}} = H^{n-1} a_{\boldsymbol{k}} (H - \hbar\omega_{\boldsymbol{k}}) = a_{\boldsymbol{k}} (H - \hbar\omega_{\boldsymbol{k}})^n \tag{15.12a}$$

$$H^n a_{\boldsymbol{k}}^\dagger = a_{\boldsymbol{k}}^\dagger (H + \hbar\omega_{\boldsymbol{k}})^n \ . \tag{15.12b}$$

From this we conclude for the exponential functions:

$$\exp\{\frac{i}{\hbar} H t\}\, a_{\boldsymbol{k}} = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{i}{\hbar}\, t \Big)^n H^n a_{\boldsymbol{k}}$$

$$= a_{\boldsymbol{k}} \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{i}{\hbar}\, t \Big)^n (H - \hbar\omega_{\boldsymbol{k}})^n =$$

$$= a_{\boldsymbol{k}} \exp\{\frac{i}{\hbar} H t\} \exp\{-i\omega_{\boldsymbol{k}} t\} \tag{15.13a}$$

$$\exp\{\frac{i}{\hbar} H t\}\, a_{\boldsymbol{k}}^\dagger = a_{\boldsymbol{k}}^\dagger \exp\{\frac{i}{\hbar} H t\} \exp\{+i\omega_{\boldsymbol{k}} t\} \tag{15.13b}$$

We derived these relations under the assumption, that the Hamilton operator H is not explicitly time-dependent. Without proof we state, that these relations hold as well with explicitly time-dependent H:

$$a_{\boldsymbol{k}}(t) = U^{-1}(t,0)\, a_{\boldsymbol{k}}\, U(t,0) = a_{\boldsymbol{k}} \exp\{-i\omega_{\boldsymbol{k}} t\} \tag{15.14a}$$

$$a_{\boldsymbol{k}}^\dagger(t) = U^{-1}(t,0)\, a_{\boldsymbol{k}}^\dagger\, U(t,0) = a_{\boldsymbol{k}}^\dagger \exp\{+i\omega_{\boldsymbol{k}} t\} \tag{15.14b}$$

$$b_{\boldsymbol{k}}(t) = U^{-1}(t,0)\, b_{\boldsymbol{k}}\, U(t,0) = b_{\boldsymbol{k}} \exp\{-i\omega_{\boldsymbol{k}} t\} \tag{15.14c}$$

$$b_{\boldsymbol{k}}^{\dagger}(t) = U^{-1}(t,0)\, b_{\boldsymbol{k}}^{\dagger}\, U(t,0) = b_{\boldsymbol{k}}^{\dagger} \exp\{+i\omega_{\boldsymbol{k}} t\} \tag{15.14d}$$

Thus the field operator in the Heisenberg picture is

$$\begin{aligned}\phi(x) &= \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}}(t) \exp\{+i\boldsymbol{k}\boldsymbol{x}\} + b_{\boldsymbol{k}}^{\dagger}(t) \exp\{-i\boldsymbol{k}\boldsymbol{x}\} \Big) \\ &= \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} + b_{\boldsymbol{k}}^{\dagger} \exp\{+ikx\} \Big) \,. \end{aligned} \tag{15.15a}$$

By the same method, the field operators

$$\phi^{\dagger}(x) \overset{(15.1)}{=} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}}^{\dagger} \exp\{+ikx\} + b_{\boldsymbol{k}} \exp\{-ikx\} \Big) \tag{15.15b}$$

$$\pi(x) \overset{(15.1)}{=} \sum_{\boldsymbol{k}} i\hbar \sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\Omega}} \Big(a_{\boldsymbol{k}}^{\dagger} \exp\{+ikx\} - b_{\boldsymbol{k}} \exp\{-ikx\} \Big) \tag{15.15c}$$

$$\pi^{\dagger}(x) \overset{(15.1)}{=} -\sum_{\boldsymbol{k}} i\hbar \sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} - b_{\boldsymbol{k}}^{\dagger} \exp\{+ikx\} \Big) \tag{15.15d}$$

can be derived. From comparison with the classical fields (10.13) and (10.14) it's obvious, that we could have arrived directly at the field operators in the Heisenberg picture, if we had simply replaced the (not time-dependent) Fourier coefficients $a_{\boldsymbol{k}}, a_{\boldsymbol{k}}^{*}, b_{\boldsymbol{k}}, b_{\boldsymbol{k}}^{*}$ in the formulas of the classical time-dependent) fields by the (again not time-dependent) Fourier operators $a_{\boldsymbol{k}}, a_{\boldsymbol{k}}^{\dagger}, b_{\boldsymbol{k}}, b_{\boldsymbol{k}}^{\dagger}$.

The time- and position-dependent operators $\phi(x) = \phi(t, \boldsymbol{x})$ etc. are suggesting a fundamental consideration: In the quantum mechanics of point-particles, time and space are treated differently, because there is a position operator (which is identical to the position vector in the position representation), but there is no time operator. Time is merely a parameter in quantum mechanics.

In relativity theory, time and position are equivalent components of a four-vector. If we want to construct a quantum theory, which is compatible

with relativity theory, i. e. in which the appropriate balance in-between time and position is preserved, then we either must make time to an observable quantity, which is — like position — represented by an operator, or we must degrade position to a mere parameter like time.

We did choose the second alternative: In the field operators $\phi(x) = \phi(t, \boldsymbol{x})$ etc., the position $\boldsymbol{x}$ is, exactly like the time t, merely a parameter, but no operator. With both t and $\boldsymbol{x}$ now reduced to parameters, the theory has become a field theory. It's not just a coincidence, that the quantum theory, which is compatible with relativity theory, is being constructed as a field theory.

15.3 Conserved Quantities

Replacing the classical Fourier-coefficients in (10.17) by the quantized Fourier-operators (15.6), we get these components of the quantized ES-tensor:

$$\begin{aligned}
\mathcal{T}^{\rho\sigma} \overset{(10.17)}{=} \sum_{\boldsymbol{k},\boldsymbol{f}} \frac{c^2\hbar^2}{2\Omega\hbar\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}} \Big[& i^2(k^\rho f^\sigma + f^\rho k^\sigma)\Big(\\
& - a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{f}} \exp\{+i(k-f)x\} + a^\dagger_{\boldsymbol{k}} b^\dagger_{\boldsymbol{f}} \exp\{+i(k+f)x\} + \\
& + b_{\boldsymbol{k}} a_{\boldsymbol{f}} \exp\{-i(k+f)x\} - b_{\boldsymbol{k}} b^\dagger_{\boldsymbol{f}} \exp\{-i(k-f)x\}\Big) - \\
& - g^{\rho\sigma} i^2 k_\mu f^\mu \Big(\\
& - a^\dagger_{\boldsymbol{f}} a_{\boldsymbol{k}} \exp\{+i(f-k)x\} + a^\dagger_{\boldsymbol{f}} b^\dagger_{\boldsymbol{k}} \exp\{+i(f+k)x\} + \\
& + b_{\boldsymbol{f}} a_{\boldsymbol{k}} \exp\{-i(f+k)x\} - b_{\boldsymbol{f}} b^\dagger_{\boldsymbol{k}} \exp\{-i(f-k)x\}\Big) + \\
& + g^{\rho\sigma} \frac{m^2c^2}{\hbar^2} \Big(\\
& a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{f}} \exp\{+i(k-f)x\} + a^\dagger_{\boldsymbol{k}} b^\dagger_{\boldsymbol{f}} \exp\{+i(k+f)x\} + \\
& + b_{\boldsymbol{k}} a_{\boldsymbol{f}} \exp\{-i(k+f)x\} + b_{\boldsymbol{k}} b^\dagger_{\boldsymbol{f}} \exp\{-i(k-f)x\}\Big)\Big] \qquad (15.16)
\end{aligned}$$

Integration over the normalization volume Ω leads to

$$T^{\rho\sigma} \equiv \int_{\Omega} \mathrm{d}^3x\, \mathcal{T}^{\rho\sigma} \overset{(10.22)}{=} \sum_{\boldsymbol{k}} \frac{c^2\hbar^2}{\hbar\omega_{\boldsymbol{k}}} k^\rho k^\sigma \left(a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b_{\boldsymbol{k}} b^\dagger_{\boldsymbol{k}}\right) \tag{15.17a}$$

$$\overset{(15.6)}{=} \sum_{\boldsymbol{k}} \frac{c^2\hbar^2}{\hbar\omega_{\boldsymbol{k}}} k^\rho k^\sigma \left(a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + \underbrace{[b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}]}_{1}\right) . \tag{15.17b}$$

Now we need to discriminate in-between discrete and continuous fields. A field is called "discrete", if it has a discrete substrate. Example: The substrate of the field of sound waves (phonons) in a solid is the discrete crystal grid of the atoms, which are constituting the solid. $\sum_{\boldsymbol{k}}$ therefore runs in case of phonon fields (and all other discrete fields) only over the first Brillouin zone. Wave vectors $\boldsymbol{k}$ of other Brillouin zones would be redundant; hence they are ignored in the summation. Consequently (15.17) is always finite in case of discrete fields.

In case of continuous fields, however, (15.17) diverges, because continuous fields have no discrete substrate. Therefore the sum in (15.17) is running up to infinitely large wave numbers. Due to $\sum_{\boldsymbol{k}}[b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}] = \infty$, there is an infinite adder to each component of $T^{\rho\sigma}$. This holds true even in the vacuum state $|0\rangle$, in which no field quantum at all is excited, i. e. in which due to

$$\langle 0|a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}|0\rangle = \langle 0|b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}}|0\rangle \overset{(14.63)}{=} 0$$

two of the three terms in (15.17b) are zero.

All elementary quantum fields (e. g. the electron/positron field, the electromagnetic field, and all other fields of the standard model of elementary particles), but no other fields, are continuous quantum fields. "Elementary quantum field" and "continuous quantum field" therefore are synonymous notions.

In [42], good reasons are listed for the assumption, that the computation of the ES-tensor according to (10.15) is correct only in case of classical and discrete Klein-Gordon fields, but not in case of elementary Klein-Gordon fields. Instead of (10.15) we postulate ad-hoc, i. e. as a law of nature, for

classical and discrete Klein-Gordon fields:

$$\mathcal{T}^{\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\,\partial^\sigma\phi + \partial^\sigma\phi^*\,\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi^*)} - g^{\rho\sigma}\mathcal{L} \tag{15.18a}$$

elementary Klein-Gordon fields:

$$\mathcal{T}^{\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\,\partial^\sigma\phi + \partial^\sigma\phi^*\,\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi^*)} - g^{\rho\sigma}\mathcal{L} - Y \tag{15.18b}$$

$Y \equiv$ the sum of all terms in (15.18a) which do not depend on the particle-number operators $a^\dagger_{\boldsymbol{k}}\,a_{\boldsymbol{k}}$ or $b^\dagger_{\boldsymbol{k}}\,b_{\boldsymbol{k}}$

(15.18a) is identical to (10.15). (15.18b), on the other hand, has not been derived but was found by guessing, i. e. it is a law of nature. Like any law of nature, (15.18) can be justified by nothing else than the fact, that it complies with all experiments and observations. It is proved in [42], that (15.18) meets this criterion, while (10.15) is disproved by astronomical observations, and furthermore doesn't comply with special relativity theory.

As an alternative to (15.18b), some authors postulate "normal order" of the operators in (15.17a), which is marked by colons:

$$:a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}: \;\equiv\; a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} \quad \text{for bosons} \tag{15.19a}$$

$$:a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}}: \;\equiv\; -a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} \quad \text{for fermions} \tag{15.19b}$$

Normal order means, that in case of elementary fields all operator products are — ignoring (15.6) — re-arranged such, that all creation operators are placed left of the annihilation operators. Furthermore a factor -1 is inserted each time when a fermion creation operator is swapped with a fermion annihilation operator. In case of Klein-Gordon fields, normal order is indeed equivalent to (15.18b). But in case of elementary fields with spontaneously broken symmetry (the Higgs field, described in chapter 29, is an example for that type of fields) only (15.18b) gives correct results. Furthermore normal order is merely a formal trick, while (15.18) is suggested by plausible physical reasons, explicated in [42].

Using (15.18), we get instead of (15.17) this ES-tensor:

$$T^{\rho\sigma} \equiv \int\limits_{\Omega} \mathrm{d}^3x\, \mathcal{T}^{\rho\sigma} = \sum_{\boldsymbol{k}} \frac{c^2\hbar^2}{\hbar\omega_{\boldsymbol{k}}}\, k^\rho k^\sigma \cdot$$

$$\cdot \begin{cases} \left(a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + 1\right) \text{ for discrete fields} & \text{(15.20a)} \\ \left(a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}}\right) \text{ for elementary fields} & \text{(15.20b)} \end{cases}$$

The vacuum expectation value of (15.20) is

$$\langle 0|T^{\rho\sigma}|0\rangle \overset{(15.20)}{=} = \begin{cases} \displaystyle\sum_{\boldsymbol{k}} \frac{c^2\hbar^2 k^\rho k^\sigma}{\hbar\omega_{\boldsymbol{k}}} \overset{(7.18)}{=} \sum_{\boldsymbol{k}} \frac{c\hbar k^\rho k^\sigma}{\sqrt{\boldsymbol{k}^2 + m^2c^2/\hbar^2}} & \text{for discrete fields.} \\ 0 \text{ for elementary fields.} \end{cases} \tag{15.21}$$

m is the mass of one quantum or anti-quantum of the Klein-Gordon field. Note that we are discussing complex Klein-Gordon fields in this section. In section 15.7 we will find different results for real Klein-Gordon fields.

The Hamilton operator $H \equiv T^{00}$ of the Klein-Gordon field is

$$H = \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} \cdot \begin{cases} (a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + 1) \text{ for discrete fields.} \\ (a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}}) \text{ for elementary fields.} \end{cases} \tag{15.22}$$

This result should be compared to the Hamilton operator (14.57) of the harmonic oscillator of point-particle quantum mechanics! The Hamilton operator of the Klein-Gordon field is the sum of the Hamilton operators of infinitely many harmonic oscillators. For each wave number $\boldsymbol{k}$, which is compatible with the boundary conditions of the normalization volume, there exist two harmonic oscillators, whose ladder operators are named $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$ resp. $b_{\boldsymbol{k}}$ and $b^\dagger_{\boldsymbol{k}}$. Note that there is no zero-point energy in case of elementary fields.

The physical momentum operator $P^j \equiv T^{0j}/c$ of the Klein-Gordon field is

$$\boxed{P^j = \sum_{\boldsymbol{k}} \hbar k^j \cdot \begin{cases} (a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} + 1) \text{ for discrete fields.} \\ (a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}}) \text{ for elementary fields.} \end{cases}} \tag{15.23}$$

Angular-momentum operator: will be supplemented in a future release of this book, if I get around to do that ...

If in classical physics a particle with charge q interacts with a cloud of N particles which have charge q' each, then there will be a force between the test particle and the cloud, which is proportional to qQ with $Q = Nq'$. In quantum field theory, however, the particle will not interact with the cloud at once, but it will interact with the constituent particles of the cloud one by one. The interaction cross section for the scattering of the test particle by one single cloud particle is proportional to qq', and the probability for such interaction to happen is proportional to N. The total charge Q of the cloud, however, never shows up in any computation of quantum field theory. Therefore the conserved charge of the quantized Klein-Gordon field

$$Q \overset{(A.91)}{=} -q \sum_{\boldsymbol{k}} (a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} - b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} - \underbrace{[b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}]}_{1}) \tag{15.24}$$

which is computed in appendix A.13, is not of much relevance for quantum field theory. In particular we don't need to worry about the unphysical commutator term, because it never becomes effective in any computation. Q is a conserved quantity. Hence particles and antiparticles can only be created or annihilated in pairs of one particle and one antiparticle. Why, then, do exist in our universe much more particles than antiparticles? I don't know, nor does anybody else. This is one of the open MEGA-questions of physics.

In section 14.1.5, we defined $|0\rangle$ as the vacuum state of a harmonic oscillator. The state with n excited quanta, which is created by applying n times the creation operator onto the vacuum state, was called

$$|n\rangle \overset{(14.68)}{=} \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \ . \tag{15.25}$$

The eigenvalue of the particle-number operator

$$a^\dagger a|n\rangle \overset{(14.63)}{=} n|n\rangle \tag{15.26}$$

is indicating, how many quanta are excited in the oscillator's state $|n\rangle$.

We will use an analogous nomenclature for the Klein-Gordon field. As we are dealing with infinitely many oscillators of the types a and b, we add two further indices. $|0\rangle$ is again the vacuum state, and

$$|n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle = \frac{(b^\dagger_{\boldsymbol{k}})^{n_{b\boldsymbol{k}}} (a^\dagger_{\boldsymbol{f}})^{n_{a\boldsymbol{f}}} (a^\dagger_{\boldsymbol{g}})^{n_{a\boldsymbol{g}}}}{\sqrt{n_{b\boldsymbol{k}}! n_{a\boldsymbol{f}}! n_{a\boldsymbol{g}}!}} |0\rangle \tag{15.27}$$

is a state of the Klein-Gordon field, in which $n_{b\boldsymbol{k}}$ quanta of type b with wave number $\boldsymbol{k}$, $n_{a\boldsymbol{f}}$ quanta of type a with wave number $\boldsymbol{f}$, and $n_{a\boldsymbol{g}}$ quanta of type a with wave number $\boldsymbol{g}$ are excited. Because of

$$b_{\boldsymbol{f}}|0\rangle = a_{\boldsymbol{g}}|0\rangle \overset{(14.61)}{=} 0 \ , \tag{15.28}$$

the application of the particle-number operators results into

$$a^\dagger_{\boldsymbol{g}} a_{\boldsymbol{g}} |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle \overset{(14.63)}{=} n_{a\boldsymbol{g}} |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle \tag{15.29}$$

$$b^\dagger_{\boldsymbol{k}} b_{\boldsymbol{k}} |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle \overset{(14.63)}{=} n_{b\boldsymbol{k}} |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle \tag{15.30}$$

$$a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle \overset{(14.63)}{=} 0 \ . \tag{15.31}$$

To save writing efforts, we will constrain our considerations to elementary Klein-Gordon fields in the sequel. It's easy to extend the formulas to discrete fields, if needed. The Hamilton operator, the momentum operator, and the charge operator are composed of particle-number operators. Therefore the state vector $|n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle$ is as well an eigenstate of these operators:

$$\begin{aligned} H|n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle &\overset{(15.22)}{=} \sum_{\boldsymbol{l}} \hbar\omega_{\boldsymbol{l}} \left(a^\dagger_{\boldsymbol{l}} a_{\boldsymbol{l}} + b^\dagger_{\boldsymbol{l}} b_{\boldsymbol{l}}\right) |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle = \\ &= \left(n_{b\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} + n_{a\boldsymbol{f}} \hbar\omega_{\boldsymbol{f}} + n_{a\boldsymbol{g}} \hbar\omega_{\boldsymbol{g}}\right) |n_{b\boldsymbol{k}} n_{a\boldsymbol{f}} n_{a\boldsymbol{g}}\rangle \end{aligned} \tag{15.32}$$

As we stipulated

$$\hbar\omega_{\boldsymbol{k}} \overset{(7.18)}{=} +\sqrt{\hbar^2\boldsymbol{k}^2 + m^2c^4} \geq 0 \ , \tag{15.33}$$

and as $n_{a\boldsymbol{k}} \geq 0$ and $n_{b\boldsymbol{k}} \geq 0$ for arbitrary $\boldsymbol{k}$, surely $H|s\rangle \geq 0$ can never become negative for arbitrary state functions $|s\rangle$. This holds as well for fields $\phi \sim \exp\{+ikx\}$, which without quantization would have negative energy due to the energy operator $i\hbar\mathrm{d}/\mathrm{d}t$ of point-particle quantum mechanics.

Application of the momentum operator results into

$$\begin{aligned} \boldsymbol{P}|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle &\overset{(15.23)}{=} \sum_{\boldsymbol{l}} \hbar\boldsymbol{l}\Big(a^\dagger_{\boldsymbol{l}}a_{\boldsymbol{l}} + b^\dagger_{\boldsymbol{l}}b_{\boldsymbol{l}}\Big)|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle \\ &= \Big(n_{b\boldsymbol{k}}\hbar\boldsymbol{k} + n_{a\boldsymbol{f}}\hbar\boldsymbol{f} + n_{a\boldsymbol{g}}\hbar\boldsymbol{g}\Big)|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle \ , \end{aligned} \tag{15.34}$$

and application of the charge operator results into

$$\begin{aligned} Q|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle &\overset{(A.91)}{=} -q\sum_{\boldsymbol{l}}\left(a^\dagger_{\boldsymbol{l}}a_{\boldsymbol{l}} - b^\dagger_{\boldsymbol{l}}b_{\boldsymbol{l}}\right)|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle \\ &= \Big(n_{b\boldsymbol{k}}q - n_{a\boldsymbol{f}}q - n_{a\boldsymbol{g}}q\Big)\,|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle \ . \end{aligned} \tag{15.35}$$

At same wave number, quanta of type a and of type b have same energy and same momentum. The energy of quanta of *either* type always is larger or equal zero. As we presupposed the validity of the frequency-wavenumber relation (7.17) of special relativity theory, the rest mass of each field-quantum is identical, no matter whether it's type is a or b. Preliminary[1] we interpret the parameter m, which shows up in the Klein-Gordon equation, as the rest mass of a single quantum. The total rest mass of the Klein-Gordon field thus is an integer multiple of m. In the state $|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle$, it's total rest mass is $m\cdot(n_{b\boldsymbol{k}} + n_{a\boldsymbol{f}} + n_{a\boldsymbol{g}})$. Each quantum of type a has the charge $-q$, and each quantum of type b has the charge $+q$. The total charge of the Klein-Gordon field is an integer multiple of $\pm q$. In the state $|n_{b\boldsymbol{k}}n_{a\boldsymbol{f}}n_{a\boldsymbol{g}}\rangle$, it's total charge is $q\cdot(n_{b\boldsymbol{k}} - n_{a\boldsymbol{f}} - n_{a\boldsymbol{g}})$.

[1] In section 22, we will renormalize the mass and the charge of the field, and will thereby significantly modify the interpretation of these parameters.

15.4 Quanta of the Klein-Gordon field

It's common to use the name "particles" for the field quanta. Particles of type b are the antiparticles of the particles of type a. Particles and antiparticles differ by the sign of all of their charges. With respect to all other properties they are identical. In particular the mass of particles and of antiparticles is positive. If a particle is not charged at all (like for example the photon, the quantum of the electromagnetic field), then it doesn't have an antiparticle.

It must not be overlooked, that this usage of the notion "particle" differs significantly from the same notion in everyday speech. Field quanta are no "things". Therefore Bose-Einstein statistics hold for the quanta of the Klein-Gordon field, while for things — even if they are arbitrarily small things — Maxwell-Boltzmann statistics hold.

The field quanta $|1_{a\boldsymbol{k}}\rangle$ or $|1_{b\boldsymbol{k}}\rangle$ with sharply defined momentum $\hbar\boldsymbol{k}$ and sharply defined energy $\hbar\omega_{\boldsymbol{k}}$ are completely de-localized. They can be found with equal probability anywhere in the normalization volume. A quantum $|1_{a\boldsymbol{x}}\rangle$ of type a or a quantum $|1_{b\boldsymbol{x}}\rangle$ of type b, which is localized at position $\boldsymbol{x}$, can be constructed by application of the field operators (15.15) with $t \equiv 0$ (i. e. the field operators in the Schrödinger picture) onto the vacuum. Respecting $a_{\boldsymbol{k}}\,|0\rangle = b_{\boldsymbol{k}}\,|0\rangle = 0$, one gets

$$|1_{a\boldsymbol{x}}\rangle \equiv \phi^\dagger(\boldsymbol{x})\,|0\rangle \overset{(15.15)}{=} \sum_{\boldsymbol{k}} \underbrace{a^\dagger_{\boldsymbol{k}}\,|0\rangle}_{|1_{a\boldsymbol{k}}\rangle} \frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \tag{15.36a}$$

$$|1_{b\boldsymbol{x}}\rangle \equiv \phi(\boldsymbol{x})\,|0\rangle \overset{(15.15)}{=} \sum_{\boldsymbol{k}} \underbrace{b^\dagger_{\boldsymbol{k}}\,|0\rangle}_{|1_{b\boldsymbol{k}}\rangle} \frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}}\ . \tag{15.36b}$$

In section 14.1.5, we normalized the state functions $|n\rangle$ of the harmonic oscillator of quantum mechanics by $\langle n|n\rangle = 1$. Accordingly, we now choose

$$\langle 0|0\rangle \equiv 1 \tag{15.37a}$$

$$\langle 1_{a\boldsymbol{f}}|1_{a\boldsymbol{k}}\rangle = \langle 1_{b\boldsymbol{f}}|1_{b\boldsymbol{k}}\rangle \equiv \delta_{\boldsymbol{f}\boldsymbol{k}} \quad , \quad \langle 1_{a\boldsymbol{f}}|1_{b\boldsymbol{k}}\rangle = 0\ , \tag{15.37b}$$

and compute

$$\langle 1_{a\boldsymbol{y}}|1_{a\boldsymbol{x}}\rangle = \langle 1_{b\boldsymbol{y}}|1_{b\boldsymbol{x}}\rangle = \sum_{\boldsymbol{k},\boldsymbol{f}} \frac{\exp\{+i(\boldsymbol{f}\boldsymbol{y}-\boldsymbol{k}\boldsymbol{x})\}}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}}\,\delta_{\boldsymbol{f}\boldsymbol{k}}$$
$$= \sum_{\boldsymbol{k}} \frac{\exp\{+i\boldsymbol{k}(\boldsymbol{y}-\boldsymbol{x})\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \quad , \quad \langle 1_{a\boldsymbol{x}}|1_{b\boldsymbol{y}}\rangle = 0\ . \tag{15.37c}$$

This result differs from the delta function

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}} \exp\{\pm i\boldsymbol{k}(\boldsymbol{y}-\boldsymbol{x})\} \overset{(7.9)}{=} \delta^{(3)}(\boldsymbol{y}-\boldsymbol{x})$$

by the factor $2\hbar\omega_{\boldsymbol{k}}$ in the denominator. The advantage of this definition will become visible in the four-dimensional extension of (15.37c), which will be computed next.

Time-dependent field quanta $|1_{a\boldsymbol{x}t}\rangle$ and $|1_{b\boldsymbol{x}t}\rangle$, which are localized at $\boldsymbol{x}$, can be constructed by application of the time-dependent field operators (15.15) onto the vacuum:

$$|1_{a\boldsymbol{x}}\rangle = |1_{a\boldsymbol{x}t}\rangle \equiv \phi^\dagger(x)\,|0\rangle = \sum_{\boldsymbol{k}} \frac{\exp\{+ikx\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \underbrace{a^\dagger_{\boldsymbol{k}}\,|0\rangle}_{|1_{a\boldsymbol{k}}\rangle} \tag{15.38a}$$

$$|1_{b\boldsymbol{x}}\rangle = |1_{b\boldsymbol{x}t}\rangle \equiv \phi(x)\,|0\rangle = \sum_{\boldsymbol{k}} \frac{\exp\{+ikx\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \underbrace{b^\dagger_{\boldsymbol{k}}\,|0\rangle}_{|1_{b\boldsymbol{k}}\rangle}\ . \tag{15.38b}$$

The particle is localized in space only, but not in time, because there is only a sum over $\boldsymbol{k}$, but no integral over k^0. The time t in (15.38) is a running parameter. The state functions $|1_{a\boldsymbol{x}}\rangle$ and $|1_{b\boldsymbol{x}}\rangle$ are Eigenfunctions of the time-dependent position operator $\boldsymbol{x}(t)$:

$$\boldsymbol{x}(t)|1_{b\boldsymbol{y}t}\rangle = \boldsymbol{y}|1_{b\boldsymbol{y}t}\rangle \tag{15.39}$$

They can be transformed into the Eigenfunctions of the time-independent position operator by

$$
\begin{aligned}
|1_{a\boldsymbol{x}}\rangle &\overset{(14.32)}{=} \exp\{-\frac{i}{\hbar}Ht\}\,|1_{a\boldsymbol{x}t}\rangle \\
&\overset{(15.38a)}{=} \sum_{\boldsymbol{k}} \frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}\}\,\exp\{+i\omega_{\boldsymbol{k}}t\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \underbrace{\exp\{-\frac{i}{\hbar}Ht\}\,|1_{a\boldsymbol{k}}\rangle}_{\exp\{-i\omega_{\boldsymbol{k}}t\}\,|1_{a\boldsymbol{k}}\rangle} \\
&= \sum_{\boldsymbol{k}} \frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}}\,|1_{a\boldsymbol{k}}\rangle\ . \qquad (15.40)
\end{aligned}
$$

This is again concordant with (15.36a).

The matrix elements of the field quanta (15.38) are

$$
\begin{aligned}
\langle 1_{a\boldsymbol{y}\tau}|1_{a\boldsymbol{x}t}\rangle &= \langle 0|\,\phi(\boldsymbol{y}\tau)\phi^{\dagger}(\boldsymbol{x}t)\,|0\rangle\ = \\
&= \sum_{\boldsymbol{f},\boldsymbol{k}} \frac{\exp\{-i(fy-kx)\}}{\Omega 2\hbar\,\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \underbrace{\langle 1_{a\boldsymbol{f}}|1_{a\boldsymbol{k}}\rangle}_{\delta_{\boldsymbol{f}\boldsymbol{k}}} \\
&= \sum_{\boldsymbol{k}} \frac{\exp\{-ik(y-x)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \qquad (15.41a) \\
\langle 1_{b\boldsymbol{y}\tau}|1_{b\boldsymbol{x}t}\rangle &= \langle 0|\,\phi^{\dagger}(\boldsymbol{y}\tau)\phi(\boldsymbol{x}t)\,|0\rangle\ = \\
&= \sum_{\boldsymbol{f},\boldsymbol{k}} \frac{\exp\{-i(fy-kx)\}}{\Omega 2\hbar\,\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \underbrace{\langle 1_{b\boldsymbol{f}}|1_{b\boldsymbol{k}}\rangle}_{\delta_{\boldsymbol{f}\boldsymbol{k}}} \\
&= \sum_{\boldsymbol{k}} \frac{\exp\{-ik(y-x)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \qquad (15.41b) \\
\langle 1_{b\boldsymbol{y}\tau}|1_{a\boldsymbol{x}t}\rangle &= 0 \qquad (15.41c)
\end{aligned}
$$

These matrix elements are Lorentz-invariant, because the argument of the exponential function — being a product of Lorentz vectors — is Lorentz-invariant, and the same holds for the product $\hbar\omega_{\boldsymbol{k}}\Omega$, as discussed at the end of chapter 7 .

15.5 The quantized Greens Function

In section 12.1 we described Greens functions as classical waves, which are spreading from sources $j(y)$ according to Huygen's principle, and then

superpose at space-time point x. Their quantized form follows from the matrix elements, which have been computed in the previous section. The matrix element

$$\begin{aligned}\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle &\overset{(15.41a)}{=} \langle 1_{ax}|1_{ay}\rangle \;= \\ &= \sum_{\boldsymbol{k}} \frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \overset{(12.11a)}{=} G^a(x-y)\end{aligned} \tag{15.42a}$$

is interpreted as the probability amplitude of a process, in which a particle of type a (the antiparticle of b) is created at time y^0 at position $\boldsymbol{y}$, then moves to $\boldsymbol{x}$, and there is annihilated at time x^0. The matrix element

$$\begin{aligned}\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle &\overset{(15.41b)}{=} \langle 1_{by}|1_{bx}\rangle \;= \\ &= \sum_{\boldsymbol{k}} \frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \overset{(12.11b)}{=} G^b(y-x)\end{aligned} \tag{15.42b}$$

is interpreted as the probability amplitude of a process, in which a particle of type b (the antiparticle of a) is created at time x^0 at position $\boldsymbol{x}$, then moves to $\boldsymbol{y}$, and there is annihilated at time y^0. Comparison of (15.42) with (12.11) and (12.14) results into the quantized propagator

$$\begin{aligned}G(x-y) &\overset{x^0\neq y^0}{=} \theta(x^0-y^0)\,G^a(x-y) + \theta(y^0-x^0)\,G^b(y-x) \\ &= \theta(x^0-y^0)\langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle + \\ &\qquad + \theta(y^0-x^0)\langle 0|\,\phi^\dagger(y)\phi(x)\,|0\rangle \\ &\equiv \langle 0|\,T\phi(x)\phi^\dagger(y)\,|0\rangle\ .\end{aligned} \tag{15.43}$$

For the time-order operator T in the last line holds the following

Definition: The time-order operator T always orders the operators within a product according to their sequence in time, starting with the latest factor (t maximal) at the left side, and ending with the earliest factor (t minimal) at the right side. A factor (-1) is to be inserted for each permutation of two fermion operators (but not for a permutation of boson operators). (15.44)

The Klein-Gordon field, which we are considering in this chapter, is a boson field.

The Huygens wave, which is spreading from the past (y^0) to the present (x^0), is due to quantization converted to a particle, which is described by the propagator $G^a(x-y)$, and is spreading from the past (y^0) to the present (x^0). The Huygens wave, which is spreading from the future (y^0) to the present (x^0), is due to quantization converted to an antiparticle, which is described by the propagator $G^b(y-x)$, and is spreading from the present (x^0) to the future (y^0). The spreading of waves forward and backwards in time in the classical term (12.3) is re-shaped to a plausible picture due to the concept of anti-matter. We will see immediately, that furthermore the waves, which are spreading — from the viewpoint of classical physics — outside the forward- or backwards-lightcones, not only are acceptable in quantum field theory, but indeed are necessary to make quantum theory and relativity theory mutually compatible.

$G(x-y)$, and $G^a(x-y)$, and $G^b(y-x)$ all are called propagators, because all of them are describing the propagation of fields in-between different points in space-time. The propagators $G^a(x-y)$ and $G^b(y-x)$ are solutions of the homogeneous equation (12.1), while the propagator $G(x-y)$ is a solution of the inhomogeneous equation (12.4). $G(x-y)$ is the *Feynman-propagator*, which is playing a central role in the description of interacting quantum fields.

15.6 Causality

For the subject of this section, the notion "microcausality" is widely used. The prefix "micro" is meaningless, microcausality is nothing other than causality. Relativity theory postulates, that an event at space-time point x and an event at space-time point y can not affect each other, if the distance between x and y is space-like. The distance is space-like, if x is not in the forward- or backwards-lightcone of y, i.e. if $(x-y)^2 < 0$.

The events, in whose causal dependence – or rather causal independence – we are interested, are the creation or annihilation of particles at different points of four-dimensional space-time. Let the Klein-Gordon field be in state $|s\rangle$. Due to $\phi(x)|s\rangle$, a particle a is annihilated and it's antiparticle

b is created at time x^0/c at position $\boldsymbol{x}$. Due to $\phi^\dagger(y)|s\rangle$, a particle a is created and it's antiparticle b is annihilated at time y^0/c at position $\boldsymbol{y}$. If the relation of the space-time points x and y is space-like, then the events at x and y can not influence one another. Therefore the expectation value of their commutator must be zero:

$$\begin{aligned}
&\langle s|\,[\phi(x),\phi^\dagger(y)]\,|s\rangle \;= \\
&= \sum_{\boldsymbol{k},\boldsymbol{f}} \frac{1}{2\Omega\hbar\sqrt{\omega_{\boldsymbol{k}}\,\omega_{\boldsymbol{f}}}} \Big(\langle s|\, \underbrace{[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{f}}]}_{\delta_{\boldsymbol{k}\boldsymbol{f}}} \,|s\rangle \exp\{-i(kx-fy)\} \\
&\qquad\qquad - \langle s|\, \underbrace{[b_{\boldsymbol{f}}, b^\dagger_{\boldsymbol{k}}]}_{\delta_{\boldsymbol{k}\boldsymbol{f}}} \,|s\rangle \exp\{-i(fy-kx)\} \Big) \\
&= \sum_{\boldsymbol{k}} \frac{1}{2\Omega\hbar\omega_{\boldsymbol{k}}} \Big(\exp\{-ik(x-y)\} - \exp\{-ik(y-x)\} \Big) \langle s|s\rangle \\
&\overset{(15.42)}{=} \Big(G^a(x-y) - G^b(y-x) \Big) \langle s|s\rangle
\end{aligned} \tag{15.45}$$

The commutator firstly is no operator but a number. Therefore it can be factored out of the product $\langle s|s\rangle$. The commutator secondly is lorentzinvariant, because the product $\Omega\hbar\omega_{\boldsymbol{k}}$ is — as discussed between (7.21) and (7.26) — lorentzinvariant, and the same holds for the scalar product $k(x-y)$ of two Lorentz four-vectors. We compute the commutator in a primed coordinate system′, in which $x'^0 - y'^0 = 0$. Note that such a coordinate system only exists because the relation of the space-time points x and y is space-like, that is to say $(x-y)^2 < 0$. If y would be in the forward- or backwards-lightcone of x, then $x'^0 - y'^0 = 0$ could not be achieved by any proper Lorentz transformation.

$$\begin{aligned}
[\phi'(x'),\phi'^\dagger(y')] &= \sum_{\boldsymbol{k}'} \frac{1}{2\Omega'\hbar\omega_{\boldsymbol{k}'}} \cdot \\
&\cdot \Big(\exp\{+i\boldsymbol{k}'(\boldsymbol{x}'-\boldsymbol{y}')\} - \exp\{-i\boldsymbol{k}'(\boldsymbol{x}'-\boldsymbol{y}')\} \Big) = 0
\end{aligned} \tag{15.46}$$

As the sum is running symmetrically over all positive and negative $\boldsymbol{k}'$, and because $\omega_{\boldsymbol{k}'} = \omega_{-\boldsymbol{k}'}$, we may exchange $\boldsymbol{k}'$ and $-\boldsymbol{k}'$ in the second summand.

Thus the commutator is indeed zero, in compliance with relativity theory.

If the Klein-Gordon field's quantization had been done with the anticommutator (14.74b), but not with the commutator (14.74a), then the result of (15.45) would not be zero, violating the causality criterion of relativity theory. The causality criterion of relativity theory is forcing us to quantize all fields with integer spin (inclusive spin zero) due to the commutator (14.74a), and to quantize all fields with half-integer spin due to the anticommutator (14.74b). We just proved this rule for the example of the Klein-Gordon field (spin 0), and we will confirm it again for the example of the Dirac field (spin $\frac{1}{2}$). We will not outline the general proof of that rule, which also is called the "spin-statistics-theorem".

The result

$$\begin{aligned}\langle s|\,[\phi(x),\phi^\dagger(y)]\,|s\rangle &\overset{(15.45)}{=}\\ &= \Big(\underbrace{G^a(x-y)}_{\neq 0} - \underbrace{G^b(y-x)}_{\neq 0}\Big)\langle s|s\rangle \overset{(x-y)^2<0}{\overset{(15.46)}{=}} 0\end{aligned} \tag{15.47}$$

is suggesting the following interpretation: The commutator's expectation value is zero, if the relation of the space-time points x and y is space-like. But it is quite remarkable, how this result is accomplished. The probability amplitude $G^a(x-y)$ for a particle of type a (which is the antiparticle of b), to spread — in apparent contradiction to relativity theory — faster than light from y to x, is different from zero. And the probability amplitude $G^b(y-x)$ for a particle of type b (which is the antiparticle of a), to spread — in apparent contradiction to relativity theory — faster than light from x to y, is different from zero. But as both probability amplitudes are exactly equal, the commutator's probability amplitude — that is to say the probability amplitude of a measurable interaction in-between the space-time points x and y — is zero, in compliance with relativity theory.

In case of an uncharged (real) Klein-Gordon field, $b = a$, $b^\dagger = a^\dagger$, i.e. this field has no anti-field. In this case, the compliance with the causality criterion is secured due to the exchange of two particles of type a between the space-time points x and y.

The particles, which are moving — seemingly not impressed by relativity

theory — with arbitrary velocity, are "virtual" particles. That is to say, they are existing on the theorist's paper, but they are not observed by a particle detector. Particles and antiparticles, which are observed by means of appropriate detectors, *always* are moving at a velocity $\leq$ the speed of light. The ontological status of virtual particles is controversial. Whether "out there" something is existing, which corresponds to (15.47), or whether that formula is merely an abstract mathematical tool for the computation of the probability of observed events, but has no immediate correlation in reality[2], can not be clarified by experiments. Observed particles and virtual particles are behaving fundamentally different.

15.7 The real Klein-Gordon Field

Thus far, all considerations in this chapter were related to the complex, charged Klein-Gordon field. The simplification to the real field isn't difficult. The field equation and the conserved quantities of the classical (not quantized) real field have been compiled in section 10.5.

The field equation (10.34) of the classical real field is identical to the field equation (10.11a) of the classical complex field. Therefore the field-operators of the quantized real field follow from (15.1) resp. (15.15) due to exchange of $b_{\boldsymbol{k}}$ by $a_{\boldsymbol{k}}$ and exchange of $b^\dagger_{\boldsymbol{k}}$ by $a^\dagger_{\boldsymbol{k}}$:

$$\phi(x) \overset{(15.15a)}{=} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} + a^\dagger_{\boldsymbol{k}} \exp\{+ikx\} \Big) \tag{15.48a}$$

The canonically conjugate momentum density (10.14) of the classical complex field is transcribed to the canonically conjugate momentum density (10.35) of the classical real field due to exchange of ϕ^* by ϕ in equation (10.14b). In case of the quantized fields, this is again corresponding to the

[2] It is extremely difficult (if not impossible), to give a plausible definition of reality, without running into circular arguments. Remarkably, the definition of the notion reality is completely superfluous for physics. Physics is not describing, *what* it's objects are, but *how* it's objects mutually are interacting, and what their structural relations are. Physics is "explaining" the structural relations of observations due to the structural relations of physical objects. Whether a physical object is "real", is irrelevant for physics.

replacements $b_{\boldsymbol{k}} \to a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}^\dagger \to a_{\boldsymbol{k}}^\dagger$:

$$\pi(x) \overset{(15.15c)}{=} \sum_{\boldsymbol{k}} i\hbar \sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2\Omega}} \left(a_{\boldsymbol{k}}^\dagger \exp\{+ikx\} - a_{\boldsymbol{k}} \exp\{-ikx\} \right) \tag{15.48b}$$

Which modifications of the ES-tensor (15.20) of the complex Klein-Gordon field are needed, to get the respective result for the real Klein-Gordon field? From the comparison of the ES-tensor (10.16a) of the complex classical Klein-Gordon field and the ES-tensor (10.36) of the classical real Klein-Gordon field we conclude, that simply $b_{\boldsymbol{k}}$ must be replaced by $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}^\dagger$ by $a_{\boldsymbol{k}}^\dagger$, and that a multiplier $1/2$ must be inserted, to get the ES-tensor of the real field:

$$T^{\rho\sigma} \equiv \int_\Omega \mathrm{d}^3x\, \mathcal{T}^{\rho\sigma} = \frac{1}{2} \sum_{\boldsymbol{k}} \frac{c^2\hbar^2}{\hbar\omega_{\boldsymbol{k}}} k^\rho k^\sigma \cdot$$

$$\cdot \begin{cases} \left(2a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + 1 \right) \text{ for discrete fields} & \text{(15.49a)} \\ 2a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} \text{ for elementary fields} & \text{(15.49b)} \end{cases}$$

Thereby we get this vacuum expectation value:

$$\langle 0|T^{\rho\sigma}|0\rangle = \\ = \begin{cases} \displaystyle\sum_{\boldsymbol{k}} \frac{c^2\hbar^2 k^\rho k^\sigma}{2\hbar\omega_{\boldsymbol{k}}} \overset{(7.18)}{=} \sum_{\boldsymbol{k}} \frac{c\hbar k^\rho k^\sigma}{2\sqrt{\boldsymbol{k}^2 + m^2c^2/\hbar^2}} \text{ for discrete fields.} \\ 0 \text{ for elementary fields.} \end{cases} \tag{15.50}$$

The vacuum expectation values of the ES-tensor differ from the respective result (15.21) of the charged field by nothing than the factor $1/2$. This difference is caused by the fact, that both the zero-point oscillations of the discrete field and the zero-point oscillations of the discrete anti-field add to (15.21), while the real field has no anti-field. For elementary Klein-Gordon fields the vacuum expectation values of the ES-tensor are zero, no matter whether the fields are complex or real.

16 The Free Dirac Field

16.1 Quantization

The Dirac field's spin is $\frac{1}{2}$. Thus the quantization method (14.74b) must be applied. The field amplitudes and the Fourier-coefficients of the Dirac equation's general solutions become operators upon quantization:

$$\psi(x) \overset{(8.83a)}{=} \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{\Omega N}} \Big({}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\} \Big) \tag{16.1a}$$

The same holds for the canonically conjugate momentum density:

$$\pi(y) \overset{(8.83b)}{=} i\hbar\psi^{\dagger}(y) = \sum_{\boldsymbol{f},s} \frac{i\hbar}{\sqrt{\Omega N}} \Big({}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}u^{\boldsymbol{f}\dagger} \exp\{+ify\} + {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\boldsymbol{f}\dagger} \exp\{-ify\} \Big) \tag{16.1b}$$

$r, s = 1, 2$ are the polarization indices of the fields. These operators must meet the canonical quantization condition

$$\begin{aligned}
&\{\psi_a(t,\boldsymbol{x}), \pi_b(t,\boldsymbol{y})\} \equiv \psi_a(t,\boldsymbol{x})\,\pi_b(t,\boldsymbol{y}) + \pi_b(t,\boldsymbol{y})\,\psi_a(t,\boldsymbol{x}) = \\
&= \sum_{\boldsymbol{k},\boldsymbol{f},r,s} \frac{i\hbar}{\Omega N} \Big[\{ {}^{r}a_{\boldsymbol{k}}, {}^{s}a^{\dagger}_{\boldsymbol{f}} \}\, {}^{r}u^{\boldsymbol{k}}_{a}\, {}^{s}u^{\boldsymbol{f}\dagger}_{b} \exp\{-i(kx - fy)\} + \\
&\quad + \{ {}^{r}a_{\boldsymbol{k}}, {}^{s}b_{-\boldsymbol{f}} \}\, {}^{r}u^{\boldsymbol{k}}_{a}\, {}^{s}v^{-\boldsymbol{f}\dagger}_{b} \exp\{-i(k^0x_0 + f^0y_0 - \boldsymbol{kx} + \boldsymbol{fy})\} + \\
&\quad + \{ {}^{r}b^{\dagger}_{\boldsymbol{k}}, {}^{s}a^{\dagger}_{-\boldsymbol{f}} \}\, {}^{r}v^{\boldsymbol{k}}_{a}\, {}^{s}u^{-\boldsymbol{f}\dagger}_{b} \exp\{+i(k^0x_0 + f^0y_0 - \boldsymbol{kx} + \boldsymbol{fy})\} + \\
&\quad + \{ {}^{r}b^{\dagger}_{\boldsymbol{k}}, {}^{s}b_{\boldsymbol{f}} \}\, {}^{r}v^{\boldsymbol{k}}_{a}\, {}^{s}v^{\boldsymbol{f}\dagger}_{b} \exp\{+i(kx - fy)\} \Big] \overset{(14.74b)}{=} i\hbar\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\delta_{ab}\ .
\end{aligned} \tag{16.2}$$

The indices a, b are marking the spinor-components of the fields, and the time argument t must be identical in both operators. As the summations

are running symmetrically over all wavenumbers $\boldsymbol{k}$ and $\boldsymbol{f}$, the sequence of terms could be changed in the second and third term on the left side. On the left side of the equation, the delta-function

$$\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \overset{(7.9)}{=} \frac{1}{\Omega}\sum_{\boldsymbol{k}} \exp\{\pm i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \tag{16.3}$$

can be identified. Consequently the anticommutators must be $\sim \delta_{\boldsymbol{kf}}$. But then the second and third term on the left side of (16.2) are time-dependent, while the right side is independent of time. Thus we must have

$$\{\,{}^{r}a_{\boldsymbol{k}}\,,{}^{s}b_{-\boldsymbol{k}}\} = \{\,{}^{r}b^{\dagger}_{\boldsymbol{k}}\,,{}^{s}a^{\dagger}_{-\boldsymbol{k}}\} = 0\ . \tag{16.4a}$$

Now we use the relations

$$\sum_{r=1}^{2}\left({}^{r}u_a^{\boldsymbol{k}}\,{}^{r}u_b^{\boldsymbol{k}\dagger} + {}^{r}v_a^{-\boldsymbol{k}}\,{}^{r}v_b^{-\boldsymbol{k}\dagger}\right) \overset{(A.61)}{=} 2\hbar\omega_{\boldsymbol{k}}\,\delta_{ab}$$

$$\sum_{r=1}^{2}\sum_{s=1}^{2}\left({}^{r}u_a^{\boldsymbol{k}\dagger}\,{}^{s}u_b^{\boldsymbol{k}} + {}^{r}v_a^{\boldsymbol{k}\dagger}\,{}^{s}v_b^{\boldsymbol{k}} + {}^{r}u_a^{-\boldsymbol{k}\dagger}\,{}^{s}u_b^{-\boldsymbol{k}} + {}^{r}v_a^{-\boldsymbol{k}\dagger}\,{}^{s}v_b^{-\boldsymbol{k}}\right) \overset{(A.61)}{=} \text{anything}\ ,$$

which have been proved in appendix A.10. Thus the condition (16.2) can be met with

$$\{\,{}^{r}a_{\boldsymbol{k}}\,,{}^{s}a^{\dagger}_{\boldsymbol{f}}\} = \delta_{\boldsymbol{kf}}\delta_{rs} \qquad , \qquad \{\,{}^{r}b^{\dagger}_{\boldsymbol{k}}\,,{}^{s}b_{\boldsymbol{f}}\} = \delta_{\boldsymbol{kf}}\delta_{rs} \tag{16.4b}$$

and

$$N = 2\hbar\omega_{\boldsymbol{k}}\ . \tag{16.4c}$$

In compliance with (16.4) we postulate these anticommutator relations for the Dirac field:

$$\{\,{}^{r}a_{\boldsymbol{k}}\,,{}^{s}a^{\dagger}_{\boldsymbol{f}}\} = \{\,{}^{r}b_{\boldsymbol{k}}\,,{}^{s}\,b^{\dagger}_{\boldsymbol{f}}\} = \delta_{\boldsymbol{kf}}\delta_{rs} \tag{16.5a}$$

$$\{\,{}^{r}a_{\boldsymbol{k}}\,,{}^{s}a_{\boldsymbol{f}}\} = \{\,{}^{r}b_{\boldsymbol{k}}\,,{}^{s}\,b_{\boldsymbol{f}}\} = \{\,{}^{r}a_{\boldsymbol{k}}\,,{}^{s}\,b_{\boldsymbol{f}}\} = \{\,{}^{r}a^{\dagger}_{\boldsymbol{k}}\,,{}^{s}\,b_{\boldsymbol{f}}\} = 0 \tag{16.5b}$$

The non-commutative algebra (16.5a) is enforced by (16.4b), but (16.5b) goes well beyond what is enforced by (16.4a). Thus (16.5) is a minimal assumption: We postulate non-commutative algebra of the Fourier-operators only if enforced by the quantization condition (16.2). Furthermore, even if we stick to the reasonable rule that Fourier-coefficients and Fourier-operators shall be dimension-less, it is quite arbitrary that we assigned to the normalization (16.4c) the numerical factor 2, instead to assigning to the condition (16.5a) the numerical factor $1/2$.

Again, as for the Klein-Gordon field, the $i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})$ in the commutator relation (16.2) has mutated into a Kronecker symbol $\delta_{\boldsymbol{kf}}$ in the commutator relation (16.5). As a new feature, the Kronecker symbol δ_{ab} of the spinor components in the commutator relation (16.2) has mutated to a Kronecker symbol δ_{rs} of the polarization indices in the commutator relation (16.5).

And again we note the remarkable side-effects of quantization: The field's normalization can't any more be chosen arbitrarily. Instead the products of the normalization factor and the Fourier operators (16.5) are uniquely fixed due to the commutator relation (16.2). In particular, all Fourier operators for arbitrary wave numbers $\boldsymbol{k}$ and arbitrary spin variables r are different from zero.

16.2 Conserved Quantities

The components of the Dirac field's energydensity-stress tensor (ES-tensor) are

$$\mathcal{T}^{\rho\sigma} \overset{(8.86)}{=} i\hbar c\,\overline{\psi}\gamma^{\rho}\mathrm{d}^{\sigma}\psi\ . \tag{16.6}$$

Inserting the quantized fields $\psi(x) =$ (16.1a), and

$$\psi^{\dagger}(x) \overset{(16.1a)}{=} \sum_{\boldsymbol{f},s} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{f}}\Omega}} \Big({}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}u^{\boldsymbol{f}\dagger} \exp\{+ifx\} + {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\boldsymbol{f}\dagger} \exp\{-ifx\} \Big)\ , \tag{16.7}$$

$$\mathrm{d}^{\sigma}\psi(x) \overset{(16.1a)}{=} \sum_{\boldsymbol{k},r} \frac{ik^{\sigma}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(-\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\} \Big)\ ,$$

one finds

$$\begin{aligned}\mathcal{T}^{\rho\sigma} = -\sum_{\boldsymbol{f},s,\boldsymbol{k},r} \frac{c\,k^\sigma}{2\Omega\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}}\Big(\\ - {}^{s}a^\dagger_{\boldsymbol{f}}\,{}^{r}a_{\boldsymbol{k}}\,{}^{s}u^{\boldsymbol{f}\dagger}\gamma^0\gamma^\rho\,{}^{r}u^{\boldsymbol{k}}\exp\{+i(f-k)x\} + \\ + {}^{s}a^\dagger_{\boldsymbol{f}}\,{}^{r}b^\dagger_{\boldsymbol{k}}\,{}^{s}u^{\boldsymbol{f}\dagger}\gamma^0\gamma^\rho\,{}^{r}v^{\boldsymbol{k}}\exp\{+i(f+k)x\} - \\ - {}^{s}b_{\boldsymbol{f}}\,{}^{r}a_{\boldsymbol{k}}\,{}^{s}v^{\boldsymbol{f}\dagger}\gamma^0\gamma^\rho\,{}^{r}u^{\boldsymbol{k}}\exp\{-if+k)x\} + \\ + {}^{s}b_{\boldsymbol{f}}\,{}^{r}b^\dagger_{\boldsymbol{k}}\,{}^{s}v^{\boldsymbol{f}\dagger}\gamma^0\gamma^\rho\,{}^{r}v^{\boldsymbol{k}}\exp\{-if-k)x\}\Big)\,. \end{aligned} \tag{16.8}$$

Integrating this result in three-dimensional position space over the normalization volume Ω leads to

$$\begin{aligned} T^{\rho\sigma} \equiv \int_\Omega \mathrm{d}^3x\,\mathcal{T}^{\rho\sigma} \overset{(8.94)}{=} \sum_{\boldsymbol{k},s}\frac{c^2\hbar^2k^\rho k^\sigma}{\hbar\omega_{\boldsymbol{k}}}\Big({}^{s}a^\dagger_{\boldsymbol{k}}\,{}^{s}a_{\boldsymbol{k}} - {}^{s}b_{\boldsymbol{k}}\,{}^{s}b^\dagger_{\boldsymbol{k}}\Big) = \\ \overset{(16.5)}{=} \sum_{\boldsymbol{k},s}\frac{c^2\hbar^2k^\rho k^\sigma}{\hbar\omega_{\boldsymbol{k}}}\Big({}^{s}a^\dagger_{\boldsymbol{k}}\,{}^{s}a_{\boldsymbol{k}} + {}^{s}b^\dagger_{\boldsymbol{k}}\,{}^{s}b_{\boldsymbol{k}} \underbrace{- {}^{s}b_{\boldsymbol{k}}\,{}^{s}b^\dagger_{\boldsymbol{k}} - {}^{s}b^\dagger_{\boldsymbol{k}}\,{}^{s}b_{\boldsymbol{k}}}_{-\{{}^{s}b_{\boldsymbol{k}},{}^{s}b^\dagger_{\boldsymbol{k}}\}\overset{(16.5)}{=}-1}\Big) \end{aligned} \tag{16.9a}$$

$$= \sum_{\boldsymbol{k},s}\frac{c^2\hbar^2k^\rho k^\sigma}{\hbar\omega_{\boldsymbol{k}}}\Big({}^{s}a^\dagger_{\boldsymbol{k}}\,{}^{s}a_{\boldsymbol{k}} - {}^{s}b^\dagger_{\boldsymbol{k}}\,{}^{s}b_{\boldsymbol{k}} \underbrace{- {}^{s}b_{\boldsymbol{k}}\,{}^{s}b^\dagger_{\boldsymbol{k}} + {}^{s}b^\dagger_{\boldsymbol{k}}\,{}^{s}b_{\boldsymbol{k}}}_{-[{}^{s}b_{\boldsymbol{k}},{}^{s}b^\dagger_{\boldsymbol{k}}]=\,?}\Big)\,. \tag{16.9b}$$

Like we have done in case of the Klein-Gordon field, we interpret the Fourier-operators $a^\dagger_{\boldsymbol{k}}$ and $b^\dagger_{\boldsymbol{k}}$ as creation operators, and the Fourier-operators $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$ as annihilation operators, of field quanta of types a and b. The particle-number operators $a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}}$ and $b^\dagger_{\boldsymbol{k}}b_{\boldsymbol{k}}$ have the eigenvalues $0, 1, 2, 3, \ldots$, but no negative eigenvalues. The eigenvalues equal the number of excited field quanta of the respective types. Therefore

$$\sum_{\boldsymbol{k},r}\hbar\omega_{\boldsymbol{k}}\Big({}^{r}a^\dagger_{\boldsymbol{k}}\,{}^{r}a_{\boldsymbol{k}} + {}^{r}b^\dagger_{\boldsymbol{k}}\,{}^{r}b_{\boldsymbol{k}}\Big)|s\rangle \geq 0 \tag{16.10}$$

is positive definite in any state $|s\rangle$ of the field. As we quantized the Dirac field by means of the anticommutator-relation (14.74b), we can identify in

(16.9a) the anticommutator (16.5) of the Fourier-operators. If we instead had postulated the commutator relation (14.74a) for the Dirac field, then we would have got a commutator-relation for the Fourier-operators as well, and then the Hamilton operator of the Dirac field would have — see (16.9b) — both positive and negative eigenvalues. The spin-statistic-theorem [40] requires, that *all* fields with half-integer spin must be quantized by means of the anti-commutator relation (14.74b). Then and only then the field's energy will never become negative.

Just as in equation (15.17) of the Klein-Gordon field, there is a constant adder in (16.9a) due to the non-vanishing commutators. And again, just like in (15.18) for the case of the Klein-Gordon field, we postulate as a law of nature that (16.9a) is valid only in case of discrete Dirac fields, but that all terms, which do not depend on the particle-number operators ${}^{r}a^{\dagger}_{\boldsymbol{k}}\,{}^{r}a_{\boldsymbol{k}}$ or ${}^{r}b^{\dagger}_{\boldsymbol{k}}\,{}^{r}b_{\boldsymbol{k}}$ must be removed in case of elementary Dirac fields:

classical and discrete Dirac fields:

$$\mathcal{T}^{\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\psi)}\,\partial^\sigma\psi + \partial^\sigma\overline{\psi}\,\frac{\partial\mathcal{L}}{\partial(\partial_\rho\overline{\psi})} - g^{\rho\sigma}\mathcal{L} \tag{16.11a}$$

elementary Dirac fields:

$$\mathcal{T}^{\rho\sigma} = \frac{\partial\mathcal{L}}{\partial(\partial_\rho\psi)}\,\partial^\sigma\psi + \partial^\sigma\overline{\psi}\,\frac{\partial\mathcal{L}}{\partial(\partial_\rho\overline{\psi})} - g^{\rho\sigma}\mathcal{L} - Y \tag{16.11b}$$

$Y \equiv$ the sum of all terms in (16.11a) which do not depend on the particle-number operators ${}^{r}a^{\dagger}_{\boldsymbol{k}}\,{}^{r}a_{\boldsymbol{k}}$ or ${}^{r}b^{\dagger}_{\boldsymbol{k}}\,{}^{r}b_{\boldsymbol{k}}$

Discrete Dirac fields seem not to exist in our universe, but we want to keep our formulas as general as possible. Thus we get for Dirac fields this ES-tensor:

$$T^{\rho\sigma} \equiv \int_\Omega \mathrm{d}^3x\, \mathcal{T}^{\rho\sigma} = \sum_{\boldsymbol{k},s} \frac{c^2\hbar^2 k^\rho k^\sigma}{\hbar\omega_{\boldsymbol{k}}} \cdot \cdot \begin{cases} \left({}^s a^\dagger_{\boldsymbol{k}}\, {}^s a_{\boldsymbol{k}} + {}^s b^\dagger_{\boldsymbol{k}}\, {}^s b_{\boldsymbol{k}} - 1 \right) \text{ for discrete fields} & \text{(16.12a)} \\ \left({}^s a^\dagger_{\boldsymbol{k}}\, {}^s a_{\boldsymbol{k}} + {}^s b^\dagger_{\boldsymbol{k}}\, {}^s b_{\boldsymbol{k}} \right) \text{ for elementary fields} & \text{(16.12b)} \end{cases}$$

In particular, the Hamilton operator $H \equiv T^{00}$ is

$$H = \sum_{\boldsymbol{k},s} \hbar\omega_{\boldsymbol{k}} \cdot \begin{cases} ({}^s a^\dagger_{\boldsymbol{k}}\, {}^s a_{\boldsymbol{k}} + {}^s b^\dagger_{\boldsymbol{k}}\, {}^s b_{\boldsymbol{k}} - 1) \text{ for discrete fields} \\ ({}^s a^\dagger_{\boldsymbol{k}}\, {}^s a_{\boldsymbol{k}} + {}^s b^\dagger_{\boldsymbol{k}}\, {}^s b_{\boldsymbol{k}}) \text{ for elementary fields,} \end{cases} \tag{16.13}$$

and the operator $P^j \equiv T^{0j}/c$ of the j-component of physical momentum is

$$P^j = \sum_{\boldsymbol{k},s} \hbar k^j \cdot \begin{cases} ({}^s a^\dagger_{\boldsymbol{k}}\, {}^s a_{\boldsymbol{k}} + {}^s b^\dagger_{\boldsymbol{k}}\, {}^s b_{\boldsymbol{k}} - 1) \text{ for discrete fields} \\ ({}^s a^\dagger_{\boldsymbol{k}}\, {}^s a_{\boldsymbol{k}} + {}^s b^\dagger_{\boldsymbol{k}}\, {}^s b_{\boldsymbol{k}}) \text{ for elementary fields.} \end{cases} \tag{16.14}$$

Angular momentum operator: Will be added in a later version of this book, if I should find the time to do that ...

For the operator of the Dirac field's conserved charge

$$Q \overset{(A.92)}{=} \sum_{\boldsymbol{k},r} q\Big({}^r a^\dagger_{\boldsymbol{k}}\, {}^r a_{\boldsymbol{k}} - {}^r b^\dagger_{\boldsymbol{k}}\, {}^r b_{\boldsymbol{k}} + \underbrace{\{ {}^r b^\dagger_{\boldsymbol{k}}, {}^r b_{\boldsymbol{k}} \}}_{1} \Big) , \tag{16.15}$$

which is computed in appendix A.14, the same remarks apply which have been made already in the lines before and after (15.24).

Considering the non vanishing commutators, which are showing up in the conserved quantities of discrete Dirac fields, a peculiar dissymmetry between field and anti-field attracts attention: In (16.9) and (16.15), it's both times the anti-field with the operators b, but not the field with the operators a, which brings about the non vanishing commutators. Might this dissymmetry be related to the dissymmetry in the Lagrangian $\mathcal{L} =$ (8.24), in which derivatives of ψ, but no derivatives of $\overline{\psi}$ show up? At the end of

section 8.6, we defined the alternative Lagrangian $\mathcal{L}' =$ (8.99), in which the fields ψ and $\overline{\psi}$ are showing up more symmetrically than in $\mathcal{L} =$ (8.24). From $\mathcal{L}'$ resulted more symmetrical canonically conjugate momentum densities (8.101), and the more symmetrical energy density

$$\mathcal{H}' \overset{(8.104)}{=} -\frac{i\hbar c}{2}\,(\mathrm{d}_0\overline{\psi})\gamma^0\psi + \frac{i\hbar c}{2}\,\overline{\psi}\gamma^0\mathrm{d}_0\psi\ , \tag{16.16a}$$

and the more symmetrical physical momentum density

$$\mathcal{P}'^j \overset{(8.105)}{=} \frac{i\hbar}{2}\,\overline{\psi}\gamma^0\mathrm{d}^j\psi - (\mathrm{d}^j\,\overline{\psi})\frac{i\hbar}{2}\gamma^0\psi \tag{16.16b}$$

of the classical (not quantized) Dirac field. But the computation of the Hamilton operator and the momentum operator, based on (16.16), which is elaborated in appendix A.16, results again into exactly the same operators (16.12a) with exactly the same non vanishing commutators.

No change at all of the conserved quantities is achieved due to the more symmetric Lagrangian $\mathcal{L}' =$ (8.99) versus $\mathcal{L} =$ (8.24)! The dissymmetry between field and anti-field in (16.9a) and (16.15) is not caused by a dissymmetry in the Lagrangian. We could have seen that already, when we quantized the Klein-Gordon field, whose Lagrangian (10.10) and canonically conjugate momenta (10.14) are as symmetrical as possible in the fields ϕ and ϕ^*. Still we found that also in the conserved quantities of the Klein-Gordon field the non vanishing commutators are always built with the operators b of the anti-fields, but not with the operators a of the fields. See (15.17b).

Instead the formal cause for the dissymmetry is this: The field operators always have the form

$$\psi = \ldots a \ldots + \ldots b^\dagger \ldots \quad , \quad \psi^\dagger = \ldots a^\dagger \ldots + \ldots b \ldots\ .$$

The operators of the conserved quantities never are products of the form $\psi \ldots \psi^\dagger \ldots$. Instead they always are products of the form

$$\ldots \psi^\dagger \ldots \psi \ldots = \ldots a^\dagger a \ldots + \underbrace{\ldots a^\dagger b^\dagger + \ldots ba \ldots}_{0} + \ldots bb^\dagger \ldots\ .$$

The mixed terms in all cases are zero, and the non vanishing commutators are in all cases caused by the terms $\ldots bb^\dagger \ldots$.

We conclude this section with the computation of the vacuum expectation values of the ES-tensor:

$$\langle 0|T^{\rho\sigma}|0\rangle \overset{(16.12)}{=} \begin{cases} \displaystyle\sum_{\boldsymbol{k}} \frac{-2c^2\hbar^2 k^\rho k^\sigma}{\hbar\omega_{\boldsymbol{k}}} \overset{(7.18)}{=} \sum_{\boldsymbol{k}} \frac{-2c\hbar k^\rho k^\sigma}{\sqrt{\boldsymbol{k}^2 + m^2c^2/\hbar^2}} & \text{for discrete fields} \\ 0 \quad \text{for elementary fields} \end{cases} \tag{16.17}$$

The factor 2 results from the summation over the polarization index, and the factor (-1) is caused by the quantization of fermion fields with the anti-commutator relation (14.74b). The vacuum expectation values of the (hypothetical) discrete Dirac field differ by the factor -2 from the vacuum expectation values (15.21) of the complex (i. e. charged) discrete Klein-Gordon field, and by a further factor 2 (i. e. in total by the factor -4) from the vacuum expectation values (15.50) of the real discrete Klein-Gordon field.

The vacuum expectation values of the ES-tensors of elementary Dirac fields are zero.

16.3 Quanta of the Dirac Field

We define the states

$$|1_{ar\boldsymbol{k}}\rangle \equiv {}^r a^\dagger_{\boldsymbol{k}}|0\rangle \qquad\qquad |1_{br\boldsymbol{k}}\rangle \equiv {}^r b^\dagger_{\boldsymbol{k}}|0\rangle \tag{16.18}$$

as one-particle-states of the Dirac field, in which one particle a or one antiparticle b with wave number $\boldsymbol{k}$ is excited in spin state r. These are eigenstates of the Hamilton operator, of the momentum operator, and of the charge operator (and of further operators). The normalization of these states is lorentz-invariant because of

$$\langle 1_{as\boldsymbol{f}}|1_{ar\boldsymbol{k}}\rangle = \langle 0|\,{}^{s}a_{\boldsymbol{f}}\,{}^{r}a^{\dagger}_{\boldsymbol{k}}|0\rangle \overset{(16.5)}{=} \underbrace{\langle 0|0\rangle}_{1}\,\delta_{rs}\,\delta_{\boldsymbol{fk}} - \underbrace{\langle 0|\,{}^{r}a^{\dagger}_{\boldsymbol{k}}\,{}^{s}a_{\boldsymbol{f}}|0\rangle}_{0}$$

$$\begin{aligned}\langle 1_{bs\boldsymbol{f}}|1_{br\boldsymbol{k}}\rangle &\overset{(16.5)}{=} \delta_{rs}\,\delta_{\boldsymbol{fk}}\\ \langle 1_{bs\boldsymbol{f}}|1_{ar\boldsymbol{k}}\rangle &\overset{(16.5)}{=} 0\ .\end{aligned} \tag{16.19}$$

One-particle-functions, which are eigenstates of the time-dependent position operator, can be created due to application of the *components* of the field operators

$$\begin{aligned}\psi(x) &\overset{(16.1a)}{=} \sum_{\boldsymbol{k},r}\frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}}\Big(\\ &\qquad {}^{r}a_{\boldsymbol{k}}\,{}^{r}u^{\boldsymbol{k}}\exp\{-ikx\} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\,{}^{r}v^{\boldsymbol{k}}\exp\{+ikx\}\Big)\\ \overline{\psi}(x) &= \sum_{\boldsymbol{k},r}\frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}}\Big(\\ &\qquad {}^{r}a^{\dagger}_{\boldsymbol{k}}\,{}^{r}\bar{u}^{\boldsymbol{k}}\exp\{+ikx\} + {}^{r}b_{\boldsymbol{k}}\,{}^{r}\bar{v}^{\boldsymbol{k}}\exp\{-ikx\}\Big)\end{aligned}$$

onto the vacuum state:

$$|1_{ax}\rangle_{\alpha} \equiv \overline{\psi}_{\alpha}(x)\,|0\rangle = \sum_{\boldsymbol{k},r}|1_{ar\boldsymbol{k}}\rangle\,{}^{r}\bar{u}^{\boldsymbol{k}}_{\alpha}\,\frac{\exp\{+ikx\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \tag{16.20a}$$

$$|1_{by}\rangle_{\beta} \equiv \psi_{\beta}(y)\,|0\rangle = \sum_{\boldsymbol{f},s}|1_{bs\boldsymbol{f}}\rangle\,{}^{s}v^{\boldsymbol{f}}_{\beta}\,\frac{\exp\{+ify\}}{\sqrt{2\hbar\omega_{\boldsymbol{f}}\Omega}} \tag{16.20b}$$

$\alpha,\beta = 1,2,3,4$ are the indices of the four spinor components. One-particle-functions, which are eigenstates of the time-independent position operator, can be created due to application of the time-independent operators in the Schrödinger-picture:

$$|1_{a\boldsymbol{x}}\rangle_\alpha \equiv \overline{\psi}_\alpha(\boldsymbol{x})\,|0\rangle = \sum_{\boldsymbol{k},r} |1_{ar\boldsymbol{k}}\rangle\, {}^r\bar{u}^{\boldsymbol{k}}_\alpha \,\frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \tag{16.20c}$$

$$|1_{b\boldsymbol{x}}\rangle_\alpha \equiv \psi_\alpha(\boldsymbol{x})\,|0\rangle = \sum_{\boldsymbol{k},r} |1_{br\boldsymbol{k}}\rangle\, {}^rv^{\boldsymbol{k}}_\alpha \,\frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}\}}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \tag{16.20d}$$

Note, that the spin of these localized particles is not well-defined. The adjoint state functions are created per definition due to the components of the adjoint spinors:

$$|1_{ax}\rangle_\alpha = \overline{\psi}_\alpha(x)\,|0\rangle \quad\Longrightarrow\quad \langle 1_{ax}|_\alpha \equiv \langle 0|\,\psi_\alpha(x) \tag{16.21a}$$

$$|1_{by}\rangle_\beta = \psi_\beta(y)\,|0\rangle \quad\Longrightarrow\quad \langle 1_{by}|_\beta \equiv \langle 0|\,\overline{\psi}_\beta(y) \tag{16.21b}$$

The state functions are defined component by component, because otherwise $\overline{\psi}(x)\,|0\rangle$ would be a row spinor, while $\psi(x)\,|0\rangle$ would be a column spinor. That would result in a meaningless difference in-between particle and antiparticle. In particular, the norm-square would be in one case a 4×4 spinor-matrix, in the other case a spinor-scalar. Therefore we arrange the following

Definition: Any $|\text{ket}\rangle$ of a spinor field is a row spinor. Any $\langle\text{bra}|$ of a spinor field is a column spinor. (16.22)

Hence any $\langle\text{bra}|\text{ket}\rangle$ of a spinor field is a spinor-matrix, but not a spinor-scalar.

Not the complete $|\text{kets}\rangle$, but their single $|\text{spinor-components}\rangle_\alpha$ therefore are to be considered as the elements of the countable infinite-dimensional Hilbert space of state-functions. The scalar-product $\langle\text{bra}|\text{ket}\rangle_{\alpha\beta}$ is a bilinear map of the Hilbert space, whose elements are the $|\text{spinor-components}\rangle_\alpha$, onto the field of complex numbers. The products $\langle\text{bra}|\text{ket}\rangle$ are 4×4-dimensional spinor-matrices, whose 16 elements are complex numbers. In the following section we will see, that the definitions (16.21) and (16.22) are mandatory, to achieve consistency with the definition (12.27) of the Feynman-propagator of the classical Dirac field.

We now are going to compute the squares of the vectors (16.20). The components

$$\begin{aligned}\langle 1_{ax}|1_{ay}\rangle_{\alpha\beta} &= \langle 0|\,\psi_\alpha(x)\overline{\psi}_\beta(y)\,|0\rangle \;= \\ &= \sum_{\boldsymbol{f},\boldsymbol{k},s,r} \underbrace{\langle 1_{as\boldsymbol{f}}|1_{ar\boldsymbol{k}}\rangle}_{\delta_{rs}\,\delta_{\boldsymbol{fk}}} {}^{s}u_\alpha^{\boldsymbol{f}}\,{}^{r}\bar{u}_\beta^{\boldsymbol{k}}\,\frac{\exp\{-i(fx-ky)\}}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \;= \\ &= \sum_{\boldsymbol{k},r} {}^{r}u_\alpha^{\boldsymbol{k}}\,{}^{r}\bar{u}_\beta^{\boldsymbol{k}}\,\frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \end{aligned} \tag{16.23}$$

are defining the 4×4 spinor matrix

$$\langle 1_{ax}|1_{ay}\rangle = \langle 0|\,\psi(x)\overline{\psi}(y)\,|0\rangle = \sum_{\boldsymbol{k},r} {}^{r}u^{\boldsymbol{k}}\,{}^{r}\bar{u}^{\boldsymbol{k}}\,\frac{\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}}\;. \tag{16.24}$$

And the components

$$\begin{aligned}\langle 1_{by}|1_{bx}\rangle_{\alpha\beta} &= \langle 0|\,\overline{\psi}_\alpha(y)\psi_\beta(x)\,|0\rangle \\ &= \sum_{\boldsymbol{f},\boldsymbol{k},s,r} \underbrace{\langle 1_{bs\boldsymbol{f}}|1_{br\boldsymbol{k}}\rangle}_{\delta_{rs}\,\delta_{\boldsymbol{fk}}} {}^{s}\bar{v}_\alpha^{\boldsymbol{f}}\,{}^{r}v_\beta^{\boldsymbol{k}}\,\frac{\exp\{-i(fy-kx)\}}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \\ &= \sum_{\boldsymbol{k},r} {}^{r}\bar{v}_\alpha^{\boldsymbol{k}}\,{}^{r}v_\beta^{\boldsymbol{k}}\,\frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} \end{aligned} \tag{16.25}$$

are defining the 4×4 spinor matrix

$$\langle 1_{by}|1_{bx}\rangle = \langle 0|\,\overline{\psi}(y)\psi(x)\,|0\rangle \overset{(16.22)}{=} \sum_{\boldsymbol{k},r} {}^{r}v^{\boldsymbol{k}}\,{}^{r}\bar{v}^{\boldsymbol{k}}\,\frac{\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}}\;. \tag{16.26}$$

Contained within these matrices are the 4×4 spinor matrices

$$\sum_{r=1}^{2} {}^{r}u^{\boldsymbol{k}}\,{}^{r}\bar{u}^{\boldsymbol{k}} \overset{(8.77a)}{=} c\gamma^\mu p_\mu + mc^2 \tag{16.27a}$$

$$\sum_{r=1}^{2} {}^{r}v^{\boldsymbol{k}}\,{}^{r}\bar{v}^{\boldsymbol{k}} \overset{(8.77b)}{=} -(-c\gamma^\mu p_\mu + mc^2)\;, \tag{16.27b}$$

which we already encountered in the investigation of the classical Dirac field

in section 8.5. They have not been modified by the field's quantization. By insertion, we find

$$\langle 1_{ax}|1_{ay}\rangle = \sum_{\boldsymbol{k}} (c\hbar\gamma^\mu k_\mu + mc^2)\, \frac{\exp\{-ik(x-y)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \tag{16.28a}$$

$$\langle 1_{by}|1_{bx}\rangle = -\sum_{\boldsymbol{k}} (-c\hbar\gamma^\mu k_\mu + mc^2)\, \frac{\exp\{-ik(y-x)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \tag{16.28b}$$

$$\langle 1_{ax}|1_{by}\rangle \overset{(16.19)}{=} 0\ . \tag{16.28c}$$

These matrix elements are lorentz-invariant. From the respective matrix elements (15.41) of the Klein-Gordon field, they differ by the spinor matrices $\pm(\pm c\hbar\gamma^\mu k_\mu + \mathbb{1}mc^2)$.

16.4 The quantized Greens-function

The 4×4 spinor matrix

$$\langle 1_{ax}|1_{ay}\rangle = \langle 0|\, \psi(x)\overline{\psi}(y)\, |0\rangle \overset{(16.28a)}{=} \sum_{\boldsymbol{k}} (c\hbar\gamma^\mu k_\mu + mc^2)\, \frac{\exp\{-ik(x-y)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \tag{16.29}$$

is interpreted as the probability amplitude for the creation of a particle at time y^0 at position $\boldsymbol{y}$ and it's subsequent annihilation at time x^0 at position $\boldsymbol{x}$. The 4×4 spinor matrix

$$\begin{aligned}\langle 1_{by}|1_{bx}\rangle &= \langle 0|\, \overline{\psi}(y)\psi(x)\, |0\rangle \overset{(16.28b)}{=} \\ &= -\sum_{\boldsymbol{k}} (-c\hbar\gamma^\mu k_\mu + mc^2)\, \frac{\exp\{-ik(y-x)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega}\end{aligned} \tag{16.30}$$

is interpreted as the probability amplitude for the creation of an antiparticle at time x^0 at position $\boldsymbol{x}$ and it's subsequent annihilation at time y^0 at position $\boldsymbol{y}$. We compare these matrices with the Feynman-propagator of the classical Dirac field, which was derived in section 12.2:

$$S(x-y) \overset{(12.27)}{=} \theta(x^0-y^0)\sum_{\boldsymbol{k}} \frac{(+\gamma^\mu c p_\mu + mc^2)\exp\{-ik(x-y)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}} + \\ + \theta(y^0-x^0)\sum_{\boldsymbol{k}} \frac{(-\gamma^\mu c p_\mu + mc^2)\exp\{-ik(y-x)\}}{\Omega 2\hbar\omega_{\boldsymbol{k}}}$$

Thus obviously

$$\boxed{\begin{aligned} S(x-y) &= \theta(x^0-y^0)\langle 0|\,\psi(x)\overline{\psi}(y)\,|0\rangle + \\ &\quad - \theta(y^0-x^0)\langle 0|\,\overline{\psi}(y)\psi(x)\,|0\rangle \\ &= \langle 0|\,T\psi(x)\overline{\psi}(y)\,|0\rangle \end{aligned}} \tag{16.31}$$

is the Feynman-propagator of the quantized Dirac field. In the last line the time-order operator is used, which has been defined in (15.44).

16.5 Causality

Like we have done for the Klein-Gordon field in section 15.6, we now want to check also for the Dirac field, that the criterion of causality of Special Relativity Theory is not violated. Thats not obvious, because in the classical Greens function we admitted waves, which are propagating faster than light. Let x and y be two space-time points with space-like distance, i.e. $(x-y)^2 < 0$. Creation and annihilation of particles at these two space-time points must not mutually interfere. Formally this does mean, that their anti-commutator must be zero. (As the Dirac field has been quantized according to (14.74b) with the anti-commutator, we must in the present investigation again use the anti-commutator.)

Let the Dirac field be in state $|s\rangle$. Due to $\psi(x)|s\rangle$, at time x^0/c at position $\boldsymbol{x}$ a particle a is annihilated and it's antiparticle b is created. Due to $\overline{\psi}(y)|s\rangle$, at time y^0/c at position $\boldsymbol{y}$ a particle a is created and it's antiparticle b is annihilated. The expectation value of their anti-commutator is

$$\begin{aligned}
&\langle s|\,\{\psi(x),\overline{\psi}(y)\}\,|s\rangle \;= \\
&= \langle s|s\rangle\Big(\sum_{\boldsymbol{k}}(c\hbar\gamma^\mu k_\mu + mc^2)\,\frac{\exp\{-ik(x-y)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega} \\
&\qquad -\sum_{\boldsymbol{k}}(-c\hbar\gamma^\mu k_\mu + mc^2)\,\frac{\exp\{-ik(y-x)\}}{2\hbar\omega_{\boldsymbol{k}}\Omega}\Big)\;. \qquad (16.32)
\end{aligned}$$

As the commutator isn't an operator, it could be extracted out of the product $\langle s|s\rangle$. And because the commutator is lorentz-invariant, it can be transformed — without changing it's value — into a primed coordinate system′, in which $x'^0 - y'^0 = 0$. Such a coordinate systems does exist, because the relation of the space-time points x and y is space-like.

$$\begin{aligned}
&\{\psi'(x'),\overline{\psi}\,'(y')\} = \\
&= \sum_{\boldsymbol{k}'}(c\hbar\gamma^0 k'_0 - c\hbar\boldsymbol{\gamma}\cdot\boldsymbol{k}' + mc^2)\,\frac{\exp\{+i\boldsymbol{k}'(\boldsymbol{x}'-\boldsymbol{y}')\}}{2\hbar\omega_{\boldsymbol{k}'}\Omega'} \\
&\qquad -\sum_{-\boldsymbol{k}'}(-c\hbar\gamma^0 k'_0 - c\hbar\boldsymbol{\gamma}\cdot\boldsymbol{k}' + mc^2)\,\frac{\exp\{-i\boldsymbol{k}'(\boldsymbol{y}'-\boldsymbol{x}')\}}{2\hbar\omega_{\boldsymbol{k}'}\Omega'}
\end{aligned}$$

In the last line, the sum over $-\boldsymbol{k}'$ was computed, respecting $\omega_{-\boldsymbol{k}'} = \omega_{\boldsymbol{k}'}$. Because of $ck'_0 = \omega_{\boldsymbol{k}'}$ and (7.9), one finds

$$\{\psi'(x'),\overline{\psi}\,'(y')\} = \gamma^0\,\delta^{(3)}(\boldsymbol{x}'-\boldsymbol{y}')\;. \qquad (16.33)$$

Therefore the commutator indeed is zero provided $\boldsymbol{x}' \neq \boldsymbol{y}'$, in compliance with Relativity Theory.

If we had quantized the Dirac field not by means of the anti-commutator (14.74b) but by means of the commutator (14.74a), then consequently we would also have to perform the present computation with the commutator, and achieve instead of (16.33) the result

$$[\psi'(x'),\overline{\psi}\,'(y')] = \sum_{\boldsymbol{k}'}\frac{(-c\hbar\boldsymbol{\gamma}\cdot\boldsymbol{k}' + mc^2)}{\hbar\omega_{\boldsymbol{k}'}}\,\frac{\exp\{+i\boldsymbol{k}'(\boldsymbol{x}'-\boldsymbol{y}')\}}{\Omega'}\;.$$

The causality criterion of Relativity Theory would be violated, because this

result can be different from zero even in case of $\boldsymbol{x}' \neq \boldsymbol{y}'$. The criterion of causality of Relativity Theory enforces the quantization of all fields with integer spin (including spin zero) by means of the commutator (14.74a), and the quantization of all fields with half-integer spin by means of the anti-commutator (14.74b). We have confirmed this rule for the example of the Klein-Gordon field (spin 0) in section 15.6, and we now confirmed it again for the example of the Dirac field (spin $\frac{1}{2}$).

17 The Free Gauge-Field $A(x)$

In section 4.5, we identified the gauge field of the electrically charged Dirac field with the potential field

$$A(x) = \begin{pmatrix} \phi(x)/c \\ \boldsymbol{A}(x) \end{pmatrix} \tag{17.1}$$

of Maxwell's electrodynamics. Furthermore we found the combined Lagrangian (4.120) of the Dirac field and it's gauge field. The Lagrangian of the free field $A(x)$ follows from (4.120) by switching the Dirac field off:

$$\mathcal{L} \overset{(4.120)}{=} -\frac{1}{4\mu_0} F_{\sigma\tau} F^{\sigma\tau} \tag{17.2a}$$

In this Lagrangian,

$$F_{\sigma\tau} \overset{(4.114)}{\equiv} \mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma \tag{17.2b}$$

is the electromagnetic field strength tensor. We had good reasons to construct the Lagrangian as the product of rotational terms: By that method, the invariance of the theory under local phase-transformations could be guaranteed. At the same time, considerable side-effects are resulting from this form of $\mathcal{L}$ for the quantization of A. These side effects become obvious already, when we compute the canonically conjugate momentum densities:

$$\begin{aligned} \pi^0 &\equiv \frac{\partial \mathcal{L}}{c\partial(\mathrm{d}_0 A_0)} = \\ &= -\frac{1}{4\mu_0} \frac{\partial}{c\partial(\mathrm{d}_0 A_0)} (\mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma)(\mathrm{d}^\sigma A^\tau - \mathrm{d}^\tau A^\sigma) = 0 \end{aligned} \tag{17.3a}$$

$$\begin{aligned}\pi^j \equiv \frac{\partial \mathcal{L}}{c\partial(\mathrm{d}_0 A_j)} &= -\frac{1}{4\mu_0}\frac{\partial}{c\partial(\mathrm{d}_0 A_j)}(\mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma)(\mathrm{d}^\sigma A^\tau - \mathrm{d}^\tau A^\sigma) \\ &= -\frac{1}{4\mu_0 c}\Big((\mathrm{d}^0 A^j - \mathrm{d}^j A^0) - (\mathrm{d}^j A^0 - \mathrm{d}^0 A^j)\Big)\cdot 2 \\ &= +\frac{1}{\mu_0 c}(-\mathrm{d}_0 A^j - \mathrm{d}_j A^0) \overset{(4.127a)}{=} +\frac{1}{\mu_0 c^2} E^j \end{aligned} \tag{17.3b}$$

The null-component of $A(x)$ can not be quantized canonically, because it's conjugate momentum density is zero. This again is caused by the fact, that $\mathcal{L}$ is a product of the rotational terms $\mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma$. Only the three space-like components of $A(x)$ can be quantized canonically. Note, that according to (17.3) the contravariant components of $\pi(x)$ are canonically conjugate to the covariant components of $A(x)$.

In section 17.3 we will — when investigating the spin of photons — supply the proof of the integer spin of the gauge field's quanta. Therefore the gauge field must be quantized according to (14.74a). We postulate (only preliminary!) the non-commuative algebra

$$\begin{aligned}[A_j(t,\boldsymbol{x}),\pi^l(t,\boldsymbol{y})] = \frac{1}{\mu_0 c^2}[A_j(t,\boldsymbol{x}), E^l(t,\boldsymbol{y})] \;= \\ = i\hbar\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}){g_j}^l \quad \text{only preliminary!}\,, \end{aligned} \tag{17.4}$$

in which both factors in the commutator are to be inserted with identical time t. This quantization rule is marked as "only preliminary", because it is inconsistent. The sources of the electric field $\boldsymbol{E}$ are the charges of the Dirac field's quanta. As we switched off the Dirac field for the computation of the free field A, the electric field $\boldsymbol{E}$ is free of sources, and thus it's divergence is zero:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$$

Therefore the divergence with respect to the coordinate $\boldsymbol{y}$ on the left side of (17.4) is zero as well:

$$\frac{\mathrm{d}}{\mathrm{d}y^l}[A_j(t,\boldsymbol{x}),\pi^l(t,\boldsymbol{y})] = \frac{1}{\mu_0 c^2}[A_j(t,\boldsymbol{x}),\underbrace{\frac{\mathrm{d}E^l(t,\boldsymbol{y})}{\mathrm{d}y^l}}_{0}] = 0 \tag{17.5a}$$

But the divergence of the right side of (17.4) is not zero:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y^l}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})g_j{}^l &\overset{(7.9)}{=} \frac{\mathrm{d}}{\mathrm{d}y^j}\frac{1}{\Omega}\sum_{\boldsymbol{k}}\exp\{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \\ &= -\frac{1}{\Omega}\sum_{\boldsymbol{k}} ik^j \exp\{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \neq 0\end{aligned} \tag{17.5b}$$

17.1 Canonical Quantization

From the fact, that the factors of $\mathcal{L}$ have the form of rotational terms, further complications are arising: A is invariant under the gauge transformation

$$A_\nu(x) \longrightarrow A'_\nu(x) = A_\nu(x) + \mathrm{d}_\nu f(x)\ , \tag{17.6}$$

where $f(x)$ can be an arbitrary analytical function. "Analytical" does mean, that

$$\mathrm{d}_\sigma \mathrm{d}_\tau f = \mathrm{d}_\tau \mathrm{d}_\sigma f\ .$$

The gauge transformation leaves the field-strength tensor — and consequently all observable quantities — invariant due to

$$F'_{\sigma\tau} = \mathrm{d}_\sigma A_\tau + \mathrm{d}_\sigma \mathrm{d}_\tau f - \mathrm{d}_\tau A_\sigma - \mathrm{d}_\tau \mathrm{d}_\sigma f = F_{\sigma\tau}\ .$$

Due to this gauge-invariance, the four-divergence

$$\mathrm{d}_\nu A'^\nu = \mathrm{d}_\nu(A^\nu + \mathrm{d}^\nu f) = \mathrm{d}_\nu A^\nu + \mathrm{d}_\nu \mathrm{d}^\nu f$$

of the field A can be chosen arbitrarily due to an appropriate definition of $\mathrm{d}_\nu \mathrm{d}^\nu f$. Choosing

$$\mathrm{d}_\nu \mathrm{d}^\nu f \equiv -\mathrm{d}_\nu A^\nu \quad\Longrightarrow\quad \mathrm{d}_\nu A'^\nu = 0\ , \tag{17.7}$$

one gets the Lorentz-gauge, which reduces the field's degrees of freedom from four to three. Specifying in addition

$$\mathrm{d}_\nu \mathrm{d}^\nu f \equiv -\mathrm{d}_\nu A^\nu \quad \text{and} \quad \mathrm{d}_0 f \equiv -A_0 \quad \Longrightarrow$$
$$\Longrightarrow \quad \boxed{A'^0 = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot \boldsymbol{A}' = 0}\,, \tag{17.8}$$

the field's degrees of freedom are reduced to two. The free gauge field $A(x)$ has only two linearly independent components, but it must be transformed as a vector-field with four space-time dimensions when the coordinate system is changed. The fact that two redundant components, which can be fixed arbitrarily due to gauge transformations, must be dragged through all equations, is the real cause for the peculiar difficulties, which we are facing in the quantization of this field.

The gauge (17.8) is called Coulomb-gauge or radiation-gauge. It's advantage is, that the field's both redundant, only seemingly independent components are fixed. In particular the null-component has disappeared, which — see (17.3a) — can anyway not be quantized canonically for purely formal reasons. The disadvantage of the Coulomb-gauge is, that the manifest Lorentz-covariance is lost, because in (17.8) not all four space-time coordinates are handled symmetrically, but the time coordinate is treated special. When we in the sequel quantize $A(x)$ in the Coulomb-gauge, then it is not obvious, whether our results are compatible with Special Relativity Theory. To prove the Lorentz-invariance of our results, we therefore will quantize the gauge field in section 17.4 once again by another method.

In addition to the unit vectors defined in section K.5, we define a further system $e^{(0)}_{\boldsymbol{k}}, e^{(1)}_{\boldsymbol{k}}, e^{(2)}_{\boldsymbol{k}}, e^{(3)}_{\boldsymbol{k}}$ of four orthonormal four-dimensional unit vectors. The orientation of the three space-like unit vectors depends on the wavenumber vector $\boldsymbol{k}$ of the field $A(x)$, while the unit vectors in time-direction in both systems are identical:

$$e_{\boldsymbol{k}}^{(\alpha)} \cdot e_{\boldsymbol{k}}^{(\beta)} = g_{\alpha\beta} \quad , \quad e_{\boldsymbol{k}}^{(0)} = e^{(0)}$$

$$e_{\boldsymbol{k}}^{(\alpha)} = \begin{pmatrix} e_{\boldsymbol{k}}^{(\alpha)0} \\ \boldsymbol{e}_{\boldsymbol{k}}^{(\alpha)} \end{pmatrix} \quad , \quad \boldsymbol{e}_{\boldsymbol{k}}^{(1)} \times \boldsymbol{e}_{\boldsymbol{k}}^{(2)} = \boldsymbol{e}_{\boldsymbol{k}}^{(3)} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|}$$

$$\text{with } \alpha = 0,1,2,3 \text{ and } \beta = 0,1,2,3 \tag{17.9a}$$

Only $e_{\boldsymbol{k}}^{(0)}$ and $e_{\boldsymbol{k}}^{(3)}$ are uniquely determined due to these relations. The remaining indeterminacy of the two other vectors is of no importance for the moment being. Furthermore we add to the definition a phase-factor φ^α, which may be chosen differently for each coordinate:

$$\epsilon_{\boldsymbol{k}}^{(\alpha)} \equiv e_{\boldsymbol{k}}^{(\alpha)} \exp\{i\varphi^\alpha\} \tag{17.9b}$$

The field equation

$$\mathrm{d}_\nu F^{\nu\rho} \overset{(4.125)}{=} \mu_0 j^\rho \quad \text{with } \rho = 0,1,2,3 \ ,$$

which we derived in section 4.5, assumes for the source-free ($j^\rho = 0$) field with (17.2b) the form

$$\mathrm{d}_\nu \mathrm{d}^\nu A^\rho - \mathrm{d}^\rho \underbrace{\mathrm{d}_\nu A^\nu}_{\overset{(17.7)}{=} 0} = 0 \quad \text{with } \rho = 0,1,2,3 \ .$$

Because of $A^0 \overset{(17.8)}{=} 0$, in the Coulomb-gauge only the three equations

$$\mathrm{d}_\nu \mathrm{d}^\nu A^j = 0 \quad \text{with } j = 1,2,3 \tag{17.10}$$

are non-trivial. Each of the three equations has the form of the Klein-Gordon equation (10.2) of a field with mass zero. Therefore a solution of equation (17.10) can be immediately transferred from the Klein-Gordon field. Using the unit vectors (17.9), the solution can be written as

$$\boldsymbol{A}(x) \overset{(10.8)}{=} \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \frac{1}{\sqrt{N\Omega}} \Big(\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(v)} c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(v)*} c_{\boldsymbol{k}}^{(v)*} \exp\{+ikx\} \Big) \ . \tag{17.11}$$

Here we are using the three-dimensional, boldface-printed unit vectors, because the time-like component A^0 is zero in the Coulomb-gauge. And the summation is running only over the both polarization vectors $\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(1)}$ and $\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(2)}$, because only these components, but not $\boldsymbol{\epsilon}_{\boldsymbol{k}}^{(3)}$, meet the condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} \overset{(17.8)}{=} 0 \quad \Longrightarrow \quad \boldsymbol{\epsilon}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{k} = 0$$

of the Coulomb-gauge. The free field $\boldsymbol{A}(x)$ has no longitudinal component. It's polarization is always transversal (elliptical or circular or linear).

We want to avoid the need to consider innumerous distinctions of cases. Therefore we will perform the quantization of the field $A(x)$ only for linearly polarized waves

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \frac{1}{\sqrt{N\Omega}} \boldsymbol{e}_{\boldsymbol{k}}^{(v)} \Big(c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)*} \exp\{+ikx\} \Big) \ . \tag{17.12}$$

Waves with elliptical polarization, which propagate in direction of the x^3-axis, can be described by adding to a wave, which is linearly polarized in x^1-direction and propagating in x^3-direction, another wave which is linearly polarized in x^2-direction and propagating in x^3-direction, where both waves have same amplitude and a phase difference of φ:

$$\boldsymbol{A}^{(\text{elliptical})} = \sqrt{\tfrac{1}{2}} \Big(\boldsymbol{e}_{\boldsymbol{k}}^{(1)} A^1 + \boldsymbol{e}_{\boldsymbol{k}}^{(2)} A^2 \exp\{i\varphi\} \Big) \tag{17.13}$$

In the special case $\varphi = \pm\pi/2$, one gets left- or right-circular polarized waves. We will construct the field-operators of elliptically- resp. circular-polarized fields by that method, after we have quantized linearly polarized fields and found their field-operators.

As the gauge field $A(x)$ is not charged, it is real. Therefore we are dealing with the Fourier-coefficients $c_{\boldsymbol{k}}^{(v)*}$, but there are no coefficients $d_{\boldsymbol{k}}^{(v)*}$

being different from $c_{\boldsymbol{k}}^{(v)*}$. As the dimension of A is momentum·charge^{-1}, and because the Fourier-coefficients as usual shall be dimensionless, the dimension of the normalization factor N must be charge^2·momentum^{-2}·volume^{-1}. We choose the definition

$$N \equiv \frac{2\hbar\omega_{\boldsymbol{k}}}{\mu_0 c^2 \hbar^2} \ , \tag{17.14}$$

by which the field becomes

$$\begin{aligned} A^j \overset{(K.13a)}{=} \boldsymbol{e}^{(j)} \cdot \boldsymbol{A} = \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \sqrt{\frac{\mu_0 c^2 \hbar^2}{2\hbar\omega_{\boldsymbol{k}}\Omega}} (\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(j)}) \cdot \\ \cdot \left(c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)*} \exp\{+ikx\} \right) \end{aligned} \tag{17.15a}$$

with $j = 1, 2, 3$. The momentum density canonically conjugate to $A_l(y)$ is

$$\begin{aligned} \pi^l(y) &\overset{(17.3b)}{=} \frac{1}{\mu_0 c^2} E^l(y) \overset{(4.127a)}{=} \frac{1}{\mu_0 c} (-\mathrm{d}_0 A^l - \mathrm{d}_l \underbrace{A^0}_{\overset{(17.8)}{=} 0}) \\ &= \frac{-1}{\mu_0 c} \sum_{\boldsymbol{f}} \sum_{u=1}^{2} \sqrt{\frac{\mu_0 c^2 \hbar}{2\omega_{\boldsymbol{f}}\Omega}} \, i f_0 (\boldsymbol{e}_{\boldsymbol{f}}^{(u)} \cdot \boldsymbol{e}^{(l)}) \cdot \\ &\quad \cdot (-c_{\boldsymbol{f}}^{(u)} \exp\{-ify\} + c_{\boldsymbol{f}}^{(u)*} \exp\{+ify\}) \ . \end{aligned} \tag{17.15b}$$

With $f_0 = \omega_{\boldsymbol{f}}/c$ and $A_j = -A^j$, the commutator of A_j and π^l becomes

$$\begin{aligned} [A_j(t,\boldsymbol{x}), \pi^l(t,\boldsymbol{y})] = \sum_{\boldsymbol{k},v,\boldsymbol{f},u} \frac{i\hbar}{2\Omega} \sqrt{\frac{\omega_{\boldsymbol{f}}}{\omega_{\boldsymbol{k}}}} (\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(j)}) (\boldsymbol{e}_{\boldsymbol{f}}^{(u)} \cdot \boldsymbol{e}^{(l)}) \cdot \\ \cdot \Big(- [c_{\boldsymbol{k}}^{(v)}, c_{\boldsymbol{f}}^{(u)}] \exp\{-i(kx + fy)\} \\ + [c_{\boldsymbol{k}}^{(v)}, c_{\boldsymbol{f}}^{(u)*}] \exp\{-i(kx - fy)\} \\ - [c_{\boldsymbol{k}}^{(v)*}, c_{\boldsymbol{f}}^{(u)}] \exp\{+i(kx - fy)\} \\ + [c_{\boldsymbol{k}}^{(v)*}, c_{\boldsymbol{f}}^{(u)*}] \exp\{+i(kx + fy)\} \Big) \ . \end{aligned} \tag{17.16}$$

We have quantized the Klein-Gordon field and the Dirac field by postulating a non-commutative algebra for their field amplitudes and their canonically conjugate momenta according to (14.74). From those relations, the algebra of

the Fourier-operators (which were interpreted as creation- and annihilation-operators of field quanta) could be computed. After our attempt, to quantize the gauge field $A(x)$ in the same manner, has failed in (17.4), we now proceed in opposite sequence. We promote the Fourier-coefficients to operators $c_{\boldsymbol{k}}^{(v)}$ and $c_{\boldsymbol{k}}^{(v)\dagger}$, and postulate for them a non-commutative algebra. From that we then will derive the algebra of the operators $\boldsymbol{A}(x)$ and $\boldsymbol{\pi}(x)$.

Considering (15.6) and (16.5), it seems reasonable to postulate this algebra:

$$\boxed{\begin{aligned} [c_{\boldsymbol{k}}^{(v)}, c_{\boldsymbol{f}}^{(u)\dagger}] &= \delta_{\boldsymbol{kf}}\, g_v{}^u \\ [c_{\boldsymbol{k}}^{(v)}, c_{\boldsymbol{f}}^{(u)}] &= [c_{\boldsymbol{k}}^{(v)\dagger}, c_{\boldsymbol{f}}^{(u)\dagger}] = 0 \\ \text{with } u &= 1,2 \text{ and } v = 1,2 \end{aligned}} \tag{17.17}$$

We emphasize, that this quantization specification can not be "derived". Like the general quantization specification (14.74), it was found by guessing, and can be justified only by the fact, that the theory, which was derived from it, turns out successful and correct in all experimental tests.

Inserting (17.17) into (17.16) results into

$$\begin{aligned} [A_j(t,\boldsymbol{x}), \pi^l(t,\boldsymbol{y})] = \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \frac{i\hbar}{2\Omega} (\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(l)}) \\ \Big(\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} + \exp\{-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \Big) \,. \end{aligned} \tag{17.18}$$

As A and π must be inserted into the commutator with $x^0 = y^0$, the time-coordinate has disappeared form the exponential functions. Furthermore the sequence of summands may be changed, because the sum is running symmetrically over all positive and negative wavenumbers $\boldsymbol{k}$:

$$[A_j(t,\boldsymbol{x}),\pi^l(t,\boldsymbol{y})] = \frac{i\hbar}{2\Omega}\sum_{\boldsymbol{k}}\sum_{v=1}^{2}\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}$$
$$\Big((\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)}) + (\boldsymbol{e}_{-\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{-\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)})\Big)$$
$$= i\hbar\underbrace{\frac{1}{\Omega}\sum_{\boldsymbol{k}}\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}}_{\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}\sum_{v=1}^{2}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)})\ . \tag{17.19}$$

Here $\boldsymbol{e}_{-\boldsymbol{k}}^{(v)} = -\boldsymbol{e}_{\boldsymbol{k}}^{(v)}$ for $v \neq 0$ has been used.

To evaluate the last factor, we expand the space-like unit vectors $\boldsymbol{e}^{(j)}$ and $\boldsymbol{e}^{(l)}$ with respect to the space-like polarization vectors:

$$\boldsymbol{e}^{(j)} \overset{\text{(K.13b)}}{=} \sum_{v=1}^{3}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \qquad \boldsymbol{e}^{(l)} \overset{\text{(K.13b)}}{=} \sum_{u=1}^{3}(\boldsymbol{e}_{\boldsymbol{k}}^{(u)}\cdot\boldsymbol{e}^{(l)})\boldsymbol{e}_{\boldsymbol{k}}^{(u)}$$

Thus the product of these both vectors is

$$\boldsymbol{e}^{(j)}\cdot\boldsymbol{e}^{(l)} \overset{\text{(K.13c)}}{=} g_j{}^l = \sum_{v=1}^{3}\sum_{u=1}^{3}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(u)}\cdot\boldsymbol{e}^{(l)})\underbrace{\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}_{\boldsymbol{k}}^{(u)}}_{g_v{}^u}$$
$$= \sum_{v=1}^{3}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)})\ .$$

From that follows

$$\sum_{v=1}^{2}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)}) = g_j{}^l - (\boldsymbol{e}_{\boldsymbol{k}}^{(3)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(3)}\cdot\boldsymbol{e}^{(l)})$$
$$= g_j{}^l - (\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\boldsymbol{e}^{(j)})(\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\cdot\boldsymbol{e}^{(l)})$$
$$= g_j{}^l + \frac{k_j k^l}{\boldsymbol{k}^2}\ . \tag{17.20}$$

This is inserted into (17.19):

$$\boxed{[A_j(t,\boldsymbol{x}),\pi^l(t,\boldsymbol{y})] = i\hbar\Delta_j{}^l(\boldsymbol{x}-\boldsymbol{y})} \tag{17.21}$$

Here the "transversal delta-function"

$$\Delta_j{}^l(\boldsymbol{x}-\boldsymbol{y}) \equiv \frac{1}{\Omega}\sum_{\boldsymbol{k}}\Big(g_j{}^l + \frac{k_j k^l}{\boldsymbol{k}^2}\Big)\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} \tag{17.22}$$

has been used, which should be compared to the "usual" delta function in the form (7.9). The divergence of the transversal delta function is zero:

$$\begin{aligned}\mathrm{d}^j\Delta_j{}^l(\boldsymbol{x}-\boldsymbol{y}) &= \frac{1}{\Omega}\sum_{\boldsymbol{k}}\Big(\mathrm{d}^l + \frac{k_j k^l}{\boldsymbol{k}^2}\mathrm{d}^j\Big)\exp\{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} = \\ &= \frac{1}{\Omega}\sum_{\boldsymbol{k}}\Big(ik^l + i\underbrace{k^j\frac{k_j}{\boldsymbol{k}^2}}_{-1}k^l\Big)\exp\{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\} = 0\end{aligned} \tag{17.23}$$

Here we returned to the divergence $-\mathrm{d}^j \equiv \mathrm{d}_j \equiv \mathrm{d}/\mathrm{d}x^j$ with respect to the coordinate $\boldsymbol{x}$. The divergence with respect to the coordinate $\boldsymbol{y}$ is zero as well. If the right-handed coordinate system x^j, x^l, x^m is rotated such that $k^m = |\boldsymbol{k}|$ and $k^j = k^l = 0$, then (17.21) simplifies to (17.4). With other orientations of the coordinate system, the term $+k_jk^l/\boldsymbol{k}^2$ in (17.21) is compensating, that the field $\boldsymbol{A}(x)$ does not have a longitudinal component. Thus (17.21) is "in a sense" identical to the general quantization rule (14.74a). The additional term $+k_jk^l/\boldsymbol{k}^2$ merely makes allowance to the fact, that due to the gauge invariance (17.6) only two of the four components of the free field $A(x)$ are independent.

17.2 Conserved Quantities

The electromagnetic field's Lagrangian is

$$\mathcal{L} \overset{(17.2)}{=} -\frac{1}{4\mu_0}(\mathrm{d}_\mu A_\nu - \mathrm{d}_\nu A_\mu)(\mathrm{d}^\mu A^\nu - \mathrm{d}^\nu A^\mu)\ . \tag{17.24}$$

The electromagnetic field is an elementary, i. e. a continuous, field. Different from the cases of the Klein-Gordon field and the Dirac field, we do not

assume that there might exist a discrete electromagnetic field, but consider exclusively the elementary field. Like in (15.18) and in (16.11), we postulate that in case of the electromagnetic field as well all terms in the ES-tensor (energydensity-stress tensor), which do not depend on the particle-number operators, must be removed.

classical electromagnetic field:

$$\mathcal{T}^{\sigma\tau} \overset{(4.32)}{=} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\sigma A_\rho)}\,\mathrm{d}^\tau A_\rho - g^{\sigma\tau}\mathcal{L}$$
$$= -\frac{1}{\mu_0}\left(\mathrm{d}^\sigma A^\rho - \mathrm{d}^\rho A^\sigma\right)\mathrm{d}^\tau A_\rho \; - g^{\sigma\tau}\mathcal{L} \qquad (17.25a)$$

quantized electromagnetic field:

$$\mathcal{T}^{\sigma\tau} = \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\sigma A_\rho)}\,\mathrm{d}^\tau A_\rho - g^{\sigma\tau}\mathcal{L} - Y$$
$$= -\frac{1}{\mu_0}\left(\mathrm{d}^\sigma A^\rho - \mathrm{d}^\rho A^\sigma\right)\mathrm{d}^\tau A_\rho \; - g^{\sigma\tau}\mathcal{L} - Y \qquad (17.25b)$$

$Y \equiv$ the sum of all terms in (17.25a), which do not depend on the particle-number operators $c_{\boldsymbol{k}}^{(u)\dagger} c_{\boldsymbol{k}}^{(u)}$

Because of $A^0 \equiv 0$, the Lagrangian may be written in the form

$$\mathcal{L} = -\frac{1}{2\mu_0}\Big((\mathrm{d}_\sigma A_j)\mathrm{d}^\sigma A^j - (\mathrm{d}_j A_l)\mathrm{d}^l A^j\Big)\,. \qquad (17.26)$$

Inserting the field operators (17.15a), the ES-tensor becomes

$$
\begin{aligned}
\mathcal{T}^{\sigma\tau} = & -Y - \frac{1}{\mu_0}\sum_{\boldsymbol{k},\boldsymbol{f}}\sum_{u,v=1}^{2}\frac{\mu_0 c^2\hbar^2}{2\hbar\Omega\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}}\, i^2\Big[\\
& -\sum_{j=1}^{3}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(j)})\cdot \\
& \quad\cdot\Big(-k^\tau c_{\boldsymbol{k}}^{(v)}\exp\{-ikx\}+k^\tau c_{\boldsymbol{k}}^{(v)\dagger}\exp\{+ikx\}\Big)\cdot \\
& \quad\cdot\Big(-f^\sigma c_{\boldsymbol{f}}^{(u)}\exp\{-ifx\}+f^\sigma c_{\boldsymbol{f}}^{(u)\dagger}\exp\{+ifx\}\Big)+ \\
& +(1-g^\sigma{}_0)\sum_{j=1}^{3}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(\sigma)})\cdot \\
& \quad\cdot\Big(-k^\tau c_{\boldsymbol{k}}^{(v)}\exp\{-ikx\}+k^\tau c_{\boldsymbol{k}}^{(v)\dagger}\exp\{+ikx\}\Big)\cdot \\
& \quad\cdot\Big(-f^j c_{\boldsymbol{f}}^{(u)}\exp\{-ifx\}+f^j c_{\boldsymbol{f}}^{(u)\dagger}\exp\{+ifx\}\Big)+ \\
& +\sum_{j=1}^{3}\frac{g^{\sigma\tau}}{2}(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(j)})\cdot \\
& \quad\cdot\Big(-k_\rho c_{\boldsymbol{k}}^{(v)}\exp\{-ikx\}+k_\rho c_{\boldsymbol{k}}^{(v)\dagger}\exp\{+ikx\}\Big)\cdot \\
& \quad\cdot\Big(-f^\rho c_{\boldsymbol{f}}^{(u)}\exp\{-ifx\}+f^\rho c_{\boldsymbol{f}}^{(u)\dagger}\exp\{+ifx\}\Big)+ \\
& +\sum_{j=1}^{3}\sum_{l=1}^{3}\frac{g^{\sigma\tau}}{2}(\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(l)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})\cdot \\
& \quad\cdot\Big(-f_j c_{\boldsymbol{f}}^{(u)}\exp\{-ifx\}+f_j c_{\boldsymbol{f}}^{(u)\dagger}\exp\{+ifx\}\Big)\cdot \\
& \quad\cdot\Big(-k^l c_{\boldsymbol{k}}^{(v)}\exp\{-ikx\}+k^l c_{\boldsymbol{k}}^{(v)\dagger}\exp\{+ikx\}\Big)\Big]\,. \qquad (17.27)
\end{aligned}
$$

Here the summation symbols have been written explicitly, because we don't sum automatically over the bracketed names (v) of the unit vectors. Note that $\boldsymbol{e}^{(\sigma)}=0$ for $\sigma=0$. As a reminder the factor $(1-g^\sigma{}_0)$ was inserted.

Now we rotate the coordinate system such, that it becomes congruent to the system which is aligned to the polarization of the field $A(x)$:

$$(\boldsymbol{e}_{\boldsymbol{k}}^{(u)}\cdot\boldsymbol{e}^{(j)}) \overset{(K.13c)}{=} g_u{}^j \quad \text{and} \quad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ k^3 \end{pmatrix} \tag{17.28}$$

That simplifies the ES-tensor to

$$\begin{aligned}
\mathcal{T}^{\sigma\tau} = & -Y - \frac{1}{\mu_0}\sum_{\boldsymbol{k},\boldsymbol{f}}\sum_{u,v=1}^{2}\frac{\mu_0 c^2\hbar^2}{2\hbar\Omega\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}}\, i^2 \Big[\\
& - f^\sigma k^\tau \Big(- c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)\dagger} \exp\{+ikx\}\Big)\cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{f}}^{(v)} \exp\{-ifx\} + c_{\boldsymbol{f}}^{(v)\dagger} \exp\{+ifx\}\Big) + \\
& + (1 - g^\sigma{}_0) g^\sigma{}_u f^v k^\tau \cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)\dagger} \exp\{+ikx\}\Big)\cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{f}}^{(u)} \exp\{-ifx\} + c_{\boldsymbol{f}}^{(u)\dagger} \exp\{+ifx\}\Big) + \\
& + \frac{g^{\sigma\tau}}{2} k_\rho f^\rho \Big(- c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)\dagger} \exp\{+ikx\}\Big)\cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{f}}^{(v)} \exp\{-ifx\} + c_{\boldsymbol{f}}^{(v)\dagger} \exp\{+ifx\}\Big) + \\
& + \frac{g^{\sigma\tau}}{2} f_v k^u \Big(- c_{\boldsymbol{f}}^{(u)} \exp\{-ifx\} + c_{\boldsymbol{f}}^{(u)\dagger} \exp\{+ifx\}\Big)\cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)\dagger} \exp\{+ikx\}\Big)\Big] .
\end{aligned} \tag{17.29}$$

$f_v = k^u = 0$ due to (17.28). Thus only two terms survive.

$$\begin{aligned}
\mathcal{T}^{\sigma\tau} = & -Y + \frac{1}{\Omega}\sum_{\boldsymbol{k},\boldsymbol{f}}\sum_{v=1}^{2}\frac{c^2\hbar^2}{2\hbar\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}}\Big(- f^\sigma k^\tau + \frac{g^{\sigma\tau}}{2} k_\rho f^\rho\Big)\cdot \\
& \cdot\Big(+ c_{\boldsymbol{k}}^{(v)} c_{\boldsymbol{f}}^{(v)} \exp\{-i(k+f)x\} - c_{\boldsymbol{k}}^{(v)} c_{\boldsymbol{f}}^{(v)\dagger} \exp\{-i(k-f)x\} \\
& \quad - c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{f}}^{(v)} \exp\{+i(k-f)x\} + c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{f}}^{(v)\dagger} \exp\{+i(k+f)x\}\Big)
\end{aligned}$$

Integrating over the total normalization volume Ω, one finds the operator

$$T^{\sigma\tau} \equiv \int_\Omega \mathrm{d}^3x\, \mathcal{T}^{\sigma\tau} \ . \tag{17.30}$$

To make use in this integral of the Kronecker-symbol in the form (16.3), we invert in two terms the signs of $\boldsymbol{f}$. This is possible, as the sum is running over all positive and negative $\boldsymbol{f}$. Using $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, one finds

$$\begin{aligned} T^{\sigma\tau} = -Y\Omega + \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \frac{c^2\hbar^2}{2\hbar\omega_{\boldsymbol{k}}} \Big(- k^\sigma k^\tau + \frac{g^{\sigma\tau}}{2}\, k_\rho k^\rho \Big) \cdot \\ \cdot \Big(+ c_{\boldsymbol{k}}^{(v)} c_{-\boldsymbol{k}}^{(v)} \exp\{-i2k^0x_0\} - c_{\boldsymbol{k}}^{(v)} c_{\boldsymbol{k}}^{(v)\dagger} - \\ - c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)} + c_{\boldsymbol{k}}^{(v)\dagger} c_{-\boldsymbol{k}}^{(v)\dagger} \exp\{+i2k^0x_0\} \Big) \ . \end{aligned} \tag{17.31}$$

The frequency $2k^0c$ of the Zitterbewegung (see the discussion at (8.93)) can be arbitrarily low for the mass-less electromagnetic field. Thus these terms can not be simply ignored on grounds of their high frequency, as we did in case of the Dirac field and the Klein-Gordon field. We still can neglect these terms for the following reason: The operator $c_{\boldsymbol{k}}^{(v)\dagger} c_{-\boldsymbol{k}}^{(v)\dagger}$ creates a photon with wave-number $\boldsymbol{k}$ and polarization v, and creates a photon with wave-number $-\boldsymbol{k}$ and polarization v. These two photons mutually compensate exactly, i. e. the term with this operator has no net effect, and can be skipped. For the same reason, the term with the operator $c_{\boldsymbol{k}}^{(v)} c_{-\boldsymbol{k}}^{(v)}$ has not net effect and can be skipped. Furthermore $k_\rho k^\rho = 0$, as the electromagnetic field is massless. Thus the operator simplifies further to

$$T^{\sigma\tau} = -Y\Omega + \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \frac{c^2\hbar^2}{2\hbar\omega_{\boldsymbol{k}}}\, k^\sigma k^\tau \left(c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)} + c_{\boldsymbol{k}}^{(v)} c_{\boldsymbol{k}}^{(v)\dagger} \right) ,$$

and thus we get:

$$\boxed{T^{\sigma\tau} \equiv \int_\Omega \mathrm{d}^3x\, \mathcal{T}^{\sigma\tau} = \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \frac{c^2\hbar^2}{\hbar\omega_{\boldsymbol{k}}}\, k^\sigma k^\tau\, c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)}} \tag{17.32}$$

Using $ck^0 = \omega_{\boldsymbol{k}}$, we get in particular the Hamilton operator

$$\boxed{H \equiv T^{00} = \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \hbar \omega_{\boldsymbol{k}} c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)}} \tag{17.33}$$

and the momentum operator

$$\boxed{P^j \equiv \frac{1}{c} T^{0j} = \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \hbar k^j c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)}} \,. \tag{17.34}$$

In section 5.7 we derived the three space-like conserved angular momenta

$$M^{jl} \overset{(5.104)}{=} \underbrace{\int_\Omega \mathrm{d}^3x \left(x^j \mathcal{P}^l - x^l \mathcal{P}^j \right)}_{\text{orbital angular momentum}} + \underbrace{\int_\Omega \mathrm{d}^3x \, \mathcal{S}^{0jl}}_{\text{spin}} \tag{17.35}$$

$$\text{with } jl = 23, 31, 12$$

for arbitrary vector fields, which are defined in four-dimensional space-time. As long as we are only considering (like we have done so far) plane waves which are de-localized over all space, the orbital angular momenta will be zero, because in the integral $\int_\Omega \mathrm{d}^3x$ over all position space, the contributions of $+x^j$ and $-x^j$ will just compensate. But even with plane waves, there exists a spin which is different from zero:

$$\mathcal{S}^{0jl} \overset{(5.99)}{=} \frac{1}{c} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_0 A_j)} A^l - \frac{1}{c} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_0 A_l)} A^j \overset{(17.3b)}{=} \pi^j A^l - \pi^l A^j$$

By insertion of the field-operators (17.15), the spin can be computed:

$$\mathcal{S}^{0jl} \overset{(17.3b)}{=} -\frac{i\hbar}{2\Omega}\sum_{\boldsymbol{k},\boldsymbol{f}}\sum_{u,v=1}^{2}\sqrt{\frac{\omega_{\boldsymbol{f}}}{\omega_{\boldsymbol{k}}}}\cdot$$
$$\cdot\Big((\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)}) - (\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(l)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})\Big)\cdot$$
$$\cdot\Big(-c_{\boldsymbol{f}}^{(u)}\exp\{-ifx\} + c_{\boldsymbol{f}}^{(u)\dagger}\exp\{+ifx\}\Big)\cdot$$
$$\cdot\Big(c_{\boldsymbol{k}}^{(v)}\exp\{-ikx\} + c_{\boldsymbol{k}}^{(v)\dagger}\exp\{+ikx\}\Big) \tag{17.36}$$

We multiply the both last lines, and exchange in two of the four terms $\boldsymbol{f}$ by $-\boldsymbol{f}$. That may be done, because the summation is running symmetrically over all positive and negative $\boldsymbol{f}$:

$$\mathcal{S}^{0jl} = -\frac{i\hbar}{2\Omega}\sum_{\boldsymbol{k},\boldsymbol{f}}\sum_{u,v=1}^{2}\Bigg($$
$$\sqrt{\frac{\omega_{\boldsymbol{f}}}{\omega_{\boldsymbol{k}}}}\Big((\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)}) - (\boldsymbol{e}_{\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(l)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})\Big)$$
$$\Big(-c_{\boldsymbol{f}}^{(u)}c_{\boldsymbol{k}}^{(v)\dagger}\exp\{-i(f^0-k^0)x^0\}\exp\{+i(\boldsymbol{f}-\boldsymbol{k})\boldsymbol{x}\}$$
$$+c_{\boldsymbol{f}}^{(u)\dagger}c_{\boldsymbol{k}}^{(v)}\exp\{+i(f^0-k^0)x^0\}\exp\{-i(\boldsymbol{f}-\boldsymbol{k})\boldsymbol{x}\}\Big)$$
$$-\sqrt{\frac{\omega_{-\boldsymbol{f}}}{\omega_{\boldsymbol{k}}}}\Big((\boldsymbol{e}_{-\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(l)}) - (\boldsymbol{e}_{-\boldsymbol{f}}^{(u)}\cdot\boldsymbol{e}^{(l)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)}\cdot\boldsymbol{e}^{(j)})\Big)$$
$$\Big(-c_{-\boldsymbol{f}}^{(u)}c_{\boldsymbol{k}}^{(v)}\exp\{-i(f^0+k^0)x^0\}\exp\{+i(-\boldsymbol{f}+\boldsymbol{k})\boldsymbol{x}\}$$
$$+c_{-\boldsymbol{f}}^{(u)\dagger}c_{\boldsymbol{k}}^{(v)\dagger}\exp\{+i(f^0+k^0)x^0\}\exp\{-i(-\boldsymbol{f}+\boldsymbol{k})\boldsymbol{x}\}\Big)\Bigg)$$

When we now integrate over the total normalization volume, then we get the Kronecker-symbol because of

$$\delta_{\boldsymbol{k}\boldsymbol{f}} \overset{(7.12)}{=} \frac{1}{\Omega}\int_{\Omega}\mathrm{d}^3x\,\exp\{-i(\boldsymbol{k}-\boldsymbol{f})\boldsymbol{x}\}\ . \tag{17.37}$$

Considering $\boldsymbol{e}_{-\boldsymbol{k}}^{(u)} = -\boldsymbol{e}_{\boldsymbol{k}}^{(u)}$, we get the spin-operator

$$
\begin{aligned}
S^{jl} \equiv \int\limits_{\Omega} \mathrm{d}^3x\, \mathcal{S}^{0jl} = -\frac{i\hbar}{2} \sum_{\boldsymbol{k}} \sum_{u,v=1}^{2} \\
\Big((\boldsymbol{e}_{\boldsymbol{k}}^{(u)} \cdot \boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(l)}) - (\boldsymbol{e}_{\boldsymbol{k}}^{(u)} \cdot \boldsymbol{e}^{(l)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(j)}) \Big) \\
\Big(- c_{\boldsymbol{k}}^{(u)} c_{\boldsymbol{k}}^{(v)\dagger} + c_{\boldsymbol{k}}^{(u)\dagger} c_{\boldsymbol{k}}^{(v)} + c_{-\boldsymbol{k}}^{(u)} c_{\boldsymbol{k}}^{(v)} \exp\{-i(2k^0)x^0\} \\
- c_{-\boldsymbol{k}}^{(u)\dagger} c_{\boldsymbol{k}}^{(v)\dagger} \exp\{+i(2k^0)x^0\} \Big) \, .
\end{aligned}
\tag{17.38}
$$

It's visible from the second line, that the sum over the indices u and v is skew-symmetric. Consequently, the both terms with exponential functions vanish for the following reason: As the summation is symmetrically over all positive and negative $\boldsymbol{k}$, and as k^0 according to (7.18) is identical for $\boldsymbol{k}$ and for $-\boldsymbol{k}$, and as the Fourier-operators with different Index (1) or (2) commute according to (17.17), we have

$$
\begin{aligned}
\sum_{\boldsymbol{k}} \left(c_{-\boldsymbol{k}}^{(1)} c_{\boldsymbol{k}}^{(2)} - c_{-\boldsymbol{k}}^{(2)} c_{\boldsymbol{k}}^{(1)} \right) \exp\{-i(2k^0)x^0\} \; = \\
= \sum_{\boldsymbol{k}} \left(c_{-\boldsymbol{k}}^{(1)} c_{\boldsymbol{k}}^{(2)} - c_{-\boldsymbol{k}}^{(1)} c_{\boldsymbol{k}}^{(2)} \right) \exp\{-i(2k^0)x^0\} = 0 \, .
\end{aligned}
\tag{17.39}
$$

Therefore the operator simplifies to

$$
\begin{aligned}
S^{jl} = \frac{i\hbar}{2} \sum_{\boldsymbol{k}} \sum_{u,v=1}^{2} \Big((\boldsymbol{e}_{\boldsymbol{k}}^{(u)} \cdot \boldsymbol{e}^{(j)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(l)}) - (\boldsymbol{e}_{\boldsymbol{k}}^{(u)} \cdot \boldsymbol{e}^{(l)})(\boldsymbol{e}_{\boldsymbol{k}}^{(v)} \cdot \boldsymbol{e}^{(j)}) \Big) \cdot \\
\cdot \left(c_{\boldsymbol{k}}^{(u)} c_{\boldsymbol{k}}^{(v)\dagger} - c_{\boldsymbol{k}}^{(u)\dagger} c_{\boldsymbol{k}}^{(v)} \right) \, .
\end{aligned}
\tag{17.40}
$$

The unit vectors $\boldsymbol{e}^{(j)}$ are aligned to the axes of the laboratory coordinate system. The unit vectors $\boldsymbol{e}_{\boldsymbol{k}}^{(u)}$ are by definition

$$
\boldsymbol{e}_{\boldsymbol{k}}^{(1)} \times \boldsymbol{e}_{\boldsymbol{k}}^{(2)} = \boldsymbol{e}_{\boldsymbol{k}}^{(3)} \overset{(17.9)}{=} \frac{\boldsymbol{k}}{|\boldsymbol{k}|}
\tag{17.41}
$$

aligned to the field's wavenumber-vector. The sequel evaluation of (17.40) will be simpler and clearer, if the laboratory system with the unit vectors

$\boldsymbol{e}^{(j)}$ is rotated such, that it becomes congruent with the system of unit vectors $\boldsymbol{e}_{\boldsymbol{k}}^{(u)}$. Then

$$\boldsymbol{e}^{(j)} \equiv \boldsymbol{e}_{\boldsymbol{k}}^{(j)} \quad \text{for } j = 1, 2, 3 \; . \tag{17.42}$$

In the rotated coordinate system, the field propagates in direction of the x^3-axis. Because u and v only assume the values 1 and 2, all spin-operators with $j = 3$ or $l = 3$ then are zero, and only $S^{12} = -S^{21}$ is different from zero.

$$S^{12} = -S^{21} = i\hbar \sum_{\boldsymbol{k}} \left(c_{\boldsymbol{k}}^{(1)} c_{\boldsymbol{k}}^{(2)\dagger} - c_{\boldsymbol{k}}^{(1)\dagger} c_{\boldsymbol{k}}^{(2)} \right) \tag{17.43}$$

Different from the energy- and momentum-operators, which we evaluated before, this spin-operator does not have the form of a particle-number operator $c_{\boldsymbol{k}}^{(u)+} c_{\boldsymbol{k}}^{(u)}$. Therefore it's significance is not obvious. This difficulty appears, because — as emphasized at (17.13) — we considered only linear polarized waves, when we quantized the field. Waves with circular polarization, which are propagating in direction of the x^3-axis, can be constructed by adding to a wave, which is linearly polarized in x^1-direction and propagating in x^3-direction, another wave, which is linearly polarized in x^2-direction and propagating in x^3-direction, with same amplitude and phase difference $\pm\pi/2$:

$$A^{(L/R)} = \sqrt{\tfrac{1}{2}} \left(A^{(1)} + A^{(2)} \exp\{\pm i\pi/2\} \right) = \sqrt{\tfrac{1}{2}} \left(A^{(1)} \pm i A^{(2)} \right)$$

Thus the creation-operators of right- and left-circular polarized photon are

$$c_{\boldsymbol{k}}^{(R)\dagger} \equiv \sqrt{\tfrac{1}{2}} \left(c_{\boldsymbol{k}}^{(1)\dagger} + i c_{\boldsymbol{k}}^{(2)\dagger} \right) \tag{17.44a}$$

$$c_{\boldsymbol{k}}^{(L)\dagger} \equiv \sqrt{\tfrac{1}{2}} \left(c_{\boldsymbol{k}}^{(1)\dagger} - i c_{\boldsymbol{k}}^{(2)\dagger} \right) \; . \tag{17.44b}$$

The inverse formulae are

$$c_{\boldsymbol{k}}^{(1)\dagger} = \sqrt{\tfrac{1}{2}} \left(c_{\boldsymbol{k}}^{(R)\dagger} + c_{\boldsymbol{k}}^{(L)\dagger} \right) \tag{17.45a}$$

$$c_{\boldsymbol{k}}^{(2)\dagger} = -i\sqrt{\tfrac{1}{2}}\left(c_{\boldsymbol{k}}^{(R)\dagger} - c_{\boldsymbol{k}}^{(L)\dagger}\right) . \tag{17.45b}$$

Using these operators, (17.43) can be written as

$$\begin{aligned} S^{12} = -S^{21} = +\hbar \sum_{\boldsymbol{k}} \tfrac{1}{2}\Big(\\ + c_{\boldsymbol{k}}^{(R)} c_{\boldsymbol{k}}^{(R)\dagger} - c_{\boldsymbol{k}}^{(R)} c_{\boldsymbol{k}}^{(L)\dagger} + c_{\boldsymbol{k}}^{(L)} c_{\boldsymbol{k}}^{(R)\dagger} - c_{\boldsymbol{k}}^{(L)} c_{\boldsymbol{k}}^{(L)\dagger} \\ + c_{\boldsymbol{k}}^{(R)\dagger} c_{\boldsymbol{k}}^{(R)} - c_{\boldsymbol{k}}^{(R)\dagger} c_{\boldsymbol{k}}^{(L)} + c_{\boldsymbol{k}}^{(L)\dagger} c_{\boldsymbol{k}}^{(R)} - c_{\boldsymbol{k}}^{(L)\dagger} c_{\boldsymbol{k}}^{(L)}\Big) . \end{aligned} \tag{17.46}$$

Inserting the definitions (17.44), and using the relations (17.17), the commutators

$$[c_{\boldsymbol{k}}^{(R)}, c_{\boldsymbol{k}}^{(R)\dagger}] = \tfrac{1}{2}\Big([c_{\boldsymbol{k}}^{(1)}, c_{\boldsymbol{k}}^{(1)\dagger}] + [c_{\boldsymbol{k}}^{(2)}, c_{\boldsymbol{k}}^{(2)\dagger}]\Big) = 1 \tag{17.47a}$$

$$[c_{\boldsymbol{k}}^{(L)}, c_{\boldsymbol{k}}^{(L)\dagger}] = \tfrac{1}{2}\Big([c_{\boldsymbol{k}}^{(1)}, c_{\boldsymbol{k}}^{(1)\dagger}] + [c_{\boldsymbol{k}}^{(2)}, c_{\boldsymbol{k}}^{(2)\dagger}]\Big) = 1 \tag{17.47b}$$

$$[c_{\boldsymbol{k}}^{(R)}, c_{\boldsymbol{k}}^{(L)\dagger}] = \tfrac{1}{2}\Big([c_{\boldsymbol{k}}^{(1)}, c_{\boldsymbol{k}}^{(1)\dagger}] - [c_{\boldsymbol{k}}^{(2)}, c_{\boldsymbol{k}}^{(2)\dagger}]\Big) = 0 \tag{17.47c}$$

$$[c_{\boldsymbol{k}}^{(L)}, c_{\boldsymbol{k}}^{(R)\dagger}] = \tfrac{1}{2}\Big([c_{\boldsymbol{k}}^{(1)}, c_{\boldsymbol{k}}^{(1)\dagger}] - [c_{\boldsymbol{k}}^{(2)}, c_{\boldsymbol{k}}^{(2)\dagger}]\Big) = 0 \tag{17.47d}$$

can be computed. Thus one finds

$$\begin{aligned} S^{12} = -S^{21} = \\ = +\hbar \sum_{\boldsymbol{k}} \tfrac{1}{2}\Big(2c_{\boldsymbol{k}}^{(R)\dagger} c_{\boldsymbol{k}}^{(R)} + \underbrace{[c_{\boldsymbol{k}}^{(R)}, c_{\boldsymbol{k}}^{(R)\dagger}] - [c_{\boldsymbol{k}}^{(L)}, c_{\boldsymbol{k}}^{(L)\dagger}]}_{0} - 2c_{\boldsymbol{k}}^{(L)\dagger} c_{\boldsymbol{k}}^{(L)}\Big) , \end{aligned}$$

and the operator for the conserved spin of a field, which is propagating in direction of the positive x^3-axis, becomes

$$\boxed{S^{12} = +\hbar \sum_{\boldsymbol{k}} \left(c_{\boldsymbol{k}}^{(R)\dagger} c_{\boldsymbol{k}}^{(R)} - c_{\boldsymbol{k}}^{(L)\dagger} c_{\boldsymbol{k}}^{(L)}\right)} . \tag{17.48}$$

17.3 Photons

For the sake of a simple and clear notation, we continue to use — according to (17.42) — a coordinate system, whose x^3-axis is rotated into the direction of the field's propagation. When the creation operators $c_{\boldsymbol{f}}^{(1)\dagger}$ and $c_{\boldsymbol{f}}^{(2)\dagger}$ are acting onto the vacuum state $|0\rangle$, they create photons, which are propagating in parallel to the x^3-axis, which are linear polarized in parallel to the x^1- resp. x^2-axis, and whose wavenumber is $\boldsymbol{f}$:

$$c_{\boldsymbol{f}}^{(1)\dagger}|0\rangle = |1_{\boldsymbol{f}1}\rangle \qquad\qquad c_{\boldsymbol{f}}^{(2)\dagger}|0\rangle = |1_{\boldsymbol{f}2}\rangle \tag{17.49}$$

The large 1 in the state-functions signifies as usual, that 1 quantum of the field is excited to this state. The normalization of these state-functions is Lorentz-invariant:

$$\begin{aligned}\langle 1_{\boldsymbol{k}u}|1_{\boldsymbol{f}v}\rangle = \langle 0|\, c_{\boldsymbol{k}}^{(u)} c_{\boldsymbol{f}}^{(v)\dagger}\,|0\rangle &= \underbrace{\langle 0|0\rangle}_{1}\,[c_{\boldsymbol{k}}^{(u)}, c_{\boldsymbol{f}}^{(v)\dagger}] + \underbrace{\langle 0|\, c_{\boldsymbol{f}}^{(v)\dagger} c_{\boldsymbol{k}}^{(u)}\,|0\rangle}_{0} \\ &\overset{(17.17)}{=} \delta_{\boldsymbol{k}\boldsymbol{f}}\, {g_v}^u\end{aligned} \tag{17.50}$$

Circular polarized photons are created by

$$\begin{aligned} c_{\boldsymbol{f}}^{(R)\dagger}|0\rangle = |1_{\boldsymbol{f}R}\rangle &= \sqrt{\tfrac{1}{2}}\Big(c_{\boldsymbol{f}}^{(1)\dagger} + i c_{\boldsymbol{f}}^{(2)\dagger}\Big)|0\rangle \\ &= \sqrt{\tfrac{1}{2}}\Big(|1_{\boldsymbol{f}1}\rangle + i|1_{\boldsymbol{f}2}\rangle\Big) \qquad (17.51a) \\ c_{\boldsymbol{f}}^{(L)\dagger}|0\rangle = |1_{\boldsymbol{f}L}\rangle &= \sqrt{\tfrac{1}{2}}\Big(c_{\boldsymbol{f}}^{(1)\dagger} - i c_{\boldsymbol{f}}^{(2)\dagger}\Big)|0\rangle \\ &= \sqrt{\tfrac{1}{2}}\Big(|1_{\boldsymbol{f}1}\rangle - i|1_{\boldsymbol{f}2}\rangle\Big)\,. \qquad (17.51b) \end{aligned}$$

The inverse relations are

$$|1_{\boldsymbol{f}1}\rangle = \sqrt{\tfrac{1}{2}}\Big(|1_{\boldsymbol{f}R}\rangle + |1_{\boldsymbol{f}L}\rangle\Big) \tag{17.52a}$$

$$|1_{\boldsymbol{f}2}\rangle = -i\sqrt{\tfrac{1}{2}}\Big(|1_{\boldsymbol{f}R}\rangle - |1_{\boldsymbol{f}L}\rangle\Big)\,. \tag{17.52b}$$

The circular polarized photons are eigenstates of the spin operator S^{12} with eigenvalues $\pm\hbar$:

$$S^{12}\,|1_{fR}\rangle = +\hbar\,|1_{fR}\rangle \tag{17.53a}$$

$$S^{12}\,|1_{fL}\rangle = -\hbar\,|1_{fL}\rangle \tag{17.53b}$$

Remarkably the photon's spin does not depend on the value of the wavenumber. It's spin-component in direction of propagation always is $+\hbar$ or $-\hbar$, it's spin-component in a direction, which is orthogonal to the direction of propagation, is always zero. Physically more correct is the wording: If the spin of a photon is measured, then it's value is always $\pm\hbar$ parallel to the photon's direction of propagation. It would not be correct to say, that each photon "has" that spin. If a photon has been reflected from a plane glass surface under the Brewster-angle, then it has a well-defined transversal polarization, for example in space direction x^2. The state function of this photon is $|1_{f2}\rangle$. The expectation value of it's spin is

$$\begin{aligned}\langle 1_{f2}|S^{12}|1_{f2}\rangle &\overset{(17.52b)}{=} \tfrac{1}{2}\Big(\langle 1_{fR}|S^{12}|1_{fR}\rangle - \langle 1_{fR}|S^{12}|1_{fL}\rangle - \\ &\qquad - \langle 1_{fL}|S^{12}|1_{fR}\rangle + \langle 1_{fL}|S^{12}|1_{fL}\rangle\Big) = \\ &= \tfrac{1}{2}\hbar\Big(+ \underbrace{\langle 1_{fR}|1_{fR}\rangle}_{1} + \underbrace{\langle 1_{fR}|1_{fL}\rangle}_{0} - \underbrace{\langle 1_{fL}|1_{fR}\rangle}_{0} - \underbrace{\langle 1_{fL}|1_{fL}\rangle}_{1}\Big) \\ &= 0\ . \end{aligned} \tag{17.54}$$

This is not to say, that a photon in state $|1_{f2}\rangle$ has spin zero. If the spin of a large set of photons is measured, which all have been prepared in the state $|1_{f2}\rangle$, then one will find the results $+\hbar$ and $-\hbar$ with equal frequency. The result zero is never observed, when the spin of arbitrarily prepared photons is measured.

There is no one-photon-eigenstate of the spin operator S^{12} with eigenvalue $\hbar = 0$. If the spin of a massive particle is J, then the spin's projection onto an arbitrary axis can assume one of the values $J, J-\hbar, J-2\hbar, \ldots, -J$. For the photon, the value zero is missing, because it's rest mass is zero. It's a consequence of the vanishing rest mass of the photon, that it is either

approaching the observer at the speed of light, or it is moving off from the observer at the speed of light. The observer can not transform himself into a coordinate system, in which he could "see the photon from the side" and measure the spin projection zero. Even though the projection of the spin onto the axis of propagation of the photon can only assume two values, the spin's value is $1\hbar$. Thus the photon is a boson, and therefore the gauge field $A(x)$ has been quantized according to (14.74a).

We define the operator

$$L^{12}|1_{\boldsymbol{k}1}\rangle = +1|1_{\boldsymbol{k}1}\rangle \qquad L^{12}|1_{\boldsymbol{k}2}\rangle = -1|1_{\boldsymbol{k}2}\rangle \ . \tag{17.55}$$

This operator is describing the measurement of the linear polarization of photons. The expectation value of the linear polarization of a photon, which has been prepared in a state with circular polarization, is

$$\begin{aligned}\langle 1_{\boldsymbol{k}R}|L^{12}|1_{\boldsymbol{k}R}\rangle &\overset{(17.51a)}{=} \tfrac{1}{2}\Big(\langle 1_{\boldsymbol{k}1}|L^{12}|1_{\boldsymbol{k}1}\rangle + i\langle 1_{\boldsymbol{k}1}|L^{12}|1_{\boldsymbol{k}2}\rangle + \\ &\quad - i\langle 1_{\boldsymbol{k}2}|L^{12}|1_{\boldsymbol{k}1}\rangle + \langle 1_{\boldsymbol{k}2}|L^{12}|1_{\boldsymbol{k}2}\rangle\Big) \\ &= \tfrac{1}{2}\Big(+ \underbrace{\langle 1_{\boldsymbol{k}1}|1_{\boldsymbol{k}1}\rangle}_{1} - i\,\underbrace{\langle 1_{\boldsymbol{k}1}|1_{\boldsymbol{k}2}\rangle}_{0} - i\,\underbrace{\langle 1_{\boldsymbol{k}2}|1_{\boldsymbol{k}1}\rangle}_{0} - \underbrace{\langle 1_{\boldsymbol{k}2}|1_{\boldsymbol{k}2}\rangle}_{1}\Big) \\ &= 0 \ . \end{aligned} \tag{17.56}$$

Circular polarized photons are either reflected or transmitted, if they impinge under the Brewster-angle onto a flat glass surface. In case of reflection, they are linearly polarized after the measurement, say in x^2-direction, and the result of the measurement is the eigenvalue -1 of the operator L^{12}. In case of transmission, the photons are transversally polarized in direction x^1 after the measurement, and the result of the measurement is the eigenvalue $+1$ of the operator L^{12}. The expectation value 0 in (17.56) does say, that both results will be observed with same frequency.

The linear polarized photons are eigenstates of the energy operator and of the momentum operator:

$$H|1_{\boldsymbol{f}1}\rangle \overset{(17.33)}{=} \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \hbar\omega_{\boldsymbol{k}} c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)} |1_{\boldsymbol{f}1}\rangle = \hbar\omega_{\boldsymbol{f}} |1_{\boldsymbol{f}1}\rangle \tag{17.57a}$$

$$P^l|1_{\boldsymbol{f}1}\rangle \overset{(17.34)}{=} \sum_{\boldsymbol{k}} \sum_{v=1}^{2} \hbar k^l c_{\boldsymbol{k}}^{(v)\dagger} c_{\boldsymbol{k}}^{(v)} |1_{\boldsymbol{f}1}\rangle = \hbar f^l |1_{\boldsymbol{f}1}\rangle \tag{17.57b}$$

Circular polarized photons are as well eigenstates of the energy operator and of the momentum operator:

$$\begin{aligned} H\,|1_{\boldsymbol{f}R}\rangle &= H\,\tfrac{1}{2}\Big(|1_{\boldsymbol{f}1}\rangle + i|1_{\boldsymbol{f}2}\rangle\Big) = \hbar\omega_{\boldsymbol{f}}\,\tfrac{1}{2}\Big(|1_{\boldsymbol{f}1}\rangle + i|1_{\boldsymbol{f}2}\rangle\Big) \\ &= \hbar\omega_{\boldsymbol{f}}\,|1_{\boldsymbol{f}R}\rangle \end{aligned} \tag{17.58a}$$

$$\begin{aligned} P^l\,|1_{\boldsymbol{f}R}\rangle &= P^l\,\tfrac{1}{2}\Big(|1_{\boldsymbol{f}1}\rangle + i|1_{\boldsymbol{f}2}\rangle\Big) = \hbar f^l\,\tfrac{1}{2}\Big(|1_{\boldsymbol{f}1}\rangle + i|1_{\boldsymbol{f}2}\rangle\Big) \\ &= \hbar f^l\,|1_{\boldsymbol{f}R}\rangle \end{aligned} \tag{17.58b}$$

Choosing a phase angle different from $\pm\pi/2$ in (17.44), elliptically polarized photons can be created, which are neither eigenstates of the spin operator S^{12} nor eigenstates of the operator L^{12} of linear polarization. Even these ellipticall polarized photons are eigenstates of the energy-operator H and of the momentum operators P^j.

All photons considered so far are completely de-localized. That means, that they can be found with equal probability at any point in the normalization volume Ω. In section 17.5, we will dwell on the description of localized photons.

17.4 Covariant Quantization

It's far from obvious, that the results of the previous sections are compatible with Special Relativity Theory. In particular the fact, that the null-component $A^0(x) \equiv 0$ is vanishing identically in the Coulomb gauge might indicate the opposite. Therefore we will now investigate a sophisticated method of quantization, which Gupta[1] and Bleuler[2] detected independently

[1] Suraj N. Gupta (∗1924)
[2] Konrad Bleuler (1912 - 1992)

in 1950. The essential trick of this method is, not to quantize immediately the gauge field $A(x)$, but to consider instead a field $\widetilde{A}(x) \not\equiv A(x)$, which is defined due to the following Lagrangian density:

$$\begin{aligned} \widetilde{\mathcal{L}} &\equiv -\frac{1}{4\mu_0}\Big(\widetilde{F}_{\sigma\tau}\widetilde{F}^{\sigma\tau} + 2(\mathrm{d}_\sigma \widetilde{A}^\sigma)\mathrm{d}_\tau \widetilde{A}^\tau\Big) \\ \widetilde{F}_{\sigma\tau} &\equiv \mathrm{d}_\sigma \widetilde{A}_\tau - \mathrm{d}_\tau \widetilde{A}_\sigma \ . \end{aligned} \tag{17.59}$$

Due to the second term, by which $\widetilde{\mathcal{L}}$ differs from $\mathcal{L} =$ (17.2), the field $\widetilde{A}(x)$ is not gauge-invariant. In particular it's null-component $\widetilde{A}^0(x) \not\equiv 0$ is different from zero, and can be quantized canonically.

In the sequel, we will perform the quantization of $\widetilde{A}(x)$ in a manifestly lorentzinvariant manner. It will turn out, that the transversal components of the field operators of the quantized field $\widetilde{A}(x)$ and the respective components of the field operators of the quantized gauge field $A(x)$ are identical. In contrast, the time-like and the longitudinal components of the field operators of $A(x)$ are zero, while the respective components of the operators of the quantized field $\widetilde{A}(x)$ in general are different from zero.

The "unphysical" components of the operators $\widetilde{A}$, due to which they differ from the operators A, will *not* be purged out later-on, because that would lead back to canonical quantization. Instead the "unphysical" parts of the state functions $|z\rangle$ of the operators $\widetilde{A}$ will be purged out by an appropriate condition. The operators $\widetilde{A}$, and the operators of the conserved quantities derived from them, will give the same results with the remaining "physical" parts of the state functions, as the canonically derived operators of the gauge field A.

To realize this program systematically, we derive from the Lagrangian by means of

$$\begin{aligned} &\frac{\partial\widetilde{\mathcal{L}}}{\partial(\mathrm{d}_\sigma \widetilde{A}_\tau)} = -\frac{1}{4\mu_0}\Big((\mathrm{d}^\sigma \widetilde{A}^\tau - \mathrm{d}^\tau \widetilde{A}^\sigma) - (\mathrm{d}^\tau \widetilde{A}^\sigma - \mathrm{d}^\sigma \widetilde{A}^\tau) + 2g^{\sigma\tau}\mathrm{d}_\nu \widetilde{A}^\nu\Big)\cdot 2 = \\ &= \frac{1}{\mu_0}(-\mathrm{d}^\sigma \widetilde{A}^\tau + \mathrm{d}^\tau \widetilde{A}^\sigma - g^{\sigma\tau}\mathrm{d}_\nu \widetilde{A}^\nu)\frac{\partial\widetilde{\mathcal{L}}}{\partial \widetilde{A}_\tau} = 0 \end{aligned}$$

the field equation

$$\mathrm{d}_\sigma \frac{\partial\widetilde{\mathcal{L}}}{\partial(\mathrm{d}_\sigma\widetilde{A}_\tau)} - \frac{\partial\widetilde{\mathcal{L}}}{\partial\widetilde{A}_\tau} = \frac{1}{\mu_0}(-\mathrm{d}_\sigma\mathrm{d}^\sigma\widetilde{A}^\tau \underbrace{+\mathrm{d}^\tau\mathrm{d}_\sigma\widetilde{A}^\sigma - \mathrm{d}^\tau\mathrm{d}_\nu\widetilde{A}^\nu}_{0}) - 0 \overset{(3.37a)}{=} 0$$

$$\Longrightarrow \quad \mathrm{d}_\sigma\mathrm{d}^\sigma\widetilde{A}^\tau = 0\ . \tag{17.60}$$

Only the field $A(x)$, but not the field $\widetilde{A}(x)$, is invariant under the gauge transformation (17.6). But due to the additional second term in the Lagrangian (17.59), the same field equation holds for $\widetilde{A}(x)$ without a gauge transformation, as for $A(x)$ with the Coulomb-gauge (17.8). Hence also for $\widetilde{A}(x)$ the Ansatz

$$\widetilde{A} \overset{(17.12)}{=} \sum_{\boldsymbol{k}}\sum_{\alpha=0}^{3}\sqrt{\frac{\mu_0 c^2\hbar}{2\omega_{\boldsymbol{k}}\Omega}}e_{\boldsymbol{k}}^{(\alpha)}\left(c_{\boldsymbol{k}}^{(\alpha)}\exp\{-ikx\} + c_{\boldsymbol{k}}^{(\alpha)*}\exp\{+ikx\}\right) \tag{17.61}$$

can be used. But as $\widetilde{A}(x)$ in general has four components which are different from zero, we now must use the thin printed unit vectors with four components, and the summations must run over all four space-time-coordinates, because the equations

$$A^0 \overset{(17.8)}{=} 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{A} \overset{(17.8)}{=} 0$$

do not hold for $\widetilde{A}(x)$. The longitudinal and the time-like components of $\widetilde{A}$ can be different from zero.

The components of the momentum density $\widetilde{\pi}(x)$, which is canonically conjugated to $\widetilde{A}(x)$, are

$$\widetilde{\pi}^\tau \overset{(17.3)}{=} \frac{\partial\widetilde{\mathcal{L}}}{c\partial(\mathrm{d}_0\widetilde{A}_\tau)} \overset{(17.60)}{=} \frac{1}{c\mu_0}(-\mathrm{d}^0\widetilde{A}^\tau + \mathrm{d}^\tau\widetilde{A}^0 - g^{0\tau}\mathrm{d}_\nu\widetilde{A}^\nu)\ . \tag{17.62}$$

Inserting (17.61), and using

$$\widetilde{A}^\lambda \overset{(K.10b)}{=} \sum_{\kappa=0}^{3} g^{\lambda\kappa}e^{(\kappa)}\cdot\widetilde{A}\ , \tag{17.63}$$

results into

$$\widetilde{\pi}^{\tau}(y) = \sum_{\boldsymbol{f}} \sum_{\beta=0}^{3} \sum_{\kappa=0}^{3} (e^{(\kappa)} \cdot e_{\boldsymbol{f}}^{(\beta)}) \sqrt{\frac{\hbar}{\mu_0 2\omega_{\boldsymbol{f}} \Omega}} \cdot$$
$$\cdot i\Big(- f^0 g^{\tau\kappa} + f^{\tau} g^{0\kappa} - g^{0\tau} f^{\kappa} \Big) \cdot$$
$$\cdot \Big(- c_{\boldsymbol{f}}^{(\beta)} \exp\{-ify\} + c_{\boldsymbol{f}}^{(\beta)*} \exp\{+ify\} \Big) \ . \tag{17.64}$$

To quantize the field, we postulate for

$$\widetilde{A}_{\mu} \overset{(K.10c)}{=} e^{(\mu)} \cdot \widetilde{A} \tag{17.65}$$

and for $\widetilde{\pi}^{\tau}$ in compliance with the general rule (14.74a) this non-commutative algebra:

$$[\widetilde{A}_{\mu}(t,\boldsymbol{x}), \widetilde{\pi}^{\tau}(t,\boldsymbol{y})] = i\hbar \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) g_{\mu}{}^{\tau} \tag{17.66a}$$
$$[\widetilde{A}_{\mu}(t,\boldsymbol{x}), \widetilde{A}^{\tau}(t,\boldsymbol{y})] = [\widetilde{\pi}_{\mu}(t,\boldsymbol{x}), \widetilde{\pi}^{\tau}(t,\boldsymbol{y})] = 0 \tag{17.66b}$$

It is proved in appendix A.15, that the following commutator-relations of the Fourier-operators follow from (17.66):

$$[c_{\boldsymbol{k}}^{(\alpha)}, c_{\boldsymbol{f}}^{(\beta)\dagger}] = -g_{\alpha\beta}\, \delta_{\boldsymbol{k}\boldsymbol{f}} \tag{17.67a}$$
$$[c_{\boldsymbol{k}}^{(\alpha)}, c_{\boldsymbol{f}}^{(\beta)}] = [c_{\boldsymbol{k}}^{(\alpha)\dagger}, c_{\boldsymbol{f}}^{(\beta)\dagger}] = 0 \tag{17.67b}$$

The transversal components ($\alpha = 1$ and $\alpha = 2$) of these commutator-relations are identical to (17.17). Note the strange negative sign of the time-like commutator

$$[c_{\boldsymbol{k}}^{(0)}, c_{\boldsymbol{f}}^{(0)\dagger}] = -g_{00}\, \delta_{\boldsymbol{k}\boldsymbol{f}} = -\delta_{\boldsymbol{k}\boldsymbol{f}} \ .$$

We will return to that immediately, but firstly we derive the operators of energy and of momentum of the quantized field $\widetilde{A}(x)$. The energy-density operator is

$$\widetilde{\mathcal{H}} \overset{(4.35)}{=} \frac{\partial \widetilde{\mathcal{L}}}{\partial(\mathrm{d}_0 \widetilde{A})}\, \mathrm{d}^0 \widetilde{A} - \widetilde{\mathcal{L}} = \widetilde{\pi}^{\sigma}(x)\, \dot{\widetilde{A}}_{\sigma}(x) - \widetilde{\mathcal{L}} \ . \tag{17.68}$$

It is shown in appendix A.17, that from (17.68) the Hamilton-operator

$$\widetilde{H} \equiv \int_{\Omega} \mathrm{d}^3x\, \widetilde{\mathcal{H}} = -\sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g^{\alpha\beta}\, \hbar\omega_{\boldsymbol{k}}\, c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)} \tag{17.69}$$

follows. The momentum density of the field $\widetilde{A}(x)$ is

$$\widetilde{\mathcal{P}}^j \overset{(4.35)}{=} \frac{1}{c} \frac{\partial \widetilde{\mathcal{L}}}{\partial(\mathrm{d}_0 \widetilde{A})}\, \mathrm{d}^j \widetilde{A} = \widetilde{\pi}^\tau(x)\, \mathrm{d}^j \widetilde{A}_\tau(x)\ . \tag{17.70}$$

In appendix A.18 it is shown, that from this equation the momentum operator

$$\widetilde{P}^j = \int_{\Omega} \mathrm{d}^3x\, \widetilde{\mathcal{P}}^j = -\sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g^{\alpha\beta} \hbar k^j\, c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)} \tag{17.71}$$

follows.

If one would exchange in these operators $\sum_{\alpha=0}^{3}$ by $\sum_{\alpha=1}^{2}$, then $\widetilde{H}$ would become identical to $H =$ (17.33), and $\widetilde{P}^j$ would become identical to $P^j =$ (17.34). Still without substantial modifications, the field $\widetilde{A}(x)$ is not suited to describe the photon field, because it was an essential pre-condition for the quantization of this field, that it's time-like and it's longitudinal components can be different from zero. Therefore this field has negative energy, if more quanta are excited with time-like polarization than with space-like polarization. That is of course physically not sensible.

A second, though more formal problem arises due to the negative sign of the time-like commutator (17.67). $|0\rangle$ is the quantized field's vacuum state. The vacuum state is characterized by

$$c_{\boldsymbol{k}}^{(\alpha)} |0\rangle = 0\ .$$

Application of the creation operator

$$c_{\boldsymbol{k}}^{(\alpha)\dagger} |0\rangle = |1_{(\alpha)\boldsymbol{k}}\rangle \tag{17.72}$$

onto the vacuum state creates a state, in which one quantum with polarization α and wavenumber $\boldsymbol{k}$ is excited. The norm of this state is

$$\langle 1_{(\beta)\boldsymbol{f}}|1_{(\alpha)\boldsymbol{k}}\rangle = \langle 0|c_{\boldsymbol{f}}^{(\beta)}c_{\boldsymbol{k}}^{(\alpha)\dagger}|0\rangle - \langle 0|c_{\boldsymbol{k}}^{(\alpha)\dagger}\underbrace{c_{\boldsymbol{f}}^{(\beta)}|0\rangle}_{0} =$$

$$= \langle 0|\,[c_{\boldsymbol{f}}^{(\beta)}, c_{\boldsymbol{k}}^{(\alpha)\dagger}]\,|0\rangle \overset{(17.67)}{=} -g^{\beta\alpha}\delta_{\boldsymbol{f}\boldsymbol{k}}\underbrace{\langle 0|0\rangle}_{1} \ . \tag{17.73}$$

If the quantum's polarization is time-like, then the norm is negative. It is quite difficult to find a reasonable interpretation of quantum fields with state functions, which may have positive or negative norm.

All these problems are caused by the additional term

$$2(\mathrm{d}_\sigma \widetilde{A}^\sigma)\mathrm{d}_\tau \widetilde{A}^\tau \ ,$$

by which the Lagrangian (17.59) of the field $\widetilde{A}$ differs from the Lagrangian (17.2) of the gauge field A. The problems can not be solved by forcing onto the field $\widetilde{A}$ the condition $\mathrm{d}_\sigma \widetilde{A}^\sigma(x) \equiv 0$, because then we would have $\widetilde{A}(x) \equiv A(x)$, and the just accomplished quantization (17.66) would be lost. It was the idea of Gupta and Bleuler, to force a restrictive condition not onto the operators, but onto the state functions. The restrictive condition is:

A state function $|z\rangle$ is admissible, if and only if it fulfills the condition

$$\langle z|\,\mathrm{d}_\sigma \widetilde{A}^\sigma\,|z\rangle = 0 \ . \tag{17.74}$$

Inadmissible state functions are to be discarded from the theory.

Note, that this is a covariant formulated condition. It does not change under an arbitrary rotation of the four-dimensional space-time coordinate system. It is the essential guideline of this quantization method, to adhere strictly to manifest Lorentz-invariance at any point of the procedure. To analyze the meaning and consequences of condition (17.74), consider the term

$$\mathrm{d}_\sigma \widetilde{A}^\sigma(x) \overset{(17.11)}{=}$$
$$= \sum_{\boldsymbol{k}} \sum_{v=0}^{3} \frac{1}{\sqrt{N\Omega}} \Big(-ik\epsilon_{\boldsymbol{k}}^{(v)} c_{\boldsymbol{k}}^{(v)} \exp\{-ikx\} + ik\epsilon_{\boldsymbol{k}}^{(v)*} c_{\boldsymbol{k}}^{(v)\dagger} \exp\{+ikx\} \Big) \ .$$

The second summand is the adjoint of the first one. Therefore the condition (17.74) is fulfilled, once it is fulfilled for the second summand:

$$\sum_{\boldsymbol{k}} \sum_{v=0}^{3} k\epsilon_{\boldsymbol{k}}^{(v)*} \langle z| \, c_{\boldsymbol{k}}^{(v)\dagger} \, |z\rangle \, \exp\{+ikx\} = 0$$
$$\Longrightarrow \qquad \sum_{v=0}^{3} k e_{\boldsymbol{k}}^{(v)} \langle z| \, c_{\boldsymbol{k}}^{(v)\dagger} \, |z\rangle = 0 \ .$$

Here all superfluous factors have been discarded. In the laboratory system, the coordinates of k are $k = (k^0, k^1, k^2, k^3)$. The condition becomes more transparent, if the coordinate system is rotated such, that it becomes congruent with the system which is aligned to the wave vector $\boldsymbol{k}$, and spanned by the unit vectors $e_{\boldsymbol{k}}^{(v)}$. Then the components of k are $k = (|\boldsymbol{k}|, 0, 0, |\boldsymbol{k}|)$ because of $k^2 = (k^0)^2 - \boldsymbol{k}^2 = 0$, and condition (17.74) simplifies — using (K.10a) — to

$$\sum_{v=0}^{3} g_{vu} k^u \langle z| \, c_{\boldsymbol{k}}^{(v)\dagger} \, |z\rangle = 0$$
$$\Longrightarrow \qquad \langle z| \, c_{\boldsymbol{k}}^{(0)\dagger} - c_{\boldsymbol{k}}^{(3)\dagger} \, |z\rangle = 0 \ . \tag{17.75}$$

Therefore a state of the field, in which an arbitrary number of photons with wave number $\boldsymbol{k}$ is excited, meets the criterion (17.74), if and only if it's form is

$$|z\rangle = \Big(n_1 c_{\boldsymbol{k}}^{(1)\dagger} + n_2 c_{\boldsymbol{k}}^{(2)\dagger} + n_{03} (c_{\boldsymbol{k}}^{(0)\dagger} - c_{\boldsymbol{k}}^{(3)\dagger}) \Big) \, |0\rangle \tag{17.76}$$
with arbitrary numbers $n_1, n_2, n_{03} \in \mathbb{N}$.

$|z\rangle$ is an eigenfunction of the Hamilton operator. In this state, the field's energy is

$$\begin{aligned}\widetilde{H}\,|z\rangle &\overset{(17.69)}{=} -\sum_{\boldsymbol{f}}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{\alpha\beta}\,\hbar\omega_{\boldsymbol{f}} \\ &\quad c_{\boldsymbol{f}}^{(\alpha)\dagger}c_{\boldsymbol{f}}^{(\alpha)}\left(n_1 c_{\boldsymbol{k}}^{(1)\dagger}+n_2 c_{\boldsymbol{k}}^{(2)\dagger}+n_{03}(c_{\boldsymbol{k}}^{(0)\dagger}-c_{\boldsymbol{k}}^{(3)\dagger})\right)|0\rangle \\ &= \hbar\omega_{\boldsymbol{k}}(-n_{03}+n_1+n_2+n_{03}) = \hbar\omega_{\boldsymbol{k}}(n_1+n_2)\ . \end{aligned} \tag{17.77}$$

The unphysical adders to the energy (and to other observables), which are caused by photons with time-like or longitudinal polarization, mutually compensate, if the field's states comply with the criterion (17.74). The problem of state functions with negative norm is solved as well. To see this, we compute the scalar product of two admissible state functions:

$$\begin{aligned}\langle z'|z\rangle &= \langle 0|\left(n'_1 c_{\boldsymbol{k}'}^{(1)}+n'_2 c_{\boldsymbol{k}'}^{(2)}+n'_{03}(c_{\boldsymbol{k}'}^{(0)}-c_{\boldsymbol{k}'}^{(3)\dagger})\right) \\ &\quad\cdot\left(n_1 c_{\boldsymbol{k}}^{(1)\dagger}+n_2 c_{\boldsymbol{k}}^{(2)\dagger}+n_{03}(c_{\boldsymbol{k}}^{(0)\dagger}-c_{\boldsymbol{k}}^{(3)\dagger})\right)|0\rangle \\ &= n'_1 n_1\langle 0|\,[c_{\boldsymbol{k}'}^{(1)},c_{\boldsymbol{k}}^{(1)\dagger}]\,|0\rangle + n'_2 n_2\langle 0|\,[c_{\boldsymbol{k}'}^{(2)},c_{\boldsymbol{k}}^{(2)\dagger}]\,|0\rangle + \\ &\quad + n'_{03}n_{03}(\,\langle 0|\,[c_{\boldsymbol{k}'}^{(0)},c_{\boldsymbol{k}}^{(0)\dagger}]\,|0\rangle + \langle 0|\,[c_{\boldsymbol{k}'}^{(3)},c_{\boldsymbol{k}}^{(3)\dagger}]\,|0\rangle) \\ &= n'_1 n_1\delta_{\boldsymbol{k}'\boldsymbol{k}} + n'_2 n_2\delta_{\boldsymbol{k}'\boldsymbol{k}} + n'_{03}n_{03}(\,-\delta_{\boldsymbol{k}'\boldsymbol{k}}+\delta_{\boldsymbol{k}'\boldsymbol{k}}) \end{aligned} \tag{17.78}$$

Here the commutators have been inserted as in the example of (17.73). All commutators which are zero, have been discarded immediately. The "unphysical" part $n_{03}(c_{\boldsymbol{k}}^{(0)\dagger}-c_{\boldsymbol{k}}^{(3)\dagger})\,|0\rangle$ of admissible state functions is orthogonal to any admissible state function, and it is even orthogonal to itself. The state functions

$$\left(n_1 c_{\boldsymbol{k}}^{(1)\dagger}+n_2 c_{\boldsymbol{k}}^{(2)\dagger}\right)|0\rangle \quad \text{and}$$
$$\left(n_1 c_{\boldsymbol{k}}^{(1)\dagger}+n_2 c_{\boldsymbol{k}}^{(2)\dagger}+n_{03}(c_{\boldsymbol{k}}^{(0)\dagger}-c_{\boldsymbol{k}}^{(3)\dagger})\right)|0\rangle$$

can by no means be discerned physically from another, because they lead to identical measurement results and identical scalar products. Therefore the state functions with same transversal parts and different longitudinal and time-like parts may be considered as an equivalence-class. One of them (for example the one with $n_{03}=0$) may be appointed to be the representative

of this class.

Due to the condition (17.74), $\widetilde{A}(x)$ and $A(x)$ can't be discerned any more. Therefore most authors skip the different notation of the fields. We as well will henceforth use the notion $A(x)$ for the field $\widetilde{A}(x)$. The fact that the field has been quantized manifestly covariant due to (17.66), then is visible in the formulas only due to the polarization indices 0 and 3.

17.5 The Photon-Propagator

In section 17.3 we described delocalized photons, which can be detected at any point in the normalization volume Ω with same probability. The analysis of localized photons has been postponed to this section, because we want to do this my means of the covariantly quantized field (see previous section).

Localized photons can be created due to application of the field operator's components

$$A^\nu(x) \overset{(17.11)}{=} \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sqrt{\frac{\mu_0 c^2 \hbar^2}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \cdot \left(\epsilon_{\boldsymbol{k}}^{(\alpha)\nu} c_{\boldsymbol{k}}^{(\alpha)} \exp\{-ikx\} + \epsilon_{\boldsymbol{k}}^{(\alpha)\nu *} c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{+ikx\} \right) \tag{17.79}$$

onto the vacuum. Because of $c_{\boldsymbol{k}}^{(v)} |0\rangle = 0$ one gets

$$|1_x\rangle^\nu \equiv A^\nu(x) |0\rangle = = \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sqrt{\frac{\mu_0 c^2 \hbar^2}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \epsilon_{\boldsymbol{k}}^{(\alpha)\nu *} \underbrace{c_{\boldsymbol{k}}^{(\alpha)\dagger} |0\rangle}_{|1_{\boldsymbol{k}\alpha}\rangle} \exp\{+ikx\} \ . \tag{17.80}$$

There is a complex phase factor in the polarization vectors, which have been defined in (17.9). Therefore the photons have an arbitrary, but well defined polarization, even if that is not documented in the simple notation $|1_x\rangle$. The vector $|1_x\rangle$ has four space-time components, because all components of the operator $A(x)$ may be different from zero. $|1_x\rangle^\nu$ could be interpreted as the component of a line-vector, whose norm-square is a matrix, or as

the component of a column-vector, whose norm-square is a scalar. With regard to definition (16.22) in case of the Dirac field one decides for the first alternative, due to the following

Definition: Any $|\text{ket}\rangle$ of a vector field is to be considered a line-vector. Any $\langle\text{bra}|$ of a vector field is to be considered a column-vector. (17.81)

$|1_x\rangle^\nu$ is the ν-component of a photon's state function, which at time x^0/c is localized (though not exactly) at $\boldsymbol{x}$. We compute the matrix element

$$\langle 1_y|1_x\rangle_{\mu\nu} = \langle 0|\, A_\mu(y) A_\nu(x)\,|0\rangle \;=$$

$$= \sum_{\boldsymbol{f},\boldsymbol{k}} \sum_{\alpha,\beta=0}^{3} \frac{\mu_0 c^2\hbar^2}{\Omega 2\hbar\,\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}}\, \epsilon^{(\beta)}_{\boldsymbol{f}\,\mu}\, \epsilon^{(\alpha)*}_{\boldsymbol{k}\,\nu} \underbrace{\langle 1_{\boldsymbol{f}\beta}|1_{\boldsymbol{k}\alpha}\rangle}_{\overset{(17.73)}{=} -g_{\alpha\beta}\,\delta_{\boldsymbol{k}\boldsymbol{f}}} \exp\{-i(fy-kx)\}$$

$$\langle 1_y|1_x\rangle_{\mu\nu} = -\sum_{\boldsymbol{k}} \frac{\mu_0 c^2\hbar^2}{2\hbar\omega_{\boldsymbol{k}}\,\Omega} \sum_{\alpha,\beta=0}^{3} g_{\alpha\beta}\, \epsilon^{(\beta)}_{\boldsymbol{k}\,\mu} \epsilon^{(\alpha)*}_{\boldsymbol{k}\,\nu} \exp\{-ik(y-x)\}\ .$$

Now we switch due to $\mu,\nu \to \sigma,\tau$ into a coordinate system, which is — same as the coordinate system α,β — aligned to the field A. In this system

$$\epsilon^{(\beta)}_{\boldsymbol{k}\,\sigma} \epsilon^{(\alpha)*}_{\boldsymbol{k}\,\tau} \overset{(17.9)}{=} g_{\sigma\tau}\, g^{\beta}{}_{\sigma}\, g^{\alpha}{}_{\tau} \tag{17.82a}$$

$$\sum_{\alpha,\beta=0}^{3} g_{\alpha\beta}\, \epsilon^{(\beta)}_{\boldsymbol{k}\,\sigma} \epsilon^{(\alpha)*}_{\boldsymbol{k}\,\tau} = g_{\sigma\tau}\ . \tag{17.82b}$$

Now the coordinate system is rotated back due to $\sigma,\tau \to \mu,\nu$. Thus one gets

$$\langle 1_y|1_x\rangle_{\mu\nu} = -g_{\mu\nu}\,\mu_0 c^2\hbar^2 \sum_{\boldsymbol{k}} \frac{\exp\{-ik(y-x)\}}{2\hbar\omega_{\boldsymbol{k}}\,\Omega} \tag{17.83}$$

Only the diagonal elements of the matrix $\langle 1_y|1_x\rangle$ are different from zero. Even in the case $x^0 = y^0$, there isn't the delta function

$$\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}) \overset{(7.9)}{=} \sum_{\boldsymbol{k}} \frac{\exp\{+i\boldsymbol{k}(\boldsymbol{y}-\boldsymbol{x})\}}{\Omega} \ .$$

in (17.83). The additional factor $\omega_{\boldsymbol{k}}$ in the denominator is prohibiting the exact localization of a photon (which anyway physically would not be realistic). At the same time, this factor in the denominator is securing the Lorentz-invariance of the matrix element (17.83), as has been discussed at the end of chapter 7.

The photon is localized only in space, but not in time, because the summation is running only over $\boldsymbol{k}$, but there is no integration over k^0. The time t here is a running parameter. The localized photon's state function $|1_{t\boldsymbol{x}}\rangle$ is an eigenfunction of the time-dependent position-operator $\boldsymbol{x}(t)$:

$$\boldsymbol{x}(t)|1_{t\boldsymbol{y}}\rangle = \boldsymbol{y}|1_{t\boldsymbol{y}}\rangle \tag{17.84}$$

The photon $|1_{t\boldsymbol{y}}\rangle$ has a well defined position and a well defined polarization; it's energy and momentum are undetermined.

In case of $y^0 > x^0$, the matrices of the localized photons with the elements

$$\langle 1_y|1_x\rangle_{\mu\nu} = \langle 0|\, A_\mu(y)A_\nu(x)\,|0\rangle = (17.83)$$

are interpreted as the probability amplitude of the event, that a photon is created at time x^0/c at position $\boldsymbol{x}$, then propagates to the position $\boldsymbol{y}$, and eventually is annihilated there at time y^0/c. By comparison of the classical propagator (12.34) with (17.83) it becomes obvious, that the quantized propagator is:

$$D_{\mu\nu}(y-x) = \langle 0|\, T\, A_\mu(y)A_\nu(x)\,|0\rangle \ = \sum_{\boldsymbol{k}} \frac{-g_{\mu\nu}\,\mu_0 c^2\hbar^2}{2\hbar\omega_{\boldsymbol{k}}\,\Omega}$$
$$\cdot \Big(\theta(y^0-x^0)\,\exp\{-ik(y-x)\} + \theta(x^0-y^0)\,\exp\{-ik(x-y)\}\Big) \tag{17.85}$$

T is the time-order operator, which has been defined in (15.44). Due to comparison with (12.33), one finds the propagator in wavenumber-space:

$$\boxed{\widetilde{D}_{\mu\nu}(k) = \frac{-ig_{\mu\nu}\,\mu_0\hbar c}{k^2 + i\epsilon'}} \tag{17.86}$$

Note: Due to condition (17.74), no observable quantity changes, if the propagator is defined by

$$\widetilde{D}^L_{\mu\nu}(k) = \frac{-i\,\mu_0\hbar c}{k^2 + i\epsilon'}\Big(g_{\mu\nu} - L\frac{k_\mu k_\nu}{k^2}\Big)$$

with constant $L \in \mathbb{R}$. Clever choice of L can simplify some computations. Therefore definitions of the propagator with $L \neq 0$ are frequently encountered in the literature. But in this book, we will exclusively use (17.86).

18 Path Integrals

In section 14.1.1 the transition from classical mechanics of point particles to quantum mechanics was achieved due to the replacement of the Poisson-brackets by the products of $i/\hbar$ and the commutators:

$$\{A, B\}_{\text{PB}} \equiv \Big(\frac{\partial A}{\partial p}\frac{\partial B}{\partial q} - \frac{\partial B}{\partial p}\frac{\partial A}{\partial q}\Big) \quad \longrightarrow$$
$$\xrightarrow{(14.6)} \quad \frac{i}{\hbar}[A\,, B] \equiv \frac{i}{\hbar}(AB - BA) \tag{18.1}$$

In the following chapters this method of "canonical quantization" then was generalized to several classical fields.

Dirac [43] indicated that it might be advantageous, to base quantum theory onto the Lagrangian instead of the Hamiltonian, which is in the center of the canonical formalism. The Lagrangian seemed to him more fundamental than the Hamiltonian for several reasons. In particular, the Lagrangian can be formulated relativistically invariant, while the Hamiltonian, being the quantity canonically conjugate to time, is not relativistically invariant from the outset.

$\langle t_N, \boldsymbol{x}_N | t_0, \boldsymbol{x}_0 \rangle$ is the quantum mechanical probability amplitude, that a particle can be observed at time t_N at position $\boldsymbol{x}_N$, if it has been observed at time t_0 at position $\boldsymbol{x}_0$. Dirac suggested, to compute this probability amplitude as follows:

$$\langle t_N, \boldsymbol{x}_N | t_0, \boldsymbol{x}_0 \rangle = \exp\Big\{\frac{i}{\hbar}S\Big\} = \exp\Big\{\frac{i}{\hbar}\int_{t_0}^{t_N} \mathrm{d}t\, L(t)\Big\} \tag{18.2}$$

In this ansatz, S is the action and L is the Lagrangian of the classical particle. Thus, different from canonical quantization, no non-commutative algebra

is postulated. The matrix element is computed by means of the classical, commutative functions S resp. L. In spite of the commutative algebra, Dirac's proposal (18.2) is not classical physics, but true quantum theory. No classical probability is computed here, but a probability-*amplitude*. Something like that does exist in quantum theory only. We will dwell on that in the next section in more detail.

Feynman [44] adopted Dirac's idea, and elaborated it to the method of path integrals. In the next section, reasons for Dirac's proposal (18.2) will be given, and the concept of path integrals will be derived for the example of non-relativistic point particles. In the subsequent sections, the path integrals of relativistic invariant scalar fields will be discussed.

18.1 Point Particles

We encountered the probability amplitude, which is showing up in Dirac's quantization proposal (18.2), already in section 14.1.3. It is the matrix element of the time-evolution operator $U(t_N, t_0)$:

$$U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \overset{(14.37)}{=} \langle t_N, \boldsymbol{x}_N | t_0, \boldsymbol{x}_0 \rangle = \langle \boldsymbol{x}_N | \, U(t_N, t_0) \, | \boldsymbol{x}_0 \rangle \tag{18.3}$$

The probability-density

$$W(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \equiv \Big| U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \Big|^2 \tag{18.4}$$

is is the square of the absolute value of the probability amplitude. And the probability, to find the particle at time t_N in the volume V, if it has been observed at time t_0 at position $\boldsymbol{x}_0$, is

$$\int\limits_V \mathrm{d}^3 x_N \, W(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \; . \tag{18.5}$$

The probability is dimension-less. The probability density's dimension is volume^{-1}, and the probability amplitude's dimension is volume$^{-1/2}$.

Consider some point of time t_j in-between t_0 and t_N. The probability amplitude, that the particle can be observed at time t_j at position $\boldsymbol{x}_j$, if it has

been observed at time t_0 at position $\boldsymbol{x}_0$, is $U(t_j, \boldsymbol{x}_j, t_0, \boldsymbol{x}_0)$. The probability amplitude, that the particle can be observed at time t_N at position $\boldsymbol{x}_N$, if it has been observed at time t_j at position $\boldsymbol{x}_j$, is $U(t_N, \boldsymbol{x}_N, t_j, \boldsymbol{x}_j)$. If one abstains from observing the particle's position at time t_j, then the probability-density to detect it at time t_N at position $\boldsymbol{x}_N$ is, according to classical physics:

$$\begin{aligned} W_{\text{classical}}(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \;&= \\ &= \int \mathrm{d}^3 x_j \, W(t_N, \boldsymbol{x}_N, t_j, \boldsymbol{x}_j) \cdot W(t_j, \boldsymbol{x}_j, t_0, \boldsymbol{x}_0) \;= \\ &= \int \mathrm{d}^3 x_j \Big| U(t_N, \boldsymbol{x}_N, t_j, \boldsymbol{x}_j) \Big|^2 \cdot \Big| U(t_j, \boldsymbol{x}_j, t_0, \boldsymbol{x}_0) \Big|^2 \end{aligned} \tag{18.6a}$$

This integral is running over all positions $\boldsymbol{x}_j$, where the particle possibly could be found, if it's position at time t_j would be observed (but actually one does not try to observe it). According to quantum mechanics, the probability-density in this case is instead

$$\begin{aligned} W(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \;&= \\ &= \Big| \int \mathrm{d}^3 x_j \, U(t_N, \boldsymbol{x}_N, t_j, \boldsymbol{x}_j) \cdot U(t_j, \boldsymbol{x}_j, t_0, \boldsymbol{x}_0) \Big|^2 \\ &\neq \int \mathrm{d}^3 x_j \Big| U(t_N, \boldsymbol{x}_N, t_j, \boldsymbol{x}_j) \Big|^2 \cdot \Big| U(t_j, \boldsymbol{x}_j, t_0, \boldsymbol{x}_0) \Big|^2 \end{aligned} \tag{18.6b}$$

In the quantum-mechanical probability-density (18.6b), there are interferences between the particles trajectories over the different possible intermediate points $\boldsymbol{x}_j$. In the classical probability-density (18.6a), there are no interferences, because the integrand is positive for all $\boldsymbol{x}_j$. (18.6b) is in accord with experimental observation. (18.6a) is an acceptable approximation, if Planck's quantum of action is negligible in comparison to the action S of the particle ($\hbar/S \ll 1$). Feynman considered the interference of the probability amplitudes to be the most characteristic feature of quantum theory. In classical physics, there exist the notion of probability (resp. probability-density), but the concept of probability amplitudes, which are capable of interference, is an exclusive quantum-theoretical idea. Therefore it seemed most adequate to Feynman, that Dirac's quantization-proposal (18.2) is

heading without any detour for that central notion of quantum theory.

18.1.1 The most general Case

In the sequel, we will give reasons for Dirac's quantization proposal (18.2), and we will state it more precisely. Consider a point particle, which has been observed at time t_0 at position $\boldsymbol{x}_0$. The probability amplitude, that it can be detected at time t_N at position $\boldsymbol{x}_N$ is

$$U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \overset{(14.37)}{=} \langle t_N, \boldsymbol{x}_N | t_0, \boldsymbol{x}_0 \rangle = \langle \boldsymbol{x}_N | \, U(t_N, t_0) \, | \boldsymbol{x}_0 \rangle \; = \\ = \langle \boldsymbol{x}_N | \, \exp\{-\frac{i}{\hbar}(t_N - t_0) H\} \, | \boldsymbol{x}_0 \rangle \; . \tag{18.7}$$

Here the simple time-evolution operator

$$U(t, t_0) = e^{-\frac{i}{\hbar}(t - t_0) H} \tag{18.8}$$

has been assumed, which is correct only if the Hamilton operator does not explicitly depend on time. We will stick to that assumption for all the reminder of this chapter. Furthermore we assume the Hamilton operator $H(\boldsymbol{x}, \boldsymbol{p})$ of the point particle to be an arbitrary, not further specified function of the position operator $\boldsymbol{x}$ and the momentum operator $\hbar \boldsymbol{k} \equiv \boldsymbol{p}$. Note, that in the case of point particles the conjugate momenta are identical to the physical momenta.

The particle's state functions $|t, \boldsymbol{x}\rangle$ are eigenfunctions of the time-dependent position operator, while the state functions $|\boldsymbol{x}\rangle$ are eigenfunctions of the time-independent position operator, see (14.35). Furthermore we will use time-independent state functions $|\boldsymbol{k}\rangle$ of the particle, which are eigenfunctions of the time-independent momentum operator $\hbar \boldsymbol{k}$.

We divide the time interval inbetween t_0 and t_N into N pieces

$$\tau \equiv \frac{t_N - t_0}{N} \; . \tag{18.9}$$

Though it isn't important for any of the following arguments, it will save us from a lot of paperwork, if we choose all N parts to have equal length. We

write the time-evolution operator in the form

$$\exp\{-\frac{i}{\hbar}(t_N - t_0)H\} = \underbrace{\exp\{-\frac{i}{\hbar}H\tau\} \ \ldots \ \exp\{-\frac{i}{\hbar}H\tau\}}_{N\times} \ , \tag{18.10}$$

and insert inbetween the N factors $N-1$ times the projector

$$1 = \int\limits_{\Omega} \mathrm{d}^3 x_j |\boldsymbol{x}_j\rangle\langle\boldsymbol{x}_j| \ . \tag{18.11}$$

j is running from 1 inbetween the factors at the very right in steps of one up to $N-1$ inbetween the factors at the very left:

$$\begin{aligned} U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \ = & \int\limits_{\Omega} \mathrm{d}^3 x_{N-1} \ldots \int\limits_{\Omega} \mathrm{d}^3 x_j \ldots \int\limits_{\Omega} \mathrm{d}^3 x_1 \\ & \prod_{j=0}^{N-1} \langle \boldsymbol{x}_{j+1}| \exp\{-\frac{i}{\hbar}H\tau\}|\boldsymbol{x}_j\rangle \end{aligned} \tag{18.12}$$

Note, that there are N matrix elements, but only $N-1$ integrals. While $\boldsymbol{x}_N$ and $\boldsymbol{x}_0$ are showing up in the matrix elements, no integral is running over them. The Hamilton operator in the exponent is to be understood as an abbreviation for the series

$$\exp\{-\frac{i}{\hbar}H\tau\} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\Big(-\frac{i}{\hbar}H\tau\Big)^n = 1 - \frac{i}{\hbar}H\tau \pm \ \ldots \ . \tag{18.13}$$

Firstly, we compute the linear matrix element

$$\langle \boldsymbol{x}_{j+1}|H|\boldsymbol{x}_j\rangle \ . \tag{18.14}$$

The Hamilton operator H in the most general case is a polynomial of the operators $\boldsymbol{x}$ and $\boldsymbol{k}$, which we will re-arrange in a tricky fashion. We demonstrate the kind of re-arrangement for two examples:

$$\boldsymbol{x}\boldsymbol{k} \longrightarrow \tfrac{1}{2}(\boldsymbol{x}\boldsymbol{k} + \underbrace{\boldsymbol{x}\boldsymbol{k} - \boldsymbol{k}\boldsymbol{x}}_{\text{commutator}} + \boldsymbol{k}\boldsymbol{x}) = \tfrac{1}{2}(\boldsymbol{x}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{x}) + f(K) \tag{18.15a}$$

$$\boldsymbol{x}^2\boldsymbol{k} \longrightarrow \tfrac{1}{4}(\boldsymbol{x}^2\boldsymbol{k} + 2\boldsymbol{x}\boldsymbol{k}\boldsymbol{x} + \boldsymbol{k}\boldsymbol{x}^2) + g(K) \tag{18.15b}$$

$f(K)$ and $g(K)$ are functions of the commutator of the operators $\boldsymbol{x}$ and $\boldsymbol{k}$. The product $\boldsymbol{k}\boldsymbol{x}$ would be arranged like (18.15a). The products $\boldsymbol{x}\boldsymbol{k}\boldsymbol{x}$ and $\boldsymbol{k}\boldsymbol{x}^2$ would be arranged like (18.15b). In general, the polynomial of position- and momentum-operators, of which H is consisting, is re-arranged such, that the terms are containing in the center the different powers of the momentum operator, and to the left and to the right of them symmetric polynomials of the position operator, which have the coefficients of a binomial series expansion. In addition there are terms, which are consisting of functions of their commutators. To illustrate this somewhat complex rule, let's consider an example:

$$\begin{aligned}
&\langle \boldsymbol{x}_{j+1}|\boldsymbol{x}^2\boldsymbol{k}|\boldsymbol{x}_j\rangle \\
&\quad = \langle \boldsymbol{x}_{j+1}|\tfrac{1}{4}(\boldsymbol{x}^2\boldsymbol{k} + 2\boldsymbol{x}\boldsymbol{k}\boldsymbol{x} + \boldsymbol{k}\boldsymbol{x}^2)|\boldsymbol{x}_j\rangle + g(K)\langle \boldsymbol{x}_{j+1}|\boldsymbol{x}_j\rangle \\
&\quad = \tfrac{1}{4}(\boldsymbol{x}_{j+1}^2 + 2\boldsymbol{x}_{j+1}\boldsymbol{x}_j + \boldsymbol{x}_j^2)\langle \boldsymbol{x}_{j+1}|\boldsymbol{k}|\boldsymbol{x}_j\rangle + \\
&\qquad + g(K)\delta^{(3)}(\boldsymbol{x}_{j+1} - \boldsymbol{x}_j)
\end{aligned} \tag{18.16}$$

The momentum operator's matrix element is

$$\langle \boldsymbol{x}_{j+1}|\boldsymbol{k}|\boldsymbol{x}_j\rangle = \sum_{\boldsymbol{k}_j}\langle \boldsymbol{x}_{j+1}|\boldsymbol{k}|\boldsymbol{k}_j\rangle\langle \boldsymbol{k}_j|\boldsymbol{x}_j\rangle = \sum_{\boldsymbol{k}_j}\boldsymbol{k}_j\langle \boldsymbol{x}_{j+1}|\boldsymbol{k}_j\rangle\langle \boldsymbol{k}_j|\boldsymbol{x}_j\rangle \ .$$

The sum is running of all wave numbers $\boldsymbol{k}_j$, which are compatible with the normalization onto the finite volume Ω (see section 7). Using

$$\begin{aligned}
&|\boldsymbol{x}_j\rangle \overset{(15.36a)}{=} \sum_{\boldsymbol{k}} \frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}_j\}}{\sqrt{\Omega}}\,|\boldsymbol{k}\rangle \quad \Longrightarrow \\
&\Longrightarrow \langle \boldsymbol{k}_j|\boldsymbol{x}_j\rangle = \sum_{\boldsymbol{k}} \frac{\exp\{-i\boldsymbol{k}\boldsymbol{x}_j\}}{\sqrt{\Omega}}\,\underbrace{\langle \boldsymbol{k}_j|\boldsymbol{k}\rangle}_{\delta_{\boldsymbol{k}_j\boldsymbol{k}}} = \frac{\exp\{-i\boldsymbol{k}_j\boldsymbol{x}_j\}}{\sqrt{\Omega}} \ ,
\end{aligned} \tag{18.17}$$

and using the delta function

$$\delta^{(3)}(\boldsymbol{x}_{j+1}-\boldsymbol{x}_j) \overset{(7.9)}{=} \frac{1}{\Omega}\sum_{\boldsymbol{k}_j}\exp\{+i\boldsymbol{k}_j(\boldsymbol{x}_{j+1}-\boldsymbol{x}_j)\}\ , \tag{18.18}$$

one gets

$$\begin{aligned}&\langle \boldsymbol{x}_{j+1}|\boldsymbol{x}^2\boldsymbol{k}|\boldsymbol{x}_j\rangle \overset{(18.16)}{=} \\ &= \frac{1}{\Omega}\sum_{\boldsymbol{k}_j}\left(\left(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2}\right)^2\boldsymbol{k}_j + g(K)\right)\exp\{+i\boldsymbol{k}_j(\boldsymbol{x}_{j+1}-\boldsymbol{x}_j)\}\ .\end{aligned} \tag{18.19}$$

Accordingly, one finds for arbitrary Hamilton operators as the linear matrix element

$$\begin{aligned}&\langle \boldsymbol{x}_{j+1}|H(\boldsymbol{x},\boldsymbol{k})|\boldsymbol{x}_j\rangle \overset{(18.16)}{=} \\ &= \frac{1}{\Omega}\sum_{\boldsymbol{k}_j}E\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2},\boldsymbol{k}_j\Big)\exp\{+i\boldsymbol{k}_j(\boldsymbol{x}_{j+1}-\boldsymbol{x}_j)\}\ ,\end{aligned} \tag{18.20}$$

and as the matrix element of the Hamilton operator in the exponential

$$\begin{aligned}&\langle \boldsymbol{x}_{j+1}|\exp\{-\frac{i}{\hbar}H(\boldsymbol{x},\boldsymbol{k})\,\tau\}|\boldsymbol{x}_j\rangle\ = \\ &= \frac{1}{\Omega}\sum_{\boldsymbol{k}_j}\exp\Big\{-\frac{i}{\hbar}E\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2},\boldsymbol{k}_j\Big)\tau\Big\}\exp\{+i\boldsymbol{k}_j(\boldsymbol{x}_{j+1}-\boldsymbol{x}_j)\} \\ &= \frac{1}{\Omega}\sum_{\boldsymbol{k}_j}\exp\Big\{\tau\,\frac{i}{\hbar}\Big(\hbar\boldsymbol{k}_j\frac{\boldsymbol{x}_{j+1}-\boldsymbol{x}_j}{\tau} - E\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2},\boldsymbol{k}_j\Big)\Big)\Big\}\ .\end{aligned} \tag{18.21}$$

The function E has exactly the same polynomial structure as the Hamilton operator. Wherever in H a power of the position operator $\boldsymbol{x}$ is showing up, there is in E the same power of the number $(\boldsymbol{x}_{j+1}+\boldsymbol{x}_j)/2$. Wherever in H a power of the wavenumber operator $\boldsymbol{k}$ is showing up, there is in E the same power of the eigenvalue $\boldsymbol{k}_j$. In addition, there may be in E a constant term $g(K)$, which is caused by the commutators which are created due to the re-arrangement (18.15).

This matrix element is inserted into the product (18.12):

$$
\begin{aligned}
&U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) \overset{(18.12)}{=} \\
&= \int_\Omega \mathrm{d}^3 x_{N-1} \frac{1}{\Omega} \sum_{\boldsymbol{k}_{N-1}} \dots \int_\Omega \mathrm{d}^3 x_1 \frac{1}{\Omega} \sum_{\boldsymbol{k}_1} \frac{1}{\Omega} \sum_{\boldsymbol{k}_0} \\
&\quad \prod_{j=0}^{N-1} \exp\left\{ \tau \frac{i}{\hbar} \left(\hbar \boldsymbol{k}_j \frac{\boldsymbol{x}_{j+1} - \boldsymbol{x}_j}{\tau} - E\Big(\frac{\boldsymbol{x}_{j+1} + \boldsymbol{x}_j}{2}, \boldsymbol{k}_j \Big) \right) \right\} \\
&= \int_\Omega \mathrm{d}^3 x_{N-1} \frac{1}{\Omega} \sum_{\boldsymbol{k}_{N-1}} \dots \int_\Omega \mathrm{d}^3 x_1 \frac{1}{\Omega} \sum_{\boldsymbol{k}_1} \frac{1}{\Omega} \sum_{\boldsymbol{k}_0} \\
&\quad \exp\Big\{ \underbrace{\frac{i}{\hbar} \sum_{j=0}^{N-1} \tau \left(\hbar \boldsymbol{k}_j \frac{\boldsymbol{x}_{j+1} - \boldsymbol{x}_j}{\tau} - E\Big(\frac{\boldsymbol{x}_{j+1} + \boldsymbol{x}_j}{2}, \boldsymbol{k}_j \Big) \right)}_{\int_{t_0}^{t_N} \mathrm{d}t \left(\boldsymbol{p}(t_j) \dot{\boldsymbol{x}}(t_j) - E\left(\frac{\boldsymbol{x}(t_{j+1}) + \boldsymbol{x}(t_j)}{2}, \boldsymbol{k}(t_j) \right) \right)} \Big\}
\end{aligned}
\tag{18.22}
$$

As there are only numbers but no operators in this equation, the product of the exponential functions could be replaced by the exponential function of a sum.

With very large N, and accordingly — see (18.9) — very small τ, the sum in the exponent becomes in good approximation the integral over the particles Lagrangian:

$$
L \overset{(3.20)}{=} \boldsymbol{p}\dot{\boldsymbol{x}} - E \tag{18.23}
$$

This integral, which is the particles action S, is known to us already from chapter 3:

$$
S \overset{(3.2)}{=} \int_{t_0}^{t_N} \mathrm{d}t \, L\Big(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), t \Big) \tag{18.24}
$$

There it was our starting point for the derivation of the equation of motion of classical point particles by means of Hamilton's principle of least action.

The product of the $N-1$ integrals, which in (18.22) are taken at times $t_1, \ldots, t_{N-1}$ over the normalization volume Ω, is called *path integral*. No particular name exists for the product of the N sums, which are computed at the times $t_0, \ldots, t_{N-1}$ over all possible wavenumbers. But there exists for this product — same as for the path integral — a special notation:

$$
\begin{aligned}
U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) &\overset{(18.22)}{=} \\
&= \int_\Omega \mathrm{d}^3 x_{N-1} \, \frac{1}{\Omega} \sum_{\boldsymbol{k}_{N-1}} \cdots \int_\Omega \mathrm{d}^3 x_1 \, \frac{1}{\Omega} \sum_{\boldsymbol{k}_1} \frac{1}{\Omega} \sum_{\boldsymbol{k}_0} \\
&\exp\Big\{ \underbrace{\frac{i}{\hbar} \sum_{j=0}^{N-1} \tau \Big(\hbar \boldsymbol{k}_j \frac{\boldsymbol{x}_{j+1} - \boldsymbol{x}_j}{\tau} - E\Big(\frac{\boldsymbol{x}_{j+1} + \boldsymbol{x}_j}{2}, \boldsymbol{k}_j \Big) \Big)}_{\int_{t_0}^{t_N} \mathrm{d}t \, L(t)} \Big\} \qquad (18.25a) \\
&\equiv \int_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t) \int \frac{\mathcal{D}\boldsymbol{k}(t)}{(2\pi)^{3N}} \, \exp\Big\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \Big\} \qquad (18.25b)
\end{aligned}
$$

The action's argument is written in square brackets, to emphasize that $[\boldsymbol{x}(t)]$ does not mean a point in position-space, but the complete path of the particle from $\boldsymbol{x}_0$ to $\boldsymbol{x}_N$ over the $N-1$ intermediate points $\boldsymbol{x}_j$. It is obvious from (18.24), that the action does not depend on a point, but on the complete path. Furthermore, to achieve a homogeneous notation, we changed in (18.25b) due to

$$
\int \frac{\mathcal{D}\boldsymbol{k}(t)}{(2\pi)^{3N}} \equiv \prod_{j=0}^{N-1} \int \frac{\mathrm{d}\boldsymbol{k}_j}{(2\pi)^3} \approx \prod_{j=0}^{N-1} \frac{1}{\Omega} \sum_{\boldsymbol{k}_j} \qquad (18.26)
$$

to the normalization over an infinite volume.

Figure 18.1 is the clumsy attempt, to illustrate the path integral graphically. To keep things simple, $N = 3$ has been assumed here, while in real applications N will be a "very large" number. At time t_0 the particle is at position $\boldsymbol{x}_0$, at time t_3 it is at position $\boldsymbol{x}_3$. At times t_1 and t_2 it is *anywhere* in the normalization volume, because in (18.25a) the integrals $\int_\Omega \mathrm{d}^3 x(t_1)$

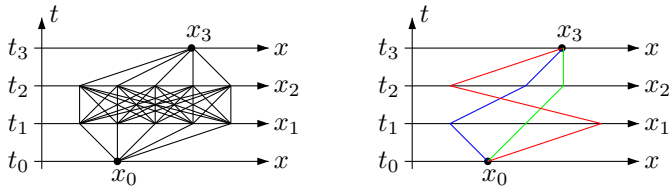


Fig. 18.1 : Some paths from x_0 to x_3

and $\int_\Omega \mathrm{d}^3x(t_2)$ are running over the complete normalization volume Ω. For simplicity, instead of the continuum of the infinitely many positions, where the particle might be at times t_1 and t_2, only 5 possible positions each have been indicated in the left sketch. Each line, which is connecting $\boldsymbol{x}_0$ with $\boldsymbol{x}_3$ over arbitrary intermediate points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, is called a "path". In the left sketch, 25 different paths inbetween $\boldsymbol{x}_0$ and $\boldsymbol{x}_3$ are indicated. In the right sketch, three of them are marked by colors for clarity. The path integral is an integral not only over the indicated 25 paths, but over the whole continuum of all paths, on which the particle can possibly travel from $\boldsymbol{x}_0$ to $\boldsymbol{x}_3$.

The path integral must not be confused with a contour integral. Therefore it's differential is not marked by d but by $\mathcal{D}$:

$$\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t) \neq \int\limits_{\boldsymbol{x}_0}^{\boldsymbol{x}_N} \mathrm{d}\boldsymbol{x} \tag{18.27}$$

The path integral is not an integral over a continuum of points, which as a whole constitute a path from $\boldsymbol{x}_0$ to $\boldsymbol{x}_N$, but over a continuum of complete paths. Each single of these paths — like for example each of the three colored paths in the right sketch of figure 18.1 — is connecting the point $\boldsymbol{x}_0$ with the point $\boldsymbol{x}_N$. This is specified by $[\boldsymbol{x}_0 \to \boldsymbol{x}_N]$ below the integral symbol. While one can develop some vague idea, what is meant by an integration over a continuum of paths, there does not exist a mathematically sound definition of the integration measure $\mathcal{D}\boldsymbol{x}(t)$, and no mathematically elaborated theory, how to handle such integrals.

The same holds for the product

$$\int \frac{\mathcal{D}\boldsymbol{k}(t)}{(2\pi)^{3N}} \equiv \frac{1}{\Omega} \sum_{\boldsymbol{k}_{N-1}} \cdots \frac{1}{\Omega} \sum_{\boldsymbol{k}_0} .$$

of the sums over $\boldsymbol{k}$. Each single summation is running over all (countable infinitely many) wavenumbers, which are compatible with the normalization onto the finite volume Ω. But nobody can compute a continuum of sums, i. e. the limit $N \to \infty$, which is indicated by $\int \mathcal{D}\boldsymbol{k}$, is not defined mathematically, and infeasible in reality.

(18.25b) is nothing other than a symbolic notation for (18.25a). Whenever one actually needs to compute a path integral, one reverts to (18.25a). As there doesn't exist an analytical method for the computation of a continuum of integrals or summations, one chooses the number of supporting points N as large as possible, but always finite.

Comparing (18.25) with Dirac's simple ansatz (18.2), one recognizes the essential extension due to Feynman: He is considering not only one single path between the points, at which the particle is found at times t_0 and t_N, but is summing up all paths, along which the particle can possibly move.

The path integral is a beautiful example for the saying, that "the extremes are mutually touching". One of the guidelines, which led Heisenberg to the detection of quantum mechanics, was the insight that the classical picture of the "orbit" is meaningless for a quantum particle, and must be purged out of the formalism. In constrast, Feynman assigns to the particle infinitely many orbits, which are weighted by the factor $\exp\{iS[\boldsymbol{x}(t)]/\hbar\}$, and arrives by that method at the same result! Feynman's infinitely many orbits (= paths) and Heisenberg's path-less formalism clearly both are irreconcilable with the classical picture of *one* path.

18.1.2 The Summation over $\boldsymbol{k}$

Until now we allowed for Hamilton-operators $H(\boldsymbol{x}, \boldsymbol{k})$, which are arbitrary polynomials of the position operator $\boldsymbol{x}$ and the momentum operator $\hbar\boldsymbol{k}$. In the special case

$$H(\boldsymbol{x},\boldsymbol{k}) \equiv \frac{\hbar^2\boldsymbol{k}^2}{2m} + V(\boldsymbol{x}) \tag{18.28}$$

one gets the probability density

$$\begin{aligned} U(t_N,\boldsymbol{x}_N,t_0,\boldsymbol{x}_0) &\overset{(18.22)}{=} \\ &= \int_\Omega \mathrm{d}^3x_{N-1}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}_{N-1}} \ldots \int_\Omega \mathrm{d}^3x_1\,\frac{1}{\Omega}\sum_{\boldsymbol{k}_1}\frac{1}{\Omega}\sum_{\boldsymbol{k}_0} \\ &\quad \exp\Big\{\frac{i}{\hbar}\sum_{j=0}^{N-1}\tau\Big(\hbar\boldsymbol{k}_j\frac{\boldsymbol{x}_{j+1}-\boldsymbol{x}_j}{\tau} - \frac{\hbar^2}{2m}\boldsymbol{k}_j^2\Big)\Big\} \\ &\quad \exp\Big\{-\frac{i}{\hbar}\sum_{j=0}^{N-1}\tau V\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2}\Big)\Big\}\,. \end{aligned} \tag{18.29}$$

The round bracket in the second exponential function is the argument of V. As $\boldsymbol{k}_j$ and $\boldsymbol{x}_j$ are no operators but eigenvalues, the exponential function could be split into two factors. The sums over $\boldsymbol{k}_j$ can be computed analytically — respecting (7.5) — by means of the formula for Gauß's integral with purely imaginary argument, which is indicated in appendix A.23. Thus one finds the probability amplitude

$$\begin{aligned} U(t_N,\boldsymbol{x}_N,t_0,\boldsymbol{x}_0) &= \int_\Omega \mathrm{d}^3x_{N-1}\,\ldots\int_\Omega \mathrm{d}^3x_1 \\ &\frac{1}{(2\pi)^{3N}}\Big(\frac{\pi 2m}{i\tau\hbar}\Big)^{\frac{3N}{2}}\exp\Big\{\sum_{j=0}^{N-1}\frac{-(\boldsymbol{x}_{j+1}-\boldsymbol{x}_j)^2 2m}{4i\tau\hbar}\Big\} \\ &\exp\Big\{-\frac{i}{\hbar}\sum_{j=0}^{N-1}\tau V\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2}\Big)\Big\}\,. \end{aligned} \tag{18.30}$$

Defining the function

$$C(N) \equiv \frac{1}{(2\pi)^{3N}}\Big(\frac{\pi 2m}{i\tau\hbar}\Big)^{\frac{3N}{2}}\,, \tag{18.31}$$

which does depend on N due to the exponents and due to $\tau = (18.9)$, one eventually gets the probability amplitude in the form

$$U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) = C(N) \int_\Omega \mathrm{d}^3 x_{N-1} \ldots \int_\Omega \mathrm{d}^3 x_1$$
$$\exp\Big\{\frac{i}{\hbar}\sum_{j=0}^{N-1}\tau\Big(\frac{m}{2}\Big(\frac{\boldsymbol{x}_{j+1}-\boldsymbol{x}_j}{\tau}\Big)^2 - V\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2}\Big)\Big)\Big\}\,. \tag{18.32}$$

The squared round bracket is a factor, while the non-squared round bracket following V is the argument of V. Again one can identify

$$\begin{aligned}\sum_{j=0}^{N-1}\tau\Big(\frac{m}{2}\Big(\frac{\boldsymbol{x}_{j+1}-\boldsymbol{x}_j}{\tau}\Big)^2 - V\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2}\Big)\Big)&\\ \approx \sum_{j=0}^{N-1}\tau\Big(E_{\mathrm{kin}}(t_j) - E_{\mathrm{pot}}(t_j)\Big)&\\ \approx \int_{t_0}^{t_N}\mathrm{d}t\, L\Big(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t)\Big) = S[\boldsymbol{x}(t)]\,,&\end{aligned} \tag{18.33}$$

and apply the symbolic notation for the path integral:

$$U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0) = C(N) \int_\Omega \mathrm{d}^3 x_{N-1} \ldots \int_\Omega \mathrm{d}^3 x_1$$
$$\exp\Big\{\underbrace{\frac{i}{\hbar}\sum_{j=0}^{N-1}\tau\Big(\frac{m}{2}\Big(\frac{\boldsymbol{x}_{j+1}-\boldsymbol{x}_j}{\tau}\Big)^2 - V\Big(\frac{\boldsymbol{x}_{j+1}+\boldsymbol{x}_j}{2}\Big)\Big)}_{\int_{t_0}^{t_N}\mathrm{d}t\, L\left(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t)\right)}\Big\} \tag{18.34a}$$
$$\equiv C(N) \int_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t) \exp\Big\{\frac{i}{\hbar} S[\boldsymbol{x}(t)]\Big\} \tag{18.34b}$$

This formula differs from (18.25) only due to the replacement of the product of the sums over $\boldsymbol{k}$ by the factor $C(N) =$ (18.31). For $N \to \infty$, this factor would diverge. This is but a further reason to restrict to a finite number N of integrals. Equation (18.34b), which is suggesting a continuum of integrals, is nothing other than a symbolic notation for equation (18.34a), which contains only a finite number of $N - 1$ discrete integrals.

In the sequel, we will use only (18.34), but not any more (18.25). In case of the simple Hamilton operator (18.28) we could compute the factor $C(N)$ analytically. We assume, that for arbitrary Hamilton operators at least a numerical computation of $C(N)$ will always be possible, and that $C(N)$ — with finite N — will always be a finite number different from zero. In actual application we will never need to compute $C(N)$ explicitly, because a factor $C(N)$, which is finite and different from zero, can be simply eliminated from the computations of matrix elements due to appropriate normalization.

18.1.3 A Toy-Model

With increasing N, i. e. with increasing number of intermediate points of time t_j, which are inserted inbetween t_0 and t_N, the paths do not become smoother. Just the contrary: The paths evolve to more and more extreme fever-charts, because $\boldsymbol{x}$ can make arbitrary jumps through the normalization volume in the time interval $t_{j+1} - t_j$. The paths are no differentiable curves, but so-called Markov chains.

If the (virtual, that is to say: not observed) particle is making wide jumps in the time interval from t_j to t_{j+1}, then it's kinetic energy, and thus the action S, may assume gigantic values. One might consider, whether the possibility $(\boldsymbol{x}_{j+1} - \boldsymbol{x}_j)/(t_{j+1} - t_j) > c$ must be prevented by means of additional restrictions. But that's not necessary, because remarkably not the paths with gigantic action, but the paths with smallest action are marking the largest contributions to $U(t_N, \boldsymbol{x}_N, t_0, \boldsymbol{x}_0)$. To make this fact obvious, we now will explicitly compute a simple toy model.

For the numerical computation it will be helpful, to introduce a constant M with the dimension of mass, and to define the dimension-less position operator

$$\boldsymbol{X}_j \equiv \sqrt{\frac{M}{2\hbar\tau}}\,\boldsymbol{x}_j \ . \tag{18.35}$$

Thus one gets

$$U(t_N,\boldsymbol{x}_N,t_0,\boldsymbol{x}_0) \overset{(18.34a)}{=} \underbrace{C\cdot\left(\frac{2\hbar\tau}{M}\right)^{\frac{N-1}{2}}}_{\widetilde{C}} \int\limits_{\Omega} \mathrm{d}^3\boldsymbol{X}_{N-1}\,\ldots\int\limits_{\Omega}\mathrm{d}^3\boldsymbol{X}_1$$
$$\exp\Big\{i\sum_{j=0}^{N-1}\left(\frac{m}{M}(\boldsymbol{X}_{j+1}-\boldsymbol{X}_j)^2-\frac{\tau}{\hbar}V\right)\Big\}\ . \tag{18.36}$$

We define a potential energy, which is independent of position:

$$V(\boldsymbol{x}) \equiv 15\,\frac{\hbar}{\tau} \tag{18.37}$$

(V stays to be time-independent, because τ is a constant, not a running time parameter.) As a most simple model we consider a particle, that is observed at time t_0 at position $\boldsymbol{x}_0$ and at time t_2 at position $\boldsymbol{x}_2$. In-between we insert only one single time-point t_1, at which the particle may be anywhere in the normalization volume Ω. Furthermore we confine the model for simplicity to one space dimension only. We choose

$$X_0 \equiv 4 \qquad\qquad X_2 \equiv 6\ . \tag{18.38}$$

Insertion into (18.36) gives the result

$$U(t_2,x_2,t_0,x_0) = \widetilde{C}\int\limits_{\Omega^{1/3}}\mathrm{d}X_1$$
$$\exp\Big\{i\Big(\frac{m}{M}(4-X_1)^2+\frac{m}{M}(6-X_1)^2-30\Big)\Big\}\ . \tag{18.39}$$

We let the PC compute this model for two particles with different masses, namely

$$m \equiv 0.8\,M \quad \longrightarrow \text{ see figure 18.2} \tag{18.40a}$$
$$m \equiv 2.4\,M \quad \longrightarrow \text{ see figure 18.3 .} \tag{18.40b}$$

Suggestion: Open this book in addition in a second window of the reader, so that you can see this text and figures 18.2 and 18.3 at the same time. The real part and the imaginary part of

$$\exp\Big\{\frac{i}{\hbar}S[x_1]\Big\} = \exp\Big\{i\Big(\frac{m}{M}(4-X_1)^2 + \frac{m}{M}(6-X_1)^2 - 30\Big)\Big\}$$

are displayed in figures 18.2 and 18.3 for paths over different intermediate positions X_1 in the respective bottom parts. The action S is minimal for the path over $X_1 = 5$. For paths over other points X_1 the action is increasing quadratically, resulting into faster and faster oscillations of $\exp\{iS[x_1]/\hbar\}$.

$U(t_2, x_2, t_0, x_0)/\widetilde{C}$ is the (not normalized) probability amplitude for the particle to move from x_0 to x_2 — along whichever intermediate point x_1. If we are asking for the relative probability of the particle not to move along an arbitrary path, but along one certain path close to one certain intermediate point x_1, then we must not integrate in (18.39) over the total normalization volume, but only over the narrow neighborhood of this special point of interest x_1. This has been done in the upper graphs respectively in the both figures 18.2 and 18.3, see the formulas indicated in the graphs. These green printed curves are displaying the relative (not normalized) probabilities for the particle to pass through that certain point x_1 on it's way from x_0 to x_2.

It's worthwhile to compare the both figures closely. They only differ in the masses of the particles, which is 3× larger for the particle of figure 18.3 as compared to the particle of figure 18.2. This difference is resulting into a significantly higher probability for the lighter particle, to deviate appreciably from the path of minimal action (i. e. the path touching $X_1 = 5$) than for the heavier particle. The smaller the ratio $\hbar/S$, the smaller become the green curves in figures 18.2 and 18.3, and the smaller will therefore the particle's deviations from the path of least action be. In the classical limit, Planck's quantum of action becomes negligible versus the action S of the system under consideration ($\hbar/S \ll 1$). In this limit, the green curve becomes a sharp needle of negligible width, and the particle will move with

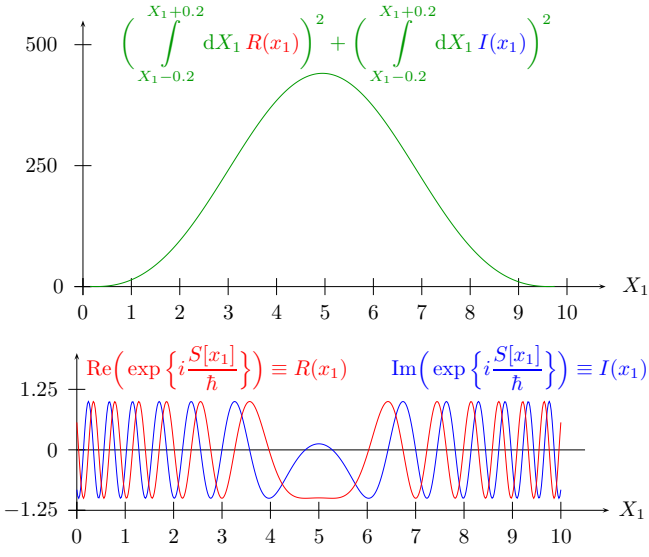


Fig. 18.2 : Model with $m = 0.8\,M$

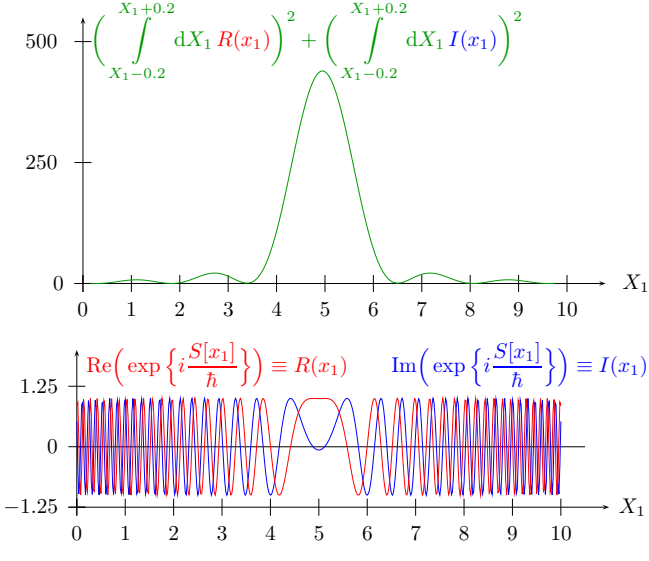


Fig. 18.3 : Model with $m = 2.4\,M$

certainty along the path with minimal action. This is exactly what is stated by Hamilton's principle, from which we derived in chapter 3 the classical equations of motion. Hamilton's principle of least action, which had to be accepted in the frame of classical physics as a not derivable law of nature, thus gets a plausible explanation due to quantum mechanics.

18.1.4 Matrix Elements of the Position Operator

Until now we computed matrix elements of the time-evolution operator. In the next section we will extend the method of path integrals to fields, and then we will need to evaluate matrix elements of field operators. As field operators (i. e. the quantized amplitudes of classical fields) are the field-theoretic counterparts of the position operators of point-particle quantum mechanics, it will be useful to clarify upfront, how matrix elements of the position operator of point-particle mechanics can be computed by means of path integrals.

For that purpose, we insert into the right side of (18.34a) the classical function $\boldsymbol{x}_a \equiv \boldsymbol{x}(t_a)$ with $t_0 < t_a < t_N$:

$$C(N) \int_\Omega \mathrm{d}^3 x_{N-1} \ldots \int_\Omega \mathrm{d}^3 x_a \ldots \int_\Omega \mathrm{d}^3 x_1 \\ \boldsymbol{x}(t_a) \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t_N} \mathrm{d}t \, L\Big(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)\Big) \right\} \tag{18.41}$$

While it was simple and convenient in the derivation of (18.34), to split the time interval from t_0 to t_N into N pieces of equal length, this was not really necessary. Therefore one can arrange one of the $N-1$ integrals in (18.41) to run over the coordinate $\boldsymbol{x}_a \equiv \boldsymbol{x}(t_a)$, no matter where exactly the point of time t_a is located in the continuum $t_0 < t_a < t_N$.

Now we make use of the fact, that in the path integral there are only $N-1$ space integrals over the intermediate points of the paths, but no integrals over the boundary points $\boldsymbol{x}_0$ and $\boldsymbol{x}_N$. If one splits the path integral at $\boldsymbol{x}_a$, then $\boldsymbol{x}_a$ becomes the boundary of two path integrals, while the integral over $\boldsymbol{x}_a$ does not belong to either of both:

$$(18.41) = \int_\Omega \mathrm{d}^3 x_a \left(C(N-a) \int_{[\boldsymbol{x}_a \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t) \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\} \right) \cdot$$
$$\cdot \boldsymbol{x}(t_a) \left(C(a) \int_{[\boldsymbol{x}_0 \to \boldsymbol{x}_a]} \mathcal{D}\boldsymbol{x}(t) \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\} \right) \qquad (18.42)$$

As there are only commutating classical functions in the path integrals, but no operators, the factors could be permuted. Now we make use of the identity of the two path integrals with the matrix elements of the time-evolution operator. Furthermore we insert the position operator of quantum mechanics into the second line of the following equation, and mark it due to a $\hat{\text{h}}$at for clarity:

$$\begin{aligned}(18.42) &= \int_\Omega \mathrm{d}^3 x_a \, \langle t_N, \boldsymbol{x}_N | t_a, \boldsymbol{x}_a \rangle \, \boldsymbol{x}(t_a) \, \langle t_a, \boldsymbol{x}_a | t_0, \boldsymbol{x}_0 \rangle \\ &= \int_\Omega \mathrm{d}^3 x_a \, \langle t_N, \boldsymbol{x}_N | \, \hat{\boldsymbol{x}}(t_a) \, | t_a, \boldsymbol{x}_a \rangle \langle t_a, \boldsymbol{x}_a | t_0, \boldsymbol{x}_0 \rangle \\ &= \langle t_N, \boldsymbol{x}_N | \, \hat{\boldsymbol{x}}(t_a) \, | t_0, \boldsymbol{x}_0 \rangle \ . \end{aligned} \qquad (18.43)$$

In the last line, the completeness relation

$$1 = \int_\Omega \mathrm{d}^3 x_a \, | t_a, \boldsymbol{x}_a \rangle \langle t_a, \boldsymbol{x}_a | \qquad (18.44)$$

of the base vectors has been used. Under the condition $t_0 < t_a < t_N$ one thus gets in total

$$\begin{aligned} &\langle t_N, \boldsymbol{x}_N | \, \hat{\boldsymbol{x}}(t_a) \, | t_0, \boldsymbol{x}_0 \rangle \ = \\ &\quad = C(N) \int_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t) \, \boldsymbol{x}(t_a) \, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\} . \end{aligned} \qquad (18.45)$$

In applications, one wants to know the normalized matrix element

$$\frac{\langle t_N, \boldsymbol{x}_N |\, \hat{\boldsymbol{x}}(t_a)\, | t_0, \boldsymbol{x}_0 \rangle}{\langle t_N, \boldsymbol{x}_N | t_0, \boldsymbol{x}_0 \rangle} = \frac{\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t)\, \boldsymbol{x}(t_a)\, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\}}{\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t)\, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\}} \tag{18.46}$$

in which the factor $C(N)$ has been canceled. The left side of this equation is the normalized matrix element of the position operator $\hat{\boldsymbol{x}}(t_a)$, computed by the formalism of canonically quantized point-particle mechanics. The right side is the same quantity, but now computed by means of a normalized path integral. All factors on the right side are classical functions with commutative algebra, including the position function $\boldsymbol{x}(t_a)$. The matrix element of the position operator $\hat{\boldsymbol{x}}(t_a)$ is identical to the path integral over the classical position function $\boldsymbol{x}(t_a)$, weighted by the factor $\exp\{iS/\hbar\}$.

If a second position function $\boldsymbol{x}(t_b)$ with $t_0 < t_b < t_N$ and $t_b \neq t_a$ is inserted into the path integral, then one gets automatically the matrix element of the time-ordered position operators:

$$\frac{\langle t_N, \boldsymbol{x}_N |\, T\hat{\boldsymbol{x}}(t_a)\hat{\boldsymbol{x}}(t_b)\, | t_0, \boldsymbol{x}_0 \rangle}{\langle t_N, \boldsymbol{x}_N | t_0, \boldsymbol{x}_0 \rangle} = \frac{\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t)\, \boldsymbol{x}(t_a)\boldsymbol{x}(t_b)\, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\}}{\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t)\, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\}} \tag{18.47a}$$

$$= \frac{\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t)\, \boldsymbol{x}(t_b)\boldsymbol{x}(t_a)\, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\}}{\int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t)\, \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\}} \tag{18.47b}$$

In the path integral, the sequence of position functions doesn't matter, as they are classical, commutating functions. But the splitting of the path integral according to (18.42)

$$(18.42) \Longrightarrow \int\limits_{\Omega} \mathrm{d}^3x_a \int\limits_{\Omega} \mathrm{d}^3x_b \left(C(N-a) \int\limits_{[\boldsymbol{x}_a \to \boldsymbol{x}_N]} \mathcal{D}\boldsymbol{x}(t) \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\} \right)$$
$$\boldsymbol{x}(t_a) \left(C(a-b) \int\limits_{[\boldsymbol{x}_b \to \boldsymbol{x}_a]} \mathcal{D}\boldsymbol{x}(t) \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\} \right)$$
$$\boldsymbol{x}(t_b) \left(C(b) \int\limits_{[\boldsymbol{x}_0 \to \boldsymbol{x}_b]} \mathcal{D}\boldsymbol{x}(t) \exp\left\{ \frac{i}{\hbar} S[\boldsymbol{x}(t)] \right\} \right) \tag{18.48}$$

into three pieces $[\boldsymbol{x}_0 \to \boldsymbol{x}_b]$, $[\boldsymbol{x}_b \to \boldsymbol{x}_a]$, and $[\boldsymbol{x}_a \to \boldsymbol{x}_N]$, is correct only, if $t_0 < t_b < t_a < t_N$. In the case $t_b > t_a$, t_a and t_b must be permuted in (18.48). Therefore one gets on the left side of (18.47) inevitably always the time-ordered product of the position operators. The same does hold, if the products of more than two position operators are computed.

18.2 The creating Functional $W[J]$

Functions $f(x) = y$ map numbers x onto numbers y. Functionals $g[f(x)] = y$ map functions $f(x)$ onto numbers y. The definition of the functional integral

$$\int\limits_{[\boldsymbol{x}(t_0) \to \boldsymbol{x}(t_N)]} \mathcal{D}\boldsymbol{x}(t)\, f[\boldsymbol{x}(t), \boldsymbol{k}(t)] \overset{(18.25)}{\equiv} \prod_{j=1}^{N-1} \int\limits_{\Omega} \mathrm{d}^3x(t_j)\, f[\boldsymbol{x}(t_j), \boldsymbol{k}(t_j)]$$
$$\int\limits_{[\boldsymbol{k}(t_0) \to \boldsymbol{k}(t_N)]} \frac{\mathcal{D}\boldsymbol{k}(t)}{(2\pi)^{3N}}\, f[\boldsymbol{x}(t), \boldsymbol{k}(t)] \overset{(18.25)}{\equiv} \prod_{j=1}^{N-1} \frac{1}{\Omega} \sum_{\boldsymbol{k}(t_j)} f[\boldsymbol{x}(t_j), \boldsymbol{k}(t_j)]$$
$$\text{with } t_N > t_{N-1} > \ldots > t_1 > t_0 \tag{18.49}$$

has been used already repeatedly in this chapter. In the sequel, we also will need the functional derivative. The derivative of a function $f(x)$ with respect to the variable x is written as $\mathrm{d}f(x)/\mathrm{d}x$. For the extension of the derivative to the functional derivative

$$\frac{\delta F[f(y)]}{\delta f(x)} \,,$$

the symbol d is replaced by the symbol δ. With respect of the analogy

$$\frac{\mathrm{d}x_i}{\mathrm{d}x_j} = \delta_{ij} \ ,$$

one *defines*

$$\frac{\delta f(y)}{\delta g(x)} \equiv \delta_{fg}\delta^{(4)}(x-y) \ . \tag{18.50}$$

Furthermore we fix by definition, that the chain rule shall be applied in the functional derivative in exactly the same manner as in the derivative of functions. Two examples:

$$\frac{\delta}{\delta f(x)} \int \mathrm{d}^4y \, f(y)\phi(y) = \phi(x)$$
$$\frac{\delta}{\delta f(x)} \exp\left\{\frac{i}{\hbar}\int \mathrm{d}^4y \, f(y)\phi(y)\right\} = \frac{i}{\hbar}\,\phi(x)\,\exp\left\{\frac{i}{\hbar}\int \mathrm{d}^4y \, f(y)\phi(y)\right\}$$

By means of an auxiliary function $J(t)$, which is not specified in detail, the creating functional $W[J(t)]$ is defined. Sometimes it is called vacuum-to-vacuum amplitude[1].

$$W[J(t)] \equiv C(N) \int\limits_\Omega \mathrm{d}^3x_{N-1} \ \ldots \int\limits_\Omega \mathrm{d}^3x_1 \exp\left\{\frac{i}{\hbar}\int\limits_{t_0}^{t_N} \mathrm{d}t \left(L + \boldsymbol{x}(t)J(t)\right)\right\} \tag{18.51}$$

$W[J(t)]$ differs from (18.34a) by the source term $\boldsymbol{x}(t)J(t)$ in the exponent. Therefore

[1] The explanation of this name can be found in [7, section 11.5].

$$\frac{\delta W[J(t)]}{\delta J(t_a)}\Big|_{J=0} = C(N)\int_\Omega \mathrm{d}^3x_{N-1}\,\ldots\int_\Omega \mathrm{d}^3x_1\,\frac{i}{\hbar}\,\boldsymbol{x}(t_a)\exp\Big\{\frac{i}{\hbar}\int_{t_0}^{t_N}\mathrm{d}t\,L\Big\}$$
$$= \frac{i}{\hbar}\cdot(18.41)\ , \tag{18.52}$$

and because of (18.46) we have in general

$$\langle t_N,\boldsymbol{x}_N|\,T\,\underbrace{\hat{\boldsymbol{x}}(t_a)\ldots\hat{\boldsymbol{x}}(t_z)}_{m\text{ operators}}\,|t_0,\boldsymbol{x}_0\rangle = \Big(\frac{\hbar}{i}\Big)^m\frac{\delta^m W[J(t)]}{\delta J(t_a)\ldots\delta J(t_z)}\Big|_{J=0}$$
$$\text{with } t_N > t_j > t_0 \text{ for } j = a\ldots z\ . \tag{18.53}$$

18.3 Klein-Gordon Field

We now want to extend the method of path integrals from point-particle mechanics to the scalar Klein-Gordon field. Our starting point is the transition amplitude

$$\langle\boldsymbol{x}(t_N)\,|\,\boldsymbol{x}(t_0)\,\rangle \overset{(18.34)}{=} C(N)\int_{[\boldsymbol{x}(t_0)\to\boldsymbol{x}(t_N)]}\mathcal{D}\boldsymbol{x}(t)\,\exp\Big\{\frac{i}{\hbar}\int_{t_0}^{t_N}\mathrm{d}t\,L\Big(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t)\Big)\Big\}$$
$$\text{with}\quad \int_{[\boldsymbol{x}(t_0)\to\boldsymbol{x}(t_N)]}\mathcal{D}\boldsymbol{x}(t) \overset{(18.34)}{\equiv} \prod_{j=1}^{N-1}\int_\Omega \mathrm{d}\boldsymbol{x}(t_j)\ . \tag{18.54}$$

Replacing the particle's $\boldsymbol{x}(t)$ by the field amplitude $\psi(t,\boldsymbol{x})$, one gets

$$\langle\psi(t_N,\boldsymbol{x})|\psi(t_0,\boldsymbol{x})\rangle\ =$$
$$= C(N)\int_{[\psi(t_0,\boldsymbol{x})\to\psi(t_N,\boldsymbol{x})]}\mathcal{D}\psi(t,\boldsymbol{x})\,\exp\Big\{\frac{i}{\hbar}\int_{t_0}^{t_N}\mathrm{d}t\int_\Omega \mathrm{d}^3x\,\mathcal{L}\Big(\psi(t,\boldsymbol{x}),\mathrm{d}_\mu\psi(t,\boldsymbol{x})\Big)\Big\}\ . \tag{18.55a}$$

Instead of the particle's Lagrangian, here the volume integral over the field's Lagrangian density has been inserted. The boundaries $[\psi(t_0,\boldsymbol{x})\to\psi(t_N,\boldsymbol{x})]$ of the path integral are indicating, that at all positions $\boldsymbol{x}$ in the normalization

volume Ω the field's configurations shall be integrated from time t_0 to time t_N. We cannot actually compute this integral, unless we have a definition of the path integral as a product of "normal" integrals. The definition in (18.54) is based on the discretization of the argument of the function $\boldsymbol{x}(t)$. In order to formulate the according definition for the path integral (18.55a), *both* arguments of the function $\psi(t, \boldsymbol{x})$ need to be discretized.

For that purpose, we split the normalization volume Ω into R partial volumes of equal size with center coordinates $\boldsymbol{x}_r$, and define

$$\int\limits_{[\psi(t_0,\boldsymbol{x}) \to \psi(t_N,\boldsymbol{x})]} \mathcal{D}\psi(t,\boldsymbol{x}) \equiv \prod_{r=1}^{R} \prod_{j=1}^{N-1} \int\limits_{\{\psi(t_j,\boldsymbol{x}_r)\}} \mathrm{d}\psi(t_j,\boldsymbol{x}_r) \ . \tag{18.55b}$$

The integral $\int \mathcal{D}\psi$ is nothing other than a symbolic notation for the product of these $R \cdot (N-1)$ integrals. R and N shall be chosen as large as possible. But they certainly must be finite, because with infinite R and/or infinite N we could write down the path integral, but we would not be able to compute it.

The boundaries $\{\psi(t_j, \boldsymbol{x}_r)\}$ are indicating, that the integration is running over all values, which ψ can possibly assume at time t_j and position $\boldsymbol{x}_r$, i. e. all complex numbers with finite modulus. The boundaries $\psi(t_0, \boldsymbol{x}_r)$ and $\psi(t_N, \boldsymbol{x}_r)$ are fix for all R volume cells. From the evaluation of point-particle mechanics we learned, that the field will preferentially evolve such in the time interval t_0 and t_N, that the integral of action will be a minimum. Therefore only those values will in the end significantly contribute to the "integral over all complex numbers" $\psi(t_j, \boldsymbol{x}_r)$, which differ only slightly from the value $\psi(t_{j-1}, \boldsymbol{x}_r)$ of the previous integration step, and which are in-between $\psi(t_0, \boldsymbol{x}_r)$ and $\psi(t_N, \boldsymbol{x}_r)$.

We will not compute the factor $C(N)$. Instead we make the plausible assumption, that it is finite and different from zero. Therefore it can be canceled due to an appropriate normalization of the state function $|\psi(t, \boldsymbol{x})\rangle$.

The useful formalism with the creating functional $W[J]$ can be transferred without further problems from the point-particles to the Klein-Gordon field:

$$W[J(x)] \overset{(18.51)}{\equiv} C(N) \int\limits_{[\psi(t_0,\boldsymbol{x}) \to \psi(t_N,\boldsymbol{x})]} \mathcal{D}\psi(x)\, \exp\Big\{\frac{i}{\hbar} \int\limits_{t_0}^{t_N} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^3x \left(\mathcal{L} + \psi(x)J(x)\right)\Big\} \tag{18.56}$$

Now the matrix elements of arbitrary products of field operators can be constructed:

$$\langle\psi(t_N,\boldsymbol{x})|\, T \underbrace{\psi(x_a)\dots\psi(x_z)}_{m \text{ operators}} |\psi(t_0,\boldsymbol{x})\rangle \overset{(18.53)}{=} \Big(\frac{\hbar}{i}\Big)^m \frac{\delta^m W[J(x)]}{\delta J(x_a)\dots\delta J(x_z)}\Big|_{J=0}$$
$$\text{with } t_N > t_j > t_0 \text{ for } j = a\dots z \tag{18.57}$$

The factors $\psi(x_a)\dots\psi(x_z)$ in the matrix element on the equation's left side are operators, while the factors $\psi(x)$ on the equation's right side resp. in (18.56) are not-quantized classical field amplitudes.

In section 15.5 we have computed the quantized propagator

$$G(x-y) \overset{(15.43)}{=} \langle 0|\, T\psi(x)\psi^\dagger(y)\, |0\rangle$$

of the Klein-Gordon field. For the computation of the scattering amplitudes of interacting fields, again we will need the matrix elements in the vacuum-state $|0\rangle$. Thus it would be clearly advantageous, if we could compute by means of path integrals the expectation values $\langle 0|\dots|0\rangle$ instead of the expectation values $\langle\psi(t_N,\boldsymbol{x})|\dots|\psi(t_0,\boldsymbol{x})\rangle$.

Actually the both types of expectation values are almost identical, provided that the point of time t_0 is sufficiently far in the past, and the point of time t_N is sufficiently far in the future. To make this clear, we need the eigen-states $|n\rangle$ of the Hamilton operator H, which solve the equation

$$H|n\rangle = E_n|n\rangle \implies \exp\Big\{-\frac{i}{\hbar}Ht\Big\}|n\rangle = \exp\Big\{-\frac{i}{\hbar}E_nt\Big\}|n\rangle\ . \tag{18.58}$$

Furthermore we assume, that the Hamilton operator does not depend explicitly on time, such that

$$|\psi(t_N,\boldsymbol{x})\rangle \overset{(14.31)}{=} U(t_N,t_0)\,|\psi(t_0,\boldsymbol{x})\rangle \overset{(14.28a)}{=} \exp\Big\{-\frac{i}{\hbar}(t_N-t_0)H\Big\}\,|\psi(t_0,\boldsymbol{x})\rangle.$$

The time-evolution operator U has been discussed in section 14.1.3. The completeness relations

$$\sum_n |n\rangle\langle n| = 1 \quad , \quad \sum_m |m\rangle\langle m| = 1$$

hold for the solutions $|n\rangle$ and $|m\rangle$ of (18.58). These two factors 1 are inserted into the matrix element:

$$\begin{aligned}
&\langle\psi(t_N,\boldsymbol{x})|\ldots|\psi(t_0,\boldsymbol{x})\rangle \;= \\
&= \sum_n \sum_m \langle\psi(t_N,\boldsymbol{x})|n\rangle\langle n|\ldots|m\rangle\langle m|\psi(t_0,\boldsymbol{x})\rangle \;= \\
&= \sum_n \sum_m \exp\Big\{\frac{i}{\hbar}\,E_n t_N\Big\} \exp\Big\{\frac{i}{\hbar}\,E_m t_0)\Big\}\langle n|\ldots|m\rangle \cdot \\
&\quad \cdot \underbrace{\langle\psi(t=0,\boldsymbol{x})|n\rangle\langle m|\psi(t=0,\boldsymbol{x})\rangle}_{K}
\end{aligned} \tag{18.59}$$

Into the last expression, two times the time-evolution operator has been inserted. The only thing we need to know about the factor K is, that it does not depend on time. It will immediately be canceled. The matrix element $\langle 0|\ldots|0\rangle$ is contained in the second-last line of (18.59), though only in one of countably infinitely many terms. But all other terms are negligible, if t_0 is sufficiently far in the past, and t_N is sufficiently far in the future. The essential trick, to make this obvious, is to shift the times slightly into the complex plane. Consider the limits

$$\begin{aligned}
&t_0 \;\to\; +\infty\cdot\exp\{i(\pi+\varphi)\} \quad , \quad t_N \;\to\; +\infty\cdot\exp\{i\varphi\} \\
&\quad \text{with } 0 < \varphi \ll 1 \in \mathbb{R}\;.
\end{aligned} \tag{18.60}$$

φ is a very small angle, by which the points of time $\pm\infty$ are shifted slightly counter-clockwise into the complex plane. Due to the imaginary part in the time factors, the exponential functions are converging fast to zero for large E_n and E_m. Therefore the term with $n = m = 0$ does dominate the

result for sufficiently large times, and all other terms are negligible. Thus the normalized expectation values become

$$\lim_{t_N\to+\infty}\lim_{t_0\to-\infty}\frac{\langle\psi(t_N,\boldsymbol{x})|\,T\overbrace{\psi(x_a)\dots\psi(x_z)}^{m\text{ operators}}\,|\psi(t_0,\boldsymbol{x})\rangle}{\langle\psi(t_N,\boldsymbol{x})\,|\,\psi(t_0,\boldsymbol{x})\rangle} =$$

$$= \frac{\langle 0|\,T\psi(x_a)\dots\psi(x_z)\,|0\rangle}{\langle 0|0\rangle} =$$

$$= \lim_{t_N\to+\infty}\lim_{t_0\to-\infty}\frac{\left(\frac{\hbar}{i}\right)^m\frac{\delta^m W[J(x)]}{\delta J(x_a)\dots\delta J(x_z)}\Big|_{J=0}}{C(N)\int\limits_{[\psi(t_0,\boldsymbol{x})\to\psi(t_N,\boldsymbol{x})]}\mathcal{D}\psi(x)\,\exp\left\{\frac{i}{\hbar}\int\limits_{t_0}^{t_N}\mathrm{d}t\int\limits_{\Omega}\mathrm{d}^3x\,\mathcal{L}\right\}}$$

$$\text{with } W[J(x)] = (18.56) \text{ and } t_N > t_j > t_0 \text{ for } j = a\dots z\ . \qquad (18.61)$$

Part 3:
Interacting Quantum Fields

19 ψ^s-Interaction of Scalar Fields

Using the Lagrangian

$$\mathcal{L} \overset{(10.10)}{=} \frac{\hbar^2 c^2}{2} (\mathrm{d}_\mu \phi) \mathrm{d}^\mu \phi - \frac{1}{2} m^2 c^4 \phi^2$$

of a free (i. e. not interacting) uncharged Klein-Gordon field, we have derived the field equation

$$\hbar^2 c^2 \mathrm{d}_\mu \mathrm{d}^\mu \phi + m^2 c^4 \phi \overset{(3.37)}{=} 0 \ . \tag{19.1}$$

By the Lagrangian

$$\boxed{\begin{array}{c} \mathcal{L} \equiv \frac{\hbar^2 c^2}{2} (\mathrm{d}_\mu \psi) \mathrm{d}^\mu \psi - \frac{1}{2} m^2 c^4 \psi\psi - \lambda \hbar^3 c^3 \psi^s \\ \text{with } \lambda \in \mathbb{R} > 0 \text{ and } s = 3 \text{ or } s = 4 \ , \end{array}} \tag{19.2}$$

an uncharged scalar field can be described, which is interacting with itself. As the dimension of the field operator according to (10.12) is $[\psi] = (\text{energy} \cdot \text{volume})^{-1/2}$, the dimension of the coupling constant λ is

$$[\lambda] = \left[\frac{\mathcal{L}}{\hbar^3 c^3 \psi^s}\right] = (\text{energy} \cdot \text{volume})^{(s-4)/2} \ . \tag{19.3}$$

While λ is dimension-less for $s = 4$, it's dimension for $s = 3$ is $(\text{energy} \cdot \text{volume})^{-1/2}$. The field-equation with ψ^s-interaction is

$$\hbar^2 c^2 \mathrm{d}_\mu \mathrm{d}^\mu \psi + m^2 c^4 \psi + s\lambda \hbar^3 c^3 \psi^{s-1} \overset{(3.37)}{=} 0 \ . \tag{19.4}$$

In general no closed solution of this equation is possible. One instead tries to approach the solutions step by step by means of perturbative methods.

The conjugate momentum density

$$\pi(x) \overset{(3.57)}{=} \frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)} = c\hbar^2 \mathrm{d}^0 \psi(x) \tag{19.5}$$

is identical for the free field and for the field with ψ^s-interaction. From (19.2) follows the Hamilton-density

$$\begin{aligned} \mathcal{H} &\overset{(4.35)}{=} \pi\psi - \mathcal{L} = \\ &= \underbrace{c\hbar^2 \mathrm{d}^0\psi - \frac{\hbar^2 c^2}{2}(\mathrm{d}_\mu \psi)\mathrm{d}^\mu \psi + \frac{1}{2} m^2 c^4 \psi\psi}_{\mathcal{H}_{(0)}} + \underbrace{\lambda \hbar^3 c^3 \psi^s}_{\mathcal{H}_{(I)}} . \end{aligned} \tag{19.6}$$

$\mathcal{H}_{(0)}$ is the Hamilton density of the free field without interaction. $\mathcal{H}_{(I)}$ (the index I stands for interaction) is that part of the Hamilton density, which is added due to the interaction. The volume integrals are

$$H = \int_\Omega \mathrm{d}^3 x\, \mathcal{H} \quad , \quad H_{(0)} = \int_\Omega \mathrm{d}^3 x\, \mathcal{H}_{(0)} \quad , \quad H_{(I)} = \int_\Omega \mathrm{d}^3 x\, \mathcal{H}_{(I)} .$$

ψ^s-theory probably is the simplest example of a quantum field theory with interaction. We start the third part of this book with it's evaluation, because it is the most convenient tool to get familiar with the computational methods of quantum field theories with interactions. ψ^4-theory has found a practical application in the description of Higgs-fields[1].

19.1 Perturbation Theory

In the computations of interacting fields, we will encounter matrix elements of time-ordered operator products like

$$\langle 0 |\, T\psi(x_1) \dots \psi(x_n)\, | 0 \rangle \ . \tag{19.7}$$

[1] named after Peter Higgs (∗ 1929), one of approximately half a dozen of theorists, who — simultaneously and independent from another — invented these fields in 1964.

$|0\rangle$ henceforth is to be interpreted as the vacuum state of the field with interaction. We will see that this vacuum — different from the vacuum of the free field — is not simply empty, but is filled by an infinite number of "vacuum bubbles". (Purists therefore use different notations for the two vacua.) T is the time-order operator, which has been defined in (15.44).

The field operators ψ in the matrix element (19.7) are solutions of the field equation (19.4). Even though they are unknown for most points of time, they can in many cases relatively easy be found by computation or by guessing for some certain points of time. As soon as the field operator is known for one point of time t_0, it can be computed for arbitrary points of time t by means of the time-evolution operator, which has been introduced in section 14.1.3:

$$\psi(t,\boldsymbol{x}) = U^{-1}(t,t_0)\psi(t_0,\boldsymbol{x})U(t,t_0) \tag{19.8}$$

$\psi(t_0,\boldsymbol{x})$ is a time-independent operator in the Schrödinger-picture. It can be expanded with respect to the creation- and annihilation-operators $a_{\boldsymbol{k}}^\dagger$ and $a_{\boldsymbol{k}}$ of the free field:

$$\psi(t_0,\boldsymbol{x}) \overset{(15.15a)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}}\Big(a_{\boldsymbol{k}}\exp\{+i\boldsymbol{k}\boldsymbol{x}\} + a_{\boldsymbol{k}}^\dagger\exp\{-i\boldsymbol{k}\boldsymbol{x}\}\Big) \tag{19.9}$$

It will turn out to be most useful, that the operators $a_{\boldsymbol{k}}^\dagger$ and $a_{\boldsymbol{k}}$ are explicitly visible in the expression for $\psi(t,\boldsymbol{x})$. $U(t,t_0)$ is unknown for the field with self-interaction, however, and therefore we get stuck at this point for the moment being. A first trial of approximation could simply be to replace the unknown operator $U(t,t_0)$ by the known operator $U_{(0)}(t,t_0)$ of the free field. This approximation leads to the field-operator in the interaction-picture:

$$\psi_{(I)}(t,\boldsymbol{x}) \equiv U_{(0)}^{-1}(t,t_0)\psi(t_0,\boldsymbol{x})U_{(0)}(t,t_0) \tag{19.10}$$

$$\overset{(15.15a)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}}\Big(a_{\boldsymbol{k}}\exp\{-ikx\} + a_{\boldsymbol{k}}^\dagger\exp\{+ikx\}\Big) \tag{19.11}$$

This approximation, however, is overly rough. Therefore we return to the exact expression (19.8), and inspect the time evolution of the field-operator

in more detail, inserting two times a factor 1:

$$\begin{aligned}
\psi(t,\boldsymbol{x}) &= U^{-1}(t,t_0)\psi(t_0,\boldsymbol{x})U(t,t_0)\\
&= U^{-1}(t,t_0)\underbrace{U_{(0)}(t,t_0)U_{(0)}^{-1}(t,t_0)}_{1}\psi(t_0,\boldsymbol{x})\underbrace{U_{(0)}(t,t_0)U_{(0)}^{-1}(t,t_0)}_{1}U(t,t_0)\\
&= \underbrace{U^{-1}(t,t_0)U_{(0)}(t,t_0)}_{U_{(I)}^{-1}(t,t_0)}\underbrace{U_{(0)}^{-1}(t,t_0)\psi(t_0,\boldsymbol{x})U_{(0)}(t,t_0)}_{\psi_{(I)}(t,\boldsymbol{x})}\underbrace{U_{(0)}^{-1}(t,t_0)U(t,t_0)}_{U_{(I)}(t,t_0)}
\end{aligned} \tag{19.12}$$

Here the time evolution operator

$$U_{(I)}(t,t_0) \equiv U_{(0)}^{-1}(t,t_0)\,U(t,t_0) \tag{19.13}$$

in the interaction-picture has been defined. We compute it's derivative with respect to time:

$$\begin{aligned}
\frac{\mathrm{d}U_{(I)}(t,t_0)}{\mathrm{d}t} &= \frac{\mathrm{d}U_{(0)}^{-1}(t,t_0)}{\mathrm{d}t}\,U(t,t_0) + U_{(0)}^{-1}(t,t_0)\,\frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}\\
&\overset{(14.43)}{=} U_{(0)}^{-1}(t,t_0)\,\frac{i}{\hbar}\underbrace{(H_{(0)}-H)}_{-H_{(I)}}U(t,t_0)\\
&= \frac{-i}{\hbar}\underbrace{U_{(0)}^{-1}(t,t_0)\,H_{(I)}\,U_{(0)}(t,t_0)}_{H_{(I)}(t)}\underbrace{U_{(0)}^{-1}(t,t_0)U(t,t_0)}_{U_{(I)}(t,t_0)}
\end{aligned} \tag{19.14}$$

The time-evolution operator $U_{(I)}(t,t_0)$ in the interaction-picture solves a Schrödinger-equation of the form (14.27) with the time-dependent Hamilton operator $H_{(I)}(t)$.

Assuming $t_1 < t_2 < \ldots < t_n$, and using the time-evolution operator $U_{(I)}(t,t_0)$ in the interaction picture, the matrix element (19.7) can be transformed:

$$\begin{aligned}
&\langle 0|\,\psi(x_n)\dots\psi(x_1)\,|0\rangle \overset{(19.12)}{=} \\
&= \langle 0|\,U_{(I)}^{-1}(t_n,t_0)\psi_{(I)}(x_n)\underbrace{U_{(I)}(t_n,t_0)\,U_{(I)}^{-1}(t_{n-1},t_0)}_{U_{(I)}(t_n,t_{n-1})}\psi_{(I)}(x_{n-1})\dots \\
&\quad\dots\psi_{(I)}(x_2)\underbrace{U_{(I)}(t_2,t_0)U_{(I)}^{-1}(t_1,t_0)}_{U_{(I)}(t_2,t_1)}\,\psi_{(I)}(x_1)U_{(I)}(t_1,t_0)\,|0\rangle
\end{aligned}$$

The conversions, which are marked by under-braces, follow from (14.29). We also transform the both outer time-evolution operators, which are not under-braced, by means of two new time parameters $t_e < t_1$ and $t_a > t_n$. The point of time t_0 may be chosen arbitrarily. We choose $t_0 = t_e$:

$$\begin{aligned}
&U_{(I)}(t_1,t_0) = U_{(I)}(t_1,t_e)\,U_{(I)}(t_e,t_0) \overset{t_0=t_e}{=} U_{(I)}(t_1,t_e) \\
&U_{(I)}^{-1}(t_n,t_0) = \Big(U_{(I)}(t_n,t_a)\,U_{(I)}(t_a,t_0)\Big)^{-1} \overset{t_0=t_e}{=} \\
&\quad = U_{(I)}^{-1}(t_a,t_e)\,U_{(I)}^{-1}(t_n,t_a) \overset{(14.29d)}{=} U_{(I)}^{-1}(t_a,t_e)\,U_{(I)}(t_a,t_n) \\
&\implies \quad \langle 0|\,\psi(x_n)\dots\psi(x_1)\,|0\rangle \;= \\
&\quad = \langle 0|\,U_{(I)}^{-1}(t_a,t_e)\,U_{(I)}(t_a,t_n)\psi_{(I)}(x_n)U_{(I)}(t_n,t_{n-1})\psi_{(I)}(x_{n-1})\dots \\
&\qquad \dots\psi_{(I)}(x_2)\,U_{(I)}(t_2,t_1)\,\psi_{(I)}(x_1)U_{(I)}(t_1,t_e)\,|0\rangle
\end{aligned} \tag{19.15}$$

The time-evolution operator at the very left side of the product is acting onto the vacuum state $\langle 0|$. What is the effect? Certainly the vacuum state will stay to be a vacuum state, but it's phase might change:

$$\langle 0|\,U_{(I)}^{-1}(t_a,t_e) = \Big(U_{(I)}(t_a,t_e)\,|0\rangle\Big)^\dagger = \Big(|0\rangle\,e^{i\varphi}\Big)^\dagger = e^{-i\varphi}\,\langle 0| \qquad \text{with } \varphi\in\mathbb{R}$$

Writing the phase factor as

$$e^{-i\varphi} = \Big(\langle 0|0\rangle e^{i\varphi}\Big)^{-1} = \Big(\langle 0|\,U_{(I)}(t_a,t_e)\,|0\rangle\Big)^{-1}\,,$$

the matrix element can be written in the form

$$\Big(\langle 0|\, U_{(I)}(t_a,t_e)\,|0\rangle\Big)^{-1}\langle 0|\, U_{(I)}(t_a,t_n)\psi_{(I)}(x_n)U_{(I)}(t_n,t_{n-1})\psi_{(I)}(x_{n-1})\dots$$
$$\dots\psi_{(I)}(x_2)\,U_{(I)}(t_2,t_1)\,\psi_{(I)}(x_1)U(t_1,t_e)\,|0\rangle\ . \tag{19.16}$$

The operators in this function are time-ordered. Their sequence may be changed, if for compensation the time-order operator is re-inserted:

$$\Big(\langle 0|\, U_{(I)}(t_a,t_e)\,|0\rangle\Big)^{-1}\langle 0|\, T\psi_{(I)}(x_n)\psi_{(I)}(x_{n-1})\dots\psi_{(I)}(x_2)\psi_{(I)}(x_1)\cdot$$
$$\cdot\underbrace{U_{(I)}(t_a,t_n)\,U_{(I)}(t_n,t_{n-1})\dots U_{(I)}(t_2,t_1)\,U_{(I)}(t_1,t_e)}_{U_{(I)}(t_a,t_e)}\,|0\rangle\ =$$
$$=\frac{\langle 0|\, T\psi_{(I)}(x_n)\psi_{(I)}(x_{n-1})\dots\psi_{(I)}(x_2)\psi_{(I)}(x_1)\,U_{(I)}(t_a,t_e)\,|0\rangle}{\langle 0|\, U_{(I)}(t_a,t_e)\,|0\rangle} \tag{19.17}$$

In (19.14) which have checked, that a Schrödinger equation with the time-dependent Hamilton operator $H_{(I)}(t)$ holds for the time-evolution operator $U_{(I)}(t_a,t_n)$. The solution of this equation has already been indicated in section 14.1.3:

$$U_{(I)}(t_a,t_e)\overset{(14.28b)}{=}T\,\exp\Big\{-\frac{i}{\hbar}\int_{t_e}^{t_a}\mathrm{d}\tau\,H_{(I)}(\tau)\Big\}$$
$$=T\sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{i}{\hbar}\int_{t_e}^{t_a}\mathrm{d}\tau\,H_{(I)}(\tau)\Big)^n \tag{19.18}$$

The operators $H_{(I)}(\tau)$ and $\psi_{(I)}(t_k,\boldsymbol{x}_k)$ get mixed in the numerator of (19.17) due to the time-order operator. Therefore the terms $U_{(I)}(t_a,t_e)\,|0\rangle$ in the numerator and denominator of (19.17) can not be canceled. In total, we have achieved this conversion of the matrix element:

$$\langle 0|\, T\psi(x_n)\dots\psi(x_1)\,|0\rangle = \sum_{j=0}^{\infty}\frac{1}{j!}\cdot$$
$$\cdot\ \frac{\langle 0|\, T\psi_{(I)}(x_n)\dots\psi_{(I)}(x_1)\Big(-\frac{i}{\hbar}\int_{t_e}^{t_a}\mathrm{d}\tau\, H_{(I)}(\tau)\Big)^j\,|0\rangle}{\langle 0|\, T\sum_{m=0}^{\infty}\frac{1}{m!}\Big(-\frac{i}{\hbar}\int_{t_e}^{t_a}\mathrm{d}\tau\, H_{(I)}(\tau)\Big)^m\,|0\rangle} \tag{19.19}$$

This expression is exact, as we did not make any approximations so far. The computation of the matrix element is significantly facilitated after this conversion. Firstly, we now are using the field operators $\psi_{(I)}(x)$ in the interaction picture. These can much easier be found than the operators $\psi(x)$. Secondly, the transformed matrix element contains the time-dependent Hamilton operator in the interaction picture

$$H_{(I)}(t) \overset{(19.14)}{=} U_{(0)}^{-1}(t,t_0)\, H_{(I)}\, U_{(0)}(t,t_0) \tag{19.20}$$
$$\text{with } H_{(I)} \overset{(19.6)}{=} H - H_{(0)}$$

as an infinite series. Thus the numerator can be computed step by step in increasing orders of j, and one does not need to master the whole problem in one single step. In fortunate cases (i. e. if the coupling constant λ in (19.6) is small versus 1), one or two steps may already be sufficient for a good approximation. One might be concerned, that the stepwise computation should be doomed to failure because one needs to have computed the denominator of (19.19) completely, before any result can be concluded from this formula. But — as we will see — that concern is not justified, because the denominator can be canceled surprisingly simple from the computations.

Thus there are good reasons, to use the operators in the interaction picture in the following computations. It would be quite annoying, always to drag the index $_{(I)}$ along. Therefore we arrange for the re-namings

$$\begin{array}{rlrl} \psi(x) & \longrightarrow \psi_{(W)}(x) & \psi_{(I)}(x) & \longrightarrow \psi(x) \\ U(t,t_0) & \longrightarrow U_{(W)}(t,t_0) & U_{(I)}(t,t_0) & \longrightarrow U(t,t_0) \\ H & \longrightarrow H_{(W)} & H_{(I)} & \longrightarrow H \quad , \end{array} \tag{19.21}$$

where the index $_{(W)}$ should be read as "interacting total system".

19.2 Wick's Theorem

If a matrix element of the type $\langle 0|\, T\psi(x_n)\dots\psi(x_1)\,|0\rangle$ shall be computed, with the $\psi(x_j)$ being field-operators in the interaction picture according to the notation (19.21), then it turns out to be most helpful that the operators in the interaction picture can be expanded with respect to the creation- and annihilation-operators:

$$\psi(x) \overset{(19.11)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}}\Big(a_{\boldsymbol{k}} \exp\{-ikx\} + a^\dagger_{\boldsymbol{k}} \exp\{+ikx\}\Big) \tag{19.22}$$

Because of $a_{\boldsymbol{k}}|0\rangle = 0$ and $\langle 0|a^\dagger_{\boldsymbol{k}} = 0$, the vacuum-expectation-value of an operator-product will certainly be zero, if the operator at the very right side is an annihilation-operator, and/or if the operator at the very left side is a creation-operator. In the sequel, a most useful method will be described, by which such products can be systematically analyzed and significantly simplified.

19.2.1 Creation- and Annihilation-Operators

Split the field-operator of the uncharged Klein-Gordon field in the interaction picture as follows:

$$\psi(x) = \psi^{\mathsf{V}}(x) + \psi^{\mathsf{E}}(x) \quad , \quad \psi^{\mathsf{E}}(x) = \Big(\psi^{\mathsf{V}}(x)\Big)^\dagger \tag{19.23a}$$

$$\psi^{\mathsf{V}}(x) \overset{(15.15)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}}\, a_{\boldsymbol{k}} \exp\{-ikx\} \tag{19.23b}$$

To enhance the readability of the formulas, we introduce the simplified notation

$$\psi^{\mathsf{V}}_x \equiv \psi^{\mathsf{V}}(x) \qquad \psi^{\mathsf{E}}_x \equiv \psi^{\mathsf{E}}(x)\ . \tag{19.23c}$$

Let's consider as an example a product of six operators:

$$\langle 0|\, T\psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, \psi^{\mathsf{V}}_y\, \psi^{\mathsf{E}}_z\, |0\rangle \quad \text{with } u^0 > v^0 > w^0 > x^0 > y^0 > z^0$$

If there would be ψ^{V}_z at the very right, or ψ^{E}_u at the very left, then the expectation value obviously would be zero. As that is not the case, the operator-product is transformed:

$$\begin{aligned}\psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, \psi^{\mathsf{V}}_y\, \psi^{\mathsf{E}}_z \;&= \\ &= \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, [\psi^{\mathsf{V}}_y\, , \psi^{\mathsf{E}}_z\,] + \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, \psi^{\mathsf{E}}_z\, \psi^{\mathsf{V}}_y\end{aligned} \tag{19.24}$$

The vacuum-expectation value of the second term is zero, because there is an annihilation-operator at the very right. In this term, the operators are not time-ordered any more. This is not a failure. The time-order-operator just demands, that the operators must be arranged at the beginning of the computation in the correct sequence, and that has been done in the first line of the last equation.

As the commutator is a number, it may be factored out:

$$\langle 0|\, \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, [\psi^{\mathsf{V}}_y\, , \psi^{\mathsf{E}}_z\,]\, |0\rangle = \langle 0|\, \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, |0\rangle\, [\psi^{\mathsf{V}}_y\, , \psi^{\mathsf{E}}_z\,]$$

If there would be ψ^{V}_x at the very right of the matrix element instead of ψ^{E}_x, then the expectation value would obviously be zero. As that's not the case, the operator-product is again transformed in the same manner as before:

$$\langle 0|\, \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, |0\rangle = \langle 0|\, \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, |0\rangle [\psi^{\mathsf{V}}_w\, , \psi^{\mathsf{E}}_x\,] + \langle 0|\, \psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{E}}_x\, \psi^{\mathsf{V}}_w\, |0\rangle$$

Again the right term is zero, because there is an annihilation-operator at the operator-products right side. The remaining matrix element is again transformed in the usual manner. Thus one eventually arrives at the result

$$\langle 0|\, T\psi^{\mathsf{V}}_u\, \psi^{\mathsf{E}}_v\, \psi^{\mathsf{V}}_w\, \psi^{\mathsf{E}}_x\, \psi^{\mathsf{V}}_y\, \psi^{\mathsf{E}}_z\, |0\rangle = [\psi^{\mathsf{V}}_u\, , \psi^{\mathsf{E}}_v\,]\, [\psi^{\mathsf{V}}_w\, , \psi^{\mathsf{E}}_x\,]\, [\psi^{\mathsf{V}}_y\, , \psi^{\mathsf{E}}_z\,]\ . \tag{19.25}$$

The vacuum-expectation-value of an odd number n of creation- and

annihilation-operators is zero in any case, because it is the product of $(n-1)/2$ commutators with the factor $\langle 0|\,\psi\,|0\rangle = 0$. This factor is zero in any case, no matter whether ψ is a creation- or an annihilation-operator.

The vacuum-expectation-value furthermore is in any case zero, if the number of operators is even, but the number of creation-operators is not equal to the number of annihilation-operators. Then several operators of one type remain, resulting in an expectation value of the type

$$\begin{aligned} &\langle 0|\,\psi^{\mathsf{V}}\ldots\psi^{\mathsf{V}}\,|0\rangle \cdot \text{commutators} = 0 \\ \text{or}\quad &\langle 0|\,\psi^{\mathsf{E}}\ldots\psi^{\mathsf{E}}\,|0\rangle \cdot \text{commutators} = 0\ . \end{aligned} \tag{19.26}$$

Thus the expectation value of a product of six creation- and annihilation-operators can be different from zero only, if it consists of three creation- and three annihilation-operators. Thereby the sequence of the operators is important. We apply the method, by which (19.25) has been derived, to the matrix element

$$\begin{aligned} \langle 0|\,T\psi_u^{\mathsf{V}}\,\psi_v^{\mathsf{V}}\,\psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}}\,|0\rangle &= [\psi_u^{\mathsf{V}}\,,\psi_w^{\mathsf{E}}\,]\,[\psi_v^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,]\,[\psi_x^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}\,] + \\ &+ [\psi_u^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,]\,[\psi_v^{\mathsf{V}}\,,\psi_w^{\mathsf{E}}\,]\,[\psi_x^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}\,] + [\psi_u^{\mathsf{V}}\,,\psi_w^{\mathsf{E}}\,]\,[\psi_v^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}\,]\,[\psi_x^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,] + \\ &+ [\psi_u^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}\,]\,[\psi_v^{\mathsf{V}}\,,\psi_w^{\mathsf{E}}\,]\,[\psi_x^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,]\ . \end{aligned} \tag{19.27}$$

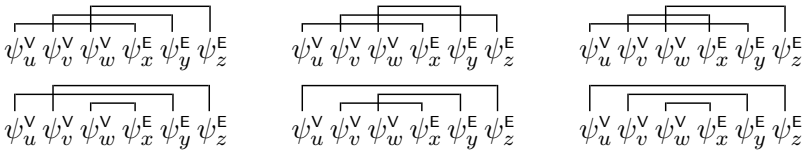
There are four terms in this case. Each term is resulting from a different contraction of the operators to commutators, which firstly are different from zero, and in which secondly the creation operator is earlier than the annihilation operator. If all three creation operators are earlier than the three annihilation operators, then the computation of the vacuum-expectation-value will result into six terms, because then six different non-vanishing contractions are possible.

In the sequel, contractions will be indicated by horizontal square brackets. The time-ordered operator-product $\psi_u^{\mathsf{V}}\,\psi_v^{\mathsf{E}}\,\psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}}$ has one non-vanishing contraction, which should be compared to (19.25):

$$\overbrace{\psi_u^{\mathsf{V}}\,\psi_v^{\mathsf{E}}\,\psi_w^{\mathsf{V}}}\,\psi_x^{\mathsf{E}}\,\overbrace{\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}}}$$

The time-ordered operator-product $\psi_u^{\mathsf{V}}\,\psi_v^{\mathsf{V}}\,\psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}}$ has four non-vanishing contractions, which should be compared to (19.27):

The time-ordered operator-product $\psi_u^{\mathsf{V}}\,\psi_v^{\mathsf{V}}\,\psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}}$ has six non-vanishing contractions:



The time-ordered operator product $\psi_u^{\mathsf{V}}\,\psi_v^{\mathsf{E}}\,\psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}}$ has only vanishing contractions. Therefore it's vacuum-expectation-value is zero.

Theorem: The vacuum-expectation-value of a time-ordered product of creation- and annihilation-operators is equal to the sum of the commutator-products, which can be constructed by all non-vanishing contractions of that operator-product. (19.28)

19.2.2 Field-Operators

In practical applications, we don't deal with the single operators ψ_x^{V} and ψ_x^{E}, but with the complete field-operator $\psi_x \equiv \psi_x^{\mathsf{V}} + \psi_x^{\mathsf{E}}$. The formulas merely become more laborious due to the extension of our results onto the complete operators, but no novel problems turn up.

The time-ordered product of two field-operators (19.23) of the uncharged Klein-Gordon field is

$$\begin{aligned} T\psi_x\psi_y = \theta(x^0-y^0)\,(\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{V}} + \psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}} + \psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{V}} + \psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{E}}) + \\ + \theta(y^0-x^0)\,(\psi_y^{\mathsf{V}}\,\psi_x^{\mathsf{V}} + \psi_y^{\mathsf{V}}\,\psi_x^{\mathsf{E}} + \psi_y^{\mathsf{E}}\,\psi_x^{\mathsf{V}} + \psi_y^{\mathsf{E}}\,\psi_x^{\mathsf{E}})\ . \end{aligned} \tag{19.29}$$

In the vacuum-expectation-value, only those two terms survive, which have creation operators at their right sides, and annihilation operators at their left sides:

$$\langle 0|\,T\psi_x\psi_y\,|0\rangle = \theta(x^0-y^0)\langle 0|\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,|0\rangle + \theta(y^0-x^0)\langle 0|\,\psi_y^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,|0\rangle \tag{19.30a}$$

Applying instead theorem (19.28), one gets

$$\langle 0|\, T\psi_x\psi_y\,|0\rangle = \theta(x^0-y^0)\,[\,\psi_x^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,] + \theta(y^0-x^0)\,[\,\psi_y^{\mathsf{V}}\,,\psi_x^{\mathsf{E}}\,]\;. \tag{19.30b}$$

This isn't a contradiction because of

$$\langle 0|\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,|0\rangle - \underbrace{\langle 0|\,\psi_y^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,|0\rangle}_{0} = \langle 0|\,[\,\psi_x^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,]\,|0\rangle = \underbrace{\langle 0|0\rangle}_{1}\,[\,\psi_x^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}\,]\;. \tag{19.31}$$

The vacuum-expectation-value (19.30) is identical to the Feynman-propagator of the uncharged Klein-Gordon field, which has been defined in section 15.5. Now we extend our consideration to a charged Klein-Gordon field, and split it into creation- and annihilation-operators:

$$\begin{aligned}
\psi_x &= \psi_x^{\mathsf{Va}} + \psi_x^{\mathsf{Eb}} \qquad , \qquad \psi_x^\dagger = \psi_x^{\mathsf{Ea}} + \psi_x^{\mathsf{Vb}} \\
\psi_x^{\mathsf{Va}} &\overset{(15.15)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}}\; a_{\boldsymbol{k}}\, \exp\{-ikx\} \\
\psi_x^{\mathsf{Eb}} &\overset{(15.15)}{=} \sum_{\boldsymbol{f}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{f}}\Omega}}\; b_{\boldsymbol{f}}^\dagger\, \exp\{+ifx\} \\
\psi_x^{\mathsf{Ea}} &= \left(\psi_x^{\mathsf{Va}}\right)^\dagger \qquad , \qquad \psi_x^{\mathsf{Vb}} = \left(\psi_x^{\mathsf{Eb}}\right)^\dagger
\end{aligned} \tag{19.32}$$

In the vacuum-expectation-value of the operator-product

$$\begin{aligned}
T\psi_x\psi_y^\dagger = \theta(x^0-y^0)\,(&\psi_x^{\mathsf{Va}}\psi_y^{\mathsf{Ea}} + \psi_x^{\mathsf{Va}}\psi_y^{\mathsf{Vb}} + \psi_x^{\mathsf{Eb}}\psi_y^{\mathsf{Ea}} + \psi_x^{\mathsf{Eb}}\psi_y^{\mathsf{Vb}}) + \\
+\,\theta(y^0-x^0)\,(&\psi_y^{\mathsf{Va}}\psi_x^{\mathsf{Ea}} + \psi_y^{\mathsf{Va}}\psi_x^{\mathsf{Vb}} + \psi_y^{\mathsf{Eb}}\psi_x^{\mathsf{Ea}} + \psi_y^{\mathsf{Eb}}\psi_x^{\mathsf{Vb}})\;,
\end{aligned}$$

only those terms survive, which have creation-operators at their right sides, and annihilation-operators at their left sides:

$$\langle 0|\, T\psi_x\psi_y^\dagger|0\rangle = \theta(x^0-y^0)\,\langle 0|\,\psi_x^{\mathsf{Va}}\psi_y^{\mathsf{Ea}}\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\psi_y^{\mathsf{Va}}\psi_x^{\mathsf{Ea}}\,|0\rangle \tag{19.33a}$$

In the vacuum-expectation-value of the operator-product

$$T\psi_x^\dagger\psi_y = \theta(x^0-y^0)\,(\psi_x^{\mathsf{Ea}}\psi_y^{\mathsf{Va}} + \psi_x^{\mathsf{Ea}}\psi_y^{\mathsf{Eb}} + \psi_x^{\mathsf{Vb}}\psi_y^{\mathsf{Va}} + \psi_x^{\mathsf{Vb}}\psi_y^{\mathsf{Eb}}) + \\ + \theta(y^0-x^0)\,(\psi_y^{\mathsf{Ea}}\psi_x^{\mathsf{Va}} + \psi_y^{\mathsf{Ea}}\psi_x^{\mathsf{Eb}} + \psi_y^{\mathsf{Vb}}\psi_x^{\mathsf{Va}} + \psi_y^{\mathsf{Vb}}\psi_x^{\mathsf{Eb}})$$

again only two terms survive:

$$\langle 0|\, T\psi_x^\dagger\psi_y|0\rangle = \theta(x^0-y^0)\,\langle 0|\,\psi_x^{\mathsf{Vb}}\psi_y^{\mathsf{Eb}}\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\psi_y^{\mathsf{Vb}}\psi_x^{\mathsf{Eb}}\,|0\rangle \tag{19.33b}$$

This result is identical to the vacuum-expectation-value of the Feynman-propagator of the charged Klein-Gordon field:

$$\begin{aligned}\langle 0|\, G(x-y)\,|0\rangle = G(x-y) \;&= \\ &\overset{(15.43)}{=} \theta(x^0-y^0)\langle 0|\psi_x\psi_y^\dagger|0\rangle + \theta(y^0-x^0)\langle 0|\psi_y^\dagger\psi_x|0\rangle \\ &\overset{(19.33)}{=} \theta(x^0-y^0)\,\langle 0|\,\psi_x^{\mathsf{Va}}\psi_y^{\mathsf{Ea}}\,|0\rangle + \theta(y^0-x^0)\langle 0|\,\psi_y^{\mathsf{Vb}}\psi_x^{\mathsf{Eb}}\,|0\rangle \\ &\overset{(12.14)}{=} \theta(x^0-y^0)G^a(x-y) + \theta(y^0-x^0)G^b(y-x)\end{aligned} \tag{19.34}$$

Let

$$\langle 0|\, T\psi_w\psi_x\psi_y\psi_z|0\rangle \quad \text{with } w^0 > x^0 > y^0 > z^0 \tag{19.35}$$

be the vacuum-expectation-value of a time-ordered product of field-operators of an uncharged Klein-Gordon field. Using the notation (19.23), the matrix element is:

$$\begin{aligned}\langle 0|\, &+ \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{V}} + \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{V}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{V}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{V}} + \\ &+ \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{V}} + \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{V}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{V}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{V}} + \\ &+ \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}} + \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}} + \\ &+ \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}} + \psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}} + \psi_w^{\mathsf{E}}\,\psi_x^{\mathsf{E}}\,\psi_y^{\mathsf{E}}\,\psi_z^{\mathsf{E}}\,|0\rangle\end{aligned} \tag{19.36}$$

The 10. term has one non-vanishing contraction, and the 13. term has two non-vanishing contractions. None of the other terms has a non-vanishing contraction. Thus according to theorem (19.28)

$$\begin{aligned}
&\langle 0|\, T\psi_w\psi_x\psi_y\psi_z|0\rangle \;= \\
&\quad = [\,\psi_w^{\mathsf{V}}\,,\psi_x^{\mathsf{E}}]\,[\,\psi_y^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}] + [\,\psi_w^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}]\,[\,\psi_x^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}] + [\,\psi_w^{\mathsf{V}}\,,\psi_z^{\mathsf{E}}]\,[\,\psi_x^{\mathsf{V}}\,,\psi_y^{\mathsf{E}}] \\
&\quad \overset{(19.31)}{=} \langle 0|\,\psi_w^{\mathsf{V}}\,\psi_x^{\mathsf{E}}|0\rangle\langle 0|\,\psi_y^{\mathsf{V}}\,\psi_z^{\mathsf{E}}|0\rangle + \langle 0|\,\psi_w^{\mathsf{V}}\,\psi_y^{\mathsf{E}}|0\rangle\langle 0|\,\psi_x^{\mathsf{V}}\,\psi_z^{\mathsf{E}}|0\rangle + \\
&\qquad + \langle 0|\,\psi_w^{\mathsf{V}}\,\psi_z^{\mathsf{E}}|0\rangle\langle 0|\,\psi_x^{\mathsf{V}}\,\psi_y^{\mathsf{E}}|0\rangle \\
&\quad \overset{(19.34)}{=} G^a(w-x)\cdot G^a(y-z) + G^a(w-y)\cdot G^a(x-z) + \\
&\qquad + G^a(w-z)\cdot G^a(x-y)\;. \qquad (19.37)
\end{aligned}$$

If we had not fixed the time-order $w^0 > x^0 > y^0 > z^0$ in (19.35), we would have had to consider the $4! = 24$ different possible time-orders of w^0, x^0, y^0, z^0. For example, instead of the first term in (19.37) we would need to consider the four possible time-orders

$$\begin{aligned}
&G^a(w-x)\cdot G^a(y-z) \quad \longrightarrow \\
&\quad \longrightarrow \quad \Big(\theta(w^0-x^0)\,G^a(w-x) + \theta(x^0-w^0)\,G^a(x-w)\Big)\cdot \\
&\qquad \cdot\Big(\theta(y^0-z^0)\,G^a(y-z) + \theta(z^0-y^0)\,G^a(z-y)\Big)\;. \qquad (19.38)
\end{aligned}$$

These products of step-functions and propagators G^a just are the Feynman-propagators

$$G(x-y) \overset{(12.14)}{=} \theta(x^0-y^0)\,G^a(x-y) + \theta(y^0-x^0)\,G^a(y-x)$$

of the uncharged Klein-Gordon field. Thus for arbitrary time-order, the result is

$$\begin{aligned}
\langle 0|\, T\psi_w\psi_x\psi_y\psi_z|0\rangle \;&=\; G(w-x)\cdot G(y-z) + \\
&+ G(w-y)\cdot G(x-z) + G(w-z)\cdot G(x-y)\;. \qquad (19.39)
\end{aligned}$$

For the vacuum-expectation-value $\langle 0|\, T\psi_x\psi_y\psi_z|0\rangle$, one gets a formula equivalent to (19.36), in which each term is consisting of three creation- or annihilation-operators. Therefore there is no non-vanishing contraction at all. The same holds for an arbitrary odd number of field-operators. The vacuum-expectation-value of a product of n field-operators of the uncharged

Klein-Gordon field can be different from zero only, if n is an even number.

Due to the results of this section, the following theorem is plausibel. We spare ourselves the rigorous proof:

Theorem: The vacuum-expectation-value of a time-ordered product of n field-operators of the uncharged Klein-Gordon field in the interaction picture is

- zero, if n is odd.
- with n even equal to the sum of the
$$(n-1)\cdot(n-3)\cdot(n-5)\cdot\ldots\cdot 1$$
different products of $(n/2)$ Feynman-propagators, to which the n operators can be combined.

(19.40a)

In case of a charged Klein-Gordon field, each field-operator ψ in the interaction picture contains the Fourier-operators a and $b^\dagger$. The adjoint field-operator $\psi^\dagger$ contains the Fourier-Operators $a^\dagger$ and b. Because of $[a,b^\dagger]=[a,b]=[a^\dagger,b^\dagger]=[a^\dagger,b]=0$, the extension of the theorem to charged fields isn't difficult:

Theorem: The vacuum-expectation-value of a time-ordered product of n field-operators $\psi(x)$ and m field-operators $\psi^\dagger(x)$ of the charged Klein-Gordon field in the interaction picture is

- zero, if $n \neq m$.
- in case $n=m$ equal to the sum of the
$$(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot\ldots\cdot 1$$
different products of the n Feynman-propagators, to which the $n+m=2n$ operators can be combined.

(19.40b)

The method of simplification of matrix elements due to contraction and normal order of the operators, which has been described in this section, is known under the name "Wick's theorem". It has been invented by Houriet and Kind [45] in 1949. When Wick[2] in the following year adopted that method, and extended it to fermion fields, he clearly named the inventors of this method in his article [46], and cited their work correctly. Houriet

[2] Gian Carlo Wick (1909 - 1992)

and Kind even had already used the contraction-brackets which also are used in this book, while Wick needed to mark contractions of operators by dots (probably, because the print shop of Physical Review was overstrained by the horizontal brackets). By today, every field-theorist knows the Wick-theorem with the Wick-brackets, while only few specialists remember the names Houriet and Kind. Why? The answer is quite simple: Houriet and Kind wrote their publication in French, while Wick published in English.

19.3 S-Matrix and LSZ-Formula

In section 15.1, we defined state functions like $|n_{b\boldsymbol{k}} m_{a\boldsymbol{f}} l_{a\boldsymbol{g}}\rangle$ of the Klein-Gordon field with sharply defined wave-numbers. For example, the eigenvalue-equation of the momentum operator

$$\boldsymbol{P}|3_{b\boldsymbol{k}}5_{a\boldsymbol{f}}\rangle \overset{(15.34)}{=} (3\hbar\boldsymbol{k} + 5\hbar\boldsymbol{f})\,|3_{b\boldsymbol{k}}5_{a\boldsymbol{f}}\rangle \tag{19.41}$$

holds for a state, in which 3 antiparticles with wavenumber $\boldsymbol{k}$ and 5 particles with wave number $\boldsymbol{f}$ are excited. As we will concentrate in the sequel onto the ψ^s-interaction of an uncharged Klein-Gordon field, we can slightly reduce the large quantity of indices due to the following definition:

$$|\boldsymbol{f}_1\boldsymbol{f}_2\boldsymbol{f}_2\boldsymbol{f}_2\boldsymbol{f}_3\ldots\rangle \equiv |1_{\boldsymbol{f}_1}3_{\boldsymbol{f}_2}1_{\boldsymbol{f}_3}\ldots\rangle \equiv |1_{a\boldsymbol{f}_1}3_{a\boldsymbol{f}_2}1_{a\boldsymbol{f}_3}\ldots\rangle$$

Furthermore we will sometimes use

$$a^\dagger_{\boldsymbol{k}}|0\rangle = |\boldsymbol{k}\rangle \;\equiv\; a^\dagger_{\boldsymbol{k}_1}\ldots a^\dagger_{\boldsymbol{k}_n}|0\rangle = |\boldsymbol{k}_1\ldots\boldsymbol{k}_n\rangle$$

as a shortcut notation for a state of n particles with (different or identical) wavenumbers $\boldsymbol{k}_j$.

The matrix elements of the momentum operator $\boldsymbol{P}$ are

$$\langle\boldsymbol{k}|\boldsymbol{P}|\boldsymbol{f}\rangle = \underbrace{\langle\boldsymbol{k}|\,U(t,0)}_{\langle t\boldsymbol{f}|}\;\underbrace{U^{-1}(t,0)\,\boldsymbol{P}\,U(t,0)}_{\boldsymbol{P}(t)}\;\underbrace{U^{-1}(t,0)\,|\boldsymbol{f}\rangle}_{|t\boldsymbol{f}\rangle} \overset{(15.37b)}{=} \hbar\boldsymbol{f}\,\delta_{\boldsymbol{k}\boldsymbol{f}}\ . \tag{19.42}$$

The eigenfunction $|t\boldsymbol{f}\rangle$ of the time-dependent momentum operator $\boldsymbol{P}(t)$ in the Heisenberg-picture is interpreted as the state function of a particle, whose momentum at time t is $\boldsymbol{f}$.

The scattering matrix, which also is called S-matrix for brevity, is defined by[3]

$$\begin{aligned} S_{\boldsymbol{f}\boldsymbol{k}} \equiv \langle t_a \boldsymbol{f} | t_e \boldsymbol{k} \rangle &= \langle \boldsymbol{f} | \, U(t_a, 0) U^{-1}(t_e, 0) \, | \boldsymbol{k} \rangle \; = \\ &= \langle \boldsymbol{f} | \, U(t_a, t_e) \, | \boldsymbol{k} \rangle = \langle \boldsymbol{f} | \, S \, | \boldsymbol{k} \rangle \; . \end{aligned} \tag{19.43a}$$

Here

$$\langle \boldsymbol{f} | \, S \, | \boldsymbol{k} \rangle \equiv \langle \boldsymbol{f}_1 \boldsymbol{f}_2 \ldots \boldsymbol{f}_n | \, S \, | \boldsymbol{k}_1 \boldsymbol{k}_2 \ldots \boldsymbol{k}_m \rangle \tag{19.43b}$$

is a shortcut notation for a scattering event with m incoming and n outgoing particles. The indices e and a signify "in" and "out". The times t_e and t_a are to be understood as follows: At time $t \leq t_e$ the incoming particles are approaching with momenta $\boldsymbol{k}_i$ a finite part of space, in which they then mutually interact during a finite time interval $t_e < t < t_a$. Eventually at time $t \geq t_a$ the outgoing particles with momenta $\boldsymbol{f}_j$ depart from the interaction range. Thus

$$\begin{aligned} S \equiv U(t_a, t_e) \; &, \text{ if interactions are possible only in} \\ &\text{the time-intervall } t_e < t < t_a \; . \end{aligned} \tag{19.43c}$$

The momenta can not be sharply defined, because otherwise the incoming and outgoing particles would be delocalized over the complete normalization volume, and would mutually interact at any time. Instead we assume that the particles properly must be described by wave-packets, whose

[3] Many authors prefer to discuss the T-matrix instead of the S-matrix. These matrices are related due to

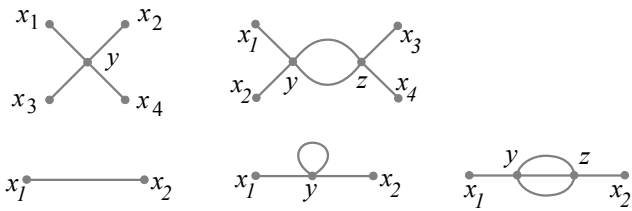
$$\langle \boldsymbol{f} | \, S \, | \boldsymbol{k} \rangle = \langle \boldsymbol{f} | \, 1 - iT \, | \boldsymbol{k} \rangle = \langle \boldsymbol{f} | \boldsymbol{k} \rangle - \langle \boldsymbol{f} | \, iT \, | \boldsymbol{k} \rangle \; .$$

Thus these both matrices essentially are differing by the trivial term $\langle \boldsymbol{f} | \boldsymbol{k} \rangle$, which is different from zero if particles cross the interaction range without any scattering. If you encounter that operator T in the literature, take care not to confuse it with the time-order operator.

momenta are concentrated "rather narrow" around $\hbar\boldsymbol{k}$ resp. $\hbar\boldsymbol{f}$, and whose positions according to Heisenberg's indeterminacy relation are delocalized over "rather large" ranges $\Delta\boldsymbol{x} \approx 1/\Delta\boldsymbol{k}$ resp. $\Delta\boldsymbol{y} \approx 1/\Delta\boldsymbol{f}$. Thus they are described by "almost" plane waves with "almost" sharply defined momenta, but only "almost". The wave packets have finite extensions, and only in the time interval from t_e to t_a their amplitudes in the interaction range are differing perceptibly from zero. Their amplitudes at the boundaries of the normalization volume Ω are zero as long as they are different from zero within the interaction range, i. e. Ω is much larger than the interaction range. To save paperwork, we keep this definition in mind, but we describe the particles in the formulas for simplicity as plane waves with sharply defined momenta.

According to (19.10), the field-operators $\psi_{(W)}$ and ψ are[4] mutually related by the fact, that $\psi_{(W)}(t_0, \boldsymbol{x}) = \psi(t_0, \boldsymbol{x})$ at (minimum) one point of time, which may be chosen arbitrarily. To proceed with the computations, we need to find the field-operator $\psi_{(W)}(t_0, \boldsymbol{x})$ for that point of time, either by solving the field equation (19.4) or by guessing. At first sight, that task seems to be very simple (and thats of course the reason, why we — like almost all textbooks on QFT — are choosing scattering events as our standard examples): At times $t \le t_e$, and at times $t \ge t_a$, the interaction term in (19.4) is zero, and thus we expect $\psi_{(W)}(t, \boldsymbol{x})$ to be equal to the well-known operator of the free field during the ranges of time $t \le t_e$ resp. $t \ge t_a$, in which the particles are not yet resp. no more mutually interacting.

But at closer evaluation, a serious problem pops up. In the following sections, we will describe matrix elements by Feynman-graphs, and we will encounter graphs like these:



(19.44)

[4] Remember the re-naming (19.21) of indices!

A pictorial interpretation of the first graph is, that two particles are approaching the interaction point y from the space-time points x_1 and x_2. After the interaction, two particles depart towards x_3 and x_4. In the second graph, there is a further interaction at space-time point z. In both cases one may assume, that no interaction between the particles will happen at times $t_e \ll y^0, z^0$ and $t_a \gg y^0, z^0$. Thats quite different for the both last graphs. They are describing the rather abstract event, that a particle is scattered "by itself" on it's way from x_1 to x_2. The comparison with experiments is proving, that such self-interactions are playing an important role, and certainly must not be neglected in the theory.

As self-interactions of the particles cannot be restricted to a time interval $t_e < t < t_a$, but may happen at any time, at first sight there seems to be no point of time at all, at which the field operator $\psi_{(W)}(x)$ can easily be guessed. To explain the solution of this problem, we need to anticipate some results of section 20.3. There we will compute the self-interaction graphs and find out, that they are diverging. The handling of those divergences is the theme of chapter 22. We will see, that the self-interaction graphs mutate to the harmless propagator-line of the third graph in (19.44), once firstly the mass parameter m of the field, and secondly the amplitude ψ of the field-operator are re-defined by an appropriate method, which is called renormalization.

The self-interactions do not disappear by that procedure from the theory, but due to renormalization of the two parameters they are integrated into the formalism such, that they are quasi automatically considered at any time, and don't need to be explicitly documented and computed in self-interaction-graphs any more. Therefore $\psi_{(W)}(t, \boldsymbol{x})$ is in the time ranges $t \leq t_e$ and $t \geq t_a$ formally identical to the known solutions of the free Klein-Gordon-equation (19.1), once the renormalized mass m and the renormalized field-operator $\psi(x)$ is inserted.

In section 20.3 we will furthermore detect, that also the second graph in (19.44) is diverging. This divergence will be cured in chapter 22 due to renormalization of the coupling constant λ. Anticipating these results, we will henceforth use the renormalized factors m, $\psi(x)$, and λ in all computations, unless the contrary is explicitly stated.

We now want to formulate $S_{\boldsymbol{fk}}$ as a function of the field-operators $\psi_{(W)}(x)$.

In appendix A.26, the following result is found by a rather lengthly — but still not mathematically rigorous — derivation:

$$\begin{aligned} S_{\boldsymbol{f}_1\dots\boldsymbol{f}_n\boldsymbol{k}_1\dots\boldsymbol{k}_m} &= \langle \boldsymbol{f}_1\dots\boldsymbol{f}_n|\, S\,|\boldsymbol{k}_1\dots\boldsymbol{k}_m\rangle \overset{(A.202)}{=} \\ &= \prod_{j=1}^{n} \frac{\widetilde{G}^{-1}(f_j)}{\sqrt{2\hbar\omega_{\boldsymbol{f}_j}\Omega}} \int_{t_e}^{t_a}\int_{\Omega} \mathrm{d}^4 y_j\, \exp\{+if_jy_j\}\cdot \\ &\quad\cdot \prod_{l=1}^{m} \frac{\widetilde{G}^{-1}(k_l)}{\sqrt{2\hbar\omega_{\boldsymbol{k}_l}\Omega}} \int_{t_e}^{t_a}\int_{\Omega} \mathrm{d}^4 x_l\, \exp\{-ik_lx_l\}\cdot \\ &\quad\cdot \langle 0|\, T\psi_{(W)}(y_1)\dots\psi_{(W)}(y_n)\,\psi_{(W)}(x_1)\dots\psi_{(W)}(x_m)\,|0\rangle \end{aligned} \tag{19.45}$$

Note, that the integrals over x and y are including the matrix element in the last line.

$$\widetilde{G}^{-1}(k) \overset{(12.7)}{=} -i\hbar c\Big(k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'\Big) \tag{19.46}$$

is the inverse of the Fourier-transformed Greens function. We check the dimensions: Any incoming or outgoing particle contributes to (19.45) the dimension

$$\begin{aligned} &\left[\frac{\widetilde{G}^{-1}}{\sqrt{\hbar\omega_{\boldsymbol{k}}\Omega}}\cdot \mathrm{d}^4x\,\psi_{(W)}(x)\right] = \\ &= \frac{\text{energy}}{\text{length}\,\sqrt{\text{energy}\cdot\text{volume}}}\cdot\frac{\text{length}^4}{\sqrt{\text{energy}\cdot\text{volume}}} = 1\ . \end{aligned} \tag{19.47}$$

Thus the probability amplitude is dimension-less. That's correct.

It will turn out quite useful, and simplify many formulas in the following sections, if t_e and t_a are formally shifted to infinity:

$$t_e\to-\infty\ ,\ t_a\to+\infty \quad\Longrightarrow\quad \int_{t_e}^{t_a}\mathrm{d}t \approx \int_{-\infty}^{+\infty}\mathrm{d}t \tag{19.48}$$

Again this stipulation clearly is not to be understood literally. We just specified that the amplitudes of the wave-packets shall be zero at the

boundaries of the normalization volume, as long as interactions between the particles are possible. With $t \to \pm\infty$ that wouldn't be the case even for an infinitely large normalization volume. Instead we assume, that t_e and t_a are — and stay — finite, but that they are chosen so large as compared to all other points of time turning up in the formulas, that (19.48) is a good approximation.

Inserting furthermore (19.19) into (19.45), we arrive at the LSZ-reduction-formula[5], which is describing the relation between the S-matrix and the vacuum-expectation-value of the field-amplitudes in the interaction picture:

$$
\begin{aligned}
S_{\boldsymbol{f}_1\dots\boldsymbol{f}_n\boldsymbol{k}_1\dots\boldsymbol{k}_m} &= \langle \boldsymbol{f}_1 \dots \boldsymbol{f}_n |\, S \,| \boldsymbol{k}_1 \dots \boldsymbol{k}_m \rangle \;= \\
&= \prod_{j=1}^{n} \frac{\widetilde{G}^{-1}(f_j)}{\sqrt{2\hbar\omega_{\boldsymbol{f}_j}\Omega}} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 y_j \, \exp\{+i f_j y_j\} \cdot \\
&\quad \cdot \prod_{l=1}^{m} \frac{\widetilde{G}^{-1}(k_l)}{\sqrt{2\hbar\omega_{\boldsymbol{k}_l}\Omega}} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 x_l \, \exp\{-i k_l x_l\} \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \\
&\quad \cdot \frac{\langle 0 |\, T\psi(y_1)\dots\psi(y_n)\,\psi(x_1)\dots\psi(x_m)\Big(-\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^j |0\rangle}{\langle 0 |\, T \sum\limits_{r=0}^{\infty} \frac{1}{r!}\Big(-\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^r |0\rangle}
\end{aligned}
\tag{19.49}
$$

Again we emphasize that the integrals over x_l and y_j are extending over the matrix element in the last line's numerator. $\widetilde{G}^{-1}(k) =$ (19.46) is zero on mass shell. How is it possible then, that there at all are S-matrix elements, which are different from zero? We will see, that Fourier-transformed propagators like (12.7) are showing up in the matrix elements $\langle 0 |\, T\psi(y_1)\dots|0\rangle$.[6] There are non-vanishing contributions to the S-matrix if and only if the poles of these propagators compensate the zero-factors of the LSZ-reduction-formula.

[5] named after it's discoverers Harry Lehmann (1924 – 1998), Kurt Symanzik (1923 – 1983), and Wolfhart Zimmermann (1928 – 2016)

[6] Actually we already detected products of Feynman-propagators in the matrix elements of field-operators in the interaction picture, see e. g. (19.39).

The delicate balance of poles and zero-factors is of course the reason, why a mathematically rigorous derivation of the LSZ-formula is so difficult and well beyond the technical level of this textbook.

While the LSZ-formula may seem to be quite complicated and intimidating, it will turn out extremely useful and easy to handle in the sequel.

20 Feynman-Graphs of ψ^s-Theory

Feynman invented a graphical representation for the matrix elements of quantum field theory, which proved most useful for various types of computations. In the first section of this chapter, the method will be introduced by means of simple tree diagrams. In the following section we will delve into the difficult subject of loop-graphs. We will see, that divergences turn up in the computation of such graphs. In chapter 22 we will clarify, how the divergences can be cured due to renormalization.

20.1 Tree-Graphs

Let's consider a scattering event with two particles coming in with wavenumbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, and two particles going out with wavenumbers $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$. We will compute the orders $S^{(0)}, S^{(1)}, S^{(2)}, \ldots$ of the scattering amplitude

$$\sum_{n=0}^{\infty} S^{(n)} \equiv S_{\boldsymbol{k}_4\boldsymbol{k}_3\boldsymbol{k}_2\boldsymbol{k}_1} = \langle \boldsymbol{k}_3\boldsymbol{k}_4 | \, S \, | \boldsymbol{k}_1\boldsymbol{k}_2 \rangle \tag{20.1}$$

step by step. For that purpose, the interaction-term

$$-\frac{i}{\hbar} \int_{-\infty}^{+\infty} \mathrm{d}\tau \, H(\tau) \overset{(19.6)}{=} (-i\lambda\hbar^2 c^2) \int_{-\infty}^{+\infty} c\mathrm{d}\tau \int_{\Omega} \mathrm{d}^3 y \, \psi^s(\tau, \boldsymbol{y}) \tag{20.2}$$

$$\text{with } s = 3 \text{ or } s = 4$$

is inserted into the LSZ-reduction-formula:

$$S^{(n)} \overset{(19.49)}{=} \frac{1}{M} \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 x_j \cdot$$
$$\cdot \exp\{-i(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4\} \frac{(-i\lambda\hbar^2 c^2)^n}{n!} \cdot$$
$$\cdot \langle 0|\, T\psi(x_1)\, \psi(x_2)\, \psi(x_3)\, \psi(x_4) \Big(\int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 y\, \psi^s(y) \Big)^n |0\rangle \qquad (20.3)$$

Here

$$M \equiv \langle 0|\, T \sum_{r=0}^{\infty} \frac{1}{r!} \Big((-i\lambda\hbar^2 c^2) \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 w\, \psi^s(w) \Big)^r |0\rangle \qquad (20.4)$$

is the denominator of the LSZ-formula (19.49), and

$$N_j \equiv \sqrt{2\hbar\omega_{\boldsymbol{k}_j}\Omega} \qquad (20.5)$$

is an abbreviated notation for the normalization factor. The series expansion of the S-matrix will converge fast, if the value of the coupling constant is small, i. e. if

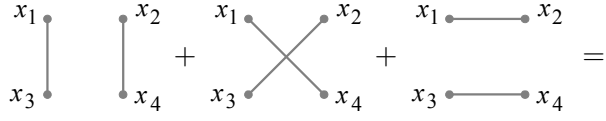
$$\frac{|\lambda\hbar^2 c^2|^{(n+1)}}{(n+1)!} \ll \frac{|\lambda\hbar^2 c^2|^n}{n!} \ .$$

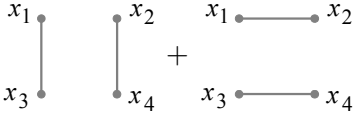
Zeroth order: $n = 0$. It's clear from the outset, that $S^{(0)} = 0$, because at $n = 0$ all factors in (20.3), which are somehow related to an interaction, are equal to one. Thus the particles will cross the interaction range without sensing one-another. Anyway we will compute $S^{(0)}$, as this is an instructive introduction to the applications of the LSZ-formula. $M = (20.4)$ is different from zero. The matrix element in the last line of (20.3), with the interaction-term being 1, is already known to us:

$$\langle 0|\, T\psi(x_1)\, \psi(x_2)\, \psi(x_3)\, \psi(x_4)\, |0\rangle \overset{(19.39)}{=} G(x_1 - x_2) \cdot G(x_3 - x_4) +$$
$$+ \ G(x_1 - x_3) \cdot G(x_2 - x_4) + G(x_1 - x_4) \cdot G(x_2 - x_3) \qquad (20.6)$$

We draw a dot for each of the four space-time-points x_1, x_2, x_3, x_4, at which the four field-operators are defined. For each propagator, which is connecting these points, we draw a line. Thereby the numerator of the scattering matrix can be symbolized in zeroth order as follows:

$$\begin{aligned}\widetilde{S}^{(0)} \equiv M \cdot S^{(0)} &\overset{(20.3)}{=} \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 x_j \cdot \\ &\cdot \exp\{-i(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4)\} \cdot \\ &\cdot \langle 0| T\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4) |0\rangle \mathrel{\widehat{=}}\end{aligned}$$

$\mathrel{\widehat{=}}$ 

$$= 2 \cdot$$ 

$$\underbrace{\qquad}_{\widetilde{S}_a^{(0)}} \qquad \underbrace{\qquad}_{\widetilde{S}_b^{(0)}} \tag{20.7}$$

As two of the three terms obviously give the same numerical result, only one of them is drawn and multiplied by the symmetry factor (in this case 2). We appended the graphs by the "corresponds to" sign $\mathrel{\widehat{=}}$, but not by the "equals" sign $=$, because the same graphs are used if this matrix-element $\langle 0| T \ldots |0\rangle$ is part of an other equation than the LSZ-formula (20.7). The same, unchanged graphs might sometimes even represent the absolute square of that matrix-element.

The last graph is representing the propagator product $G(x_1 - x_2) \cdot G(x_3 - x_4)$, showing up in (20.6) as one of three possible contractions of the matrix element. This graph does not fit to the pictorial interpretation of the graphs, with two particles $\psi(x_1)$ and $\psi(x_2)$ coming in, and two particles $\psi(x_3)$ and $\psi(x_4)$ going out. Still this is a valid contraction of the matrix element. Thus we must not skip that graph, unless we can prove that it is zero.

Furthermore the notation

$$\widetilde{S}^{(n)} \equiv M \cdot S^{(n)} \tag{20.8}$$

has been defined for the numerator of the LSZ-formula (20.3). We will investigate the denominator M in (20.20)ff. Firstly we compute

$$\widetilde{S}_a^{(0)} = 2 \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 x_j \, G(x_3 - x_1)\, G(x_4 - x_2) \cdot \\ \cdot \exp\{-i(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4)\} \; . \tag{20.9}$$

$\widetilde{S}_a^{(0)} \equiv \widetilde{S}_{a31}^{(0)} \cdot \widetilde{S}_{a42}^{(0)}$ can obviously be split into the factor

$$\frac{\widetilde{S}_{a31}^{(0)}}{\sqrt{2}} = \prod_{j=3,1} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 x_j \, G(x_3 - x_1)\, e^{-i(k_1 x_1 - k_3 x_3)} \; ,$$

and a factor $\widetilde{S}_{a42}^{(0)}$, which differs from $\widetilde{S}_{a31}^{(0)}$ only by the replacement of the indices 3 by 4 and 1 by 2. Using the Fourier-transformation

$$G(x_3 - x_1) \overset{(7.15)}{=} \frac{1}{\Omega} \sum_{\boldsymbol{f}} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}f^0}{2\pi} \, \widetilde{G}(f) \exp\{-if(x_3 - x_1)\} \; , \tag{20.10}$$

one gets

$$\widetilde{S}_{a31}^{(0)} = \frac{\sqrt{2}}{\Omega} \sum_{\boldsymbol{f}} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}f^0}{2\pi} \prod_{j=3,1} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 x_j \cdot \\ \cdot \exp\{-i(f - k_3)x_3 - i(k_1 - f)x_1)\} \, \widetilde{G}(f) \; . \tag{20.11}$$

Inserting

$$\int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 x_1 \, \exp\{-i(k_1 - f)x_1)\} \overset{(7.16b)}{=} 2\pi\Omega \, \delta(f^0 - k_1^0) \, \delta_{\boldsymbol{f}\boldsymbol{k}_1} \; , \tag{20.12}$$

the sum over $\boldsymbol{f}$ and the integral over f^0 can easily be computed. The result

is

$$\widetilde{S}^{(0)}_{a31} = \frac{\sqrt{2}}{\Omega 2\pi}\, 2\pi\Omega \prod_{j=3,1} \frac{\widetilde{G}^{-1}(k_j)}{N_j}\, \widetilde{G}(k_1) \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 x_3\, \exp\{-i(k_1 - k_3)x_3\}\,. \tag{20.13}$$

The integral over x_3 is the delta function and Kronecker symbol (20.12), which guarantees the conservation of energy and momentum of the particle moving in at x_1 and moving out at x_3. The Fourier-transformed Greens-function can be canceled:

$$\begin{aligned} \widetilde{S}^{(0)}_{a31} &= \sqrt{2}\, \frac{\widetilde{G}^{-1}(k_3)}{N_1 N_3}\, 2\pi\Omega\, \delta(k_3^0 - k_1^0)\, \delta_{\boldsymbol{k}_3 \boldsymbol{k}_1} \\ \widetilde{S}^{(0)}_{a} &= 2\, \frac{\widetilde{G}^{-1}(k_3)\, \widetilde{G}^{-1}(k_4)}{N_1 N_2 N_3 N_4}\, (2\pi\Omega)^2\, \delta(k_3^0 - k_1^0)\, \delta_{\boldsymbol{k}_3 \boldsymbol{k}_1}\, \delta(k_4^0 - k_2^0)\, \delta_{\boldsymbol{k}_4 \boldsymbol{k}_2} = 0 \end{aligned} \tag{20.14}$$

Only two of the four zero-factors $\widetilde{G}^{-1}$ could be canceled, because there are only two propagators in each of the graphs (20.7). Therefore $\widetilde{S}^{(0)}_a = 0$. Four propagators would be required, to compensate the four zero-factors $\widetilde{G}^{-1}$ in the numerator. And clearly $\widetilde{S}^{(0)}_b = 0$ as well, for the same reason. This result is no surprise. In zeroth order, with the interaction factor in (20.3) being $\sim \left(\lambda\psi^s(y)\right)^0$, the both incoming particles don't sense each other, and consequently there is no scattering.

There is a further, generally valid reason, why diagrams like $\widetilde{S}^{(0)}_b$, which is displayed in (20.7), always are zero: When we compute them, we will find a result with the constraint $\delta(k_1^0 - k_2^0)\, \delta_{\boldsymbol{k}_1 \boldsymbol{k}_2}$, and consequently we must have $\boldsymbol{k}_2 = \boldsymbol{k}_1$. But with zero relative momentum of the two incoming particles, there can be no scattering event. Thus we may safely discard in future all diagrams, in which the field operators $\psi(x_1)$ and $\psi(x_2)$ of the two incoming particles are contracted to a propagator $G(x_2 - x_1)$.

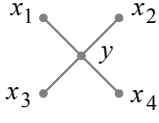
Computation of $\widetilde{S}^{(1)}$: We have seen already in zeroth order, that minimum four propagators are needed to compensate the zero-factors in the numerator of the LSZ-formula. Eight field-operators are required to build the four

propagators. The both incoming and the both outgoing particles in total deliver four field-operators. The missing additional four field-operators must be supplied by the interaction term. Therefore it would be wasting of time to evaluate the interaction (20.2) in first order with $s = 3$, which delivers only three operators. Instead we investigate the scattering in first order of ψ^4-theory, i. e. we insert the interaction-term

$$\mathcal{O}(\lambda^1) \quad \Longrightarrow \quad T\frac{1}{1!}\Big((-i\lambda\hbar^2c^2)\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4y\,\psi^4(y)\Big)^1 \tag{20.15}$$

into the LSZ-formula:

$$\begin{aligned}
\widetilde{S}^{(1)} = M\cdot S^{(1)} \overset{(20.3)}{=} & \prod_{j=1}^{4}\frac{\widetilde{G}^{-1}(k_j)}{N_j}\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega}\mathrm{d}^4x_j \\
& \cdot \exp\{-i(k_1x_1+k_2x_2-k_3x_3-k_4x_4)\}\,(-i\lambda\hbar^2c^2)\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega}\mathrm{d}^4y \\
& \cdot \langle 0|\,T\psi(x_1)\,\psi(x_2)\,\psi(x_3)\,\psi(x_4)\,\psi(y)\,\psi(y)\,\psi(y)\,\psi(y)\,|0\rangle \\
= & \prod_{j=1}^{4}\frac{\widetilde{G}^{-1}(k_j)}{N_j}\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega}\mathrm{d}^4x_j\,(-i\lambda\hbar^2c^2)\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega}\mathrm{d}^4y \\
& \cdot \exp\{-i(k_1x_1+k_2x_2-k_3x_3-k_4x_4)\} \\
& \cdot 4!\,\Big(G(x_1-y)\,G(x_2-y)\,G(x_3-y)\,G(x_4-y)\Big) \mathrel{\widehat{=}} \\
\mathrel{\widehat{=}}\ 24\ \cdot & \quad
\end{aligned}$$



(20.16)

The vacuum-expectation-value of the time-ordered operator product equals according to theorem (19.40a) the sum of the 105 products of 4 Feynman-propagators each, to which the eight field-operators can be contracted. If two of the operators $\psi(y)$ are combined to the propagator $G(y-y) = 1$, then they don't contribute to the compensation of the four zero-factors $\widetilde{G}^{-1}(k_j)$. There can be a non-vanishing contribution to the S-matrix only, if each of the operators $\psi(x_1)\ldots\psi(x_4)$ is combined to a propagator with

one of the operators $\psi(y)$. Only terms of that type have been indicated in (20.16), all 81 other terms have been skipped from the outset. There are 4 alternatives to combine the operator $\psi(x_1)$ with one of the operators $\psi(y)$, 3 alternatives to combine $\psi(x_2)$ with one of the remaining operators $\psi(y)$, and so on. In total, there are 4! different combinations possible. Thus $\widetilde{S}^{(1)}$ is consisting of 4! terms. As the value of each of the terms obviously is identical, only one graph will be computed, and multiplied by the symmetry factor $4! = 24$.

Remark: We will encounter the symmetry-factor $4! = 24$ quite often in ψ^4-theory, same as the symmetry-factor $3! = 9$ in ψ^3-theory. To save paperwork, many authors therefore define the coupling constant $\lambda/s!$ instead of our definition λ. The present author considers the explicit visibility of the symmetry-factors a didactic advantage, which very well does justify the marginal additional writing effort.

In the Feynman-diagram (20.16), several propagator-lines meet at space-time-point y. Such an inner point is called vertex (Latin: pivot point, plural: vertices). This notion is not used for the outer points, which are final points of only one propagator each. Insertion of the Fourier-transformation (20.10) and the definition

$$y_j \equiv x_j - y \qquad \Longrightarrow \qquad \begin{aligned} \mathrm{d}^4 x_j &= \mathrm{d}^4 y_j \\ x_j &= y_j + y \end{aligned} \tag{20.17}$$

results into

$$\begin{aligned} \widetilde{S}^{(1)} = 24 \cdot (-i\lambda\hbar^2 c^2) \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 y \, \exp\{-i(k_1 + k_2 - k_3 - k_4)y\} \cdot \\ \cdot \prod_{j=1}^{2} \frac{\widetilde{G}^{-1}(k_j)}{N_j\,\Omega} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 y_j \sum_{f_j} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}f_j^0}{2\pi} \, \widetilde{G}(f_j) \exp\{-i(f_j + k_j)y_j\} \cdot \\ \cdot \prod_{l=3}^{4} \frac{\widetilde{G}^{-1}(k_l)}{N_l\,\Omega} \int\limits_{-\infty}^{+\infty} \int\limits_{\Omega} \mathrm{d}^4 y_l \sum_{f_l} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}f_l^0}{2\pi} \, \widetilde{G}(f_l) \exp\{-i(f_l - k_l)y_l\} = \end{aligned}$$

$$= 24 \cdot (-i\lambda\hbar^2c^2) \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4y \, \exp\{-i(k_1+k_2-k_3-k_4)y\} \cdot$$
$$\cdot \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \, \widetilde{G}(-k_1) \, \widetilde{G}(-k_2) \, \widetilde{G}(k_3) \, \widetilde{G}(k_4) \; . \tag{20.18}$$

Here (20.12) has been used. Because of $\widetilde{G}(k) \overset{(12.7)}{=} \widetilde{G}(-k)$, the four zero-factors can be canceled. The integral over y again gives a delta function and a Kronecker symbol, which guarantee the conservation of energy and momentum:

$$\widetilde{S}^{(1)} = 24 \cdot (-i\lambda\hbar^2c^2) \prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}} \cdot$$
$$\cdot \, 2\pi\Omega \, \delta(k_1^0 + k_2^0 - k_3^0 - k_4^0) \, \delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)} \; . \tag{20.19}$$

Now we need to investigate the denominator

$$M \overset{(20.4)}{=} \langle 0| \, T \sum_{r=0}^{\infty} \frac{1}{r!} \Big(- i\lambda\hbar^2c^2 \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4y \, \psi^4(y) \Big)^r \, |0\rangle \tag{20.20}$$

of the LSZ-reduction-formula. The zeroth-order term is

$$M^{(0)} = \langle 0|0\rangle = 1 \; . \tag{20.21}$$

The first-order term

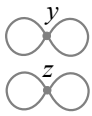
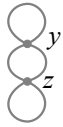
$$M^{(1)} = -i\lambda\hbar^2c^2 \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4y \, \langle 0| \, \psi^4(y) \, |0\rangle \mathrel{\widehat{=}} 3 \cdot$$

(20.22)

can be represented by a Feynman-graph, in which two propagators begin and end at the same vertex y. There are three alternatives for the pair-wise combination of the four field-operators $\psi(y)$ to this graph. Therefore the symmetry factor is three.

The matrix element of second order

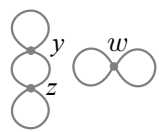
$$M^{(2)} = \frac{(-i\lambda\hbar^2c^2)^2}{2!} \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4y \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4z \,\langle 0|\, T\,\psi^4(y)\,\psi^4(z)\,|0\rangle \;\widehat{=}$$

$$\widehat{=}\; 9 \cdot$$  $$+\; 72 \cdot$$  $$+\; 24 \cdot$$  (20.23)

is represented by the sum of three graphs. According to theorem (19.40a), there are in total $7 \cdot 5 \cdot 3 = 105$ alternatives for the combination of the 8 field-operators ψ to 4 propagators. The upper part of the first graph can — exactly like (20.22) — be realized by 3 different ways, and the lower part alike. Thus there are in total 9 alternatives for the combination of the $4+4$ field-operators to the first graph. To combine four $\psi(y)$ and four $\psi(z)$ in the second graph to the middle loop, there are $4 \cdot 4 \cdot 3 \cdot 3/2! = 72$ alternatives. Afterwards only one way remains for the construction of the upper and the lower loop. To combine the four $\psi(y)$ and the four $\psi(z)$ in the third graph to the four propagators $G(y-z)$, there are $4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2/4! = 24$ alternatives. Note, that the exchange of vertices $y \leftrightarrow z$ would merely be a rotation of the graphs by 180 degrees, but would not change their topology, and therefore would not represent new variants. For the topology of a graph it only matters, which of the eight operators $\psi(y)$ resp. $\psi(z)$ is combined to a propagator with which of the seven other operators $\psi(y)$ resp. $\psi(z)$, but not whether a vacuum bubble is drawn in a graph's upper or lower part.

The third order matrix element is

$$M^{(3)} = \frac{(-i\lambda\hbar^2c^2)^3}{3!} \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4y \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4z \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4w$$
$$\cdot\, \langle 0|\, T\,\psi^4(y)\,\psi^4(z)\,\psi^4(w)\,|0\rangle \;\widehat{=}$$

$$\widehat{=}\; \underbrace{3 \cdot 72 \cdot 3}_{648} \cdot$$  $$+ \ldots\ .$$ (20.24)

We draw and consider only one of the many graphs of this matrix element.

The symmetry factor of the left structure is 72, see (20.23). The symmetry factor of the right structure is 3, see (20.22). There is an additional factor 3, because it matters whether the vertex y, z, or w is built into the right structure. Thus the total symmetry factor is $72 \cdot 3 \cdot 3 = 648$.

We will not compute explicitly the orders $M^{(n)}$ indicated so far, nor the infinitely many terms $M^{(n)}$ of higher order. Instead there is an elegant method, to cancel M versus an identical sum in the numerator of the LSZ-formula. What we want to compute eventually is the scattering amplitude

$$S = \sum_{n=0}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{\widetilde{S}^{(n)}}{M} = \frac{\sum_{n=0}^{\infty} \widetilde{S}^{(n)}}{\sum_{j=0}^{\infty} M^{(j)}} \ . \tag{20.25}$$

Let's compile systematically the graphs and symmetry-factors of $M^{(n)}$ and of $\widetilde{S}^{(n)}$ order by order.

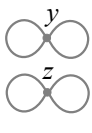
$$M^{(0)} \widehat{=} m^{(0)} \cdot \ 1 \tag{20.26a}$$

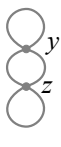
Of course we know from (20.21) the symmetry-factor $m^{(0)} = 1$. And 1 is not really a Feynman-graph, but simply the numerical value of the matrix element $\langle 0|0\rangle$. But we want to set up a systematic description with no need to compute all symmetry-factors explicitly, and with one or several graphs (or at least the 1 replacing a graph) showing up in every entry. In first order we have

$$M^{(1)} \widehat{=} m^{(1)} \cdot$$ 

$$\ . \tag{20.26b}$$

The second-order terms are

$$M^{(2)} \widehat{=} m_a^{(2)} \cdot$$ 

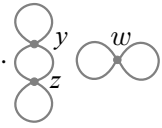
$$+ \ m_b^{(2)} \cdot$$ 

$$+ \ m_c^{(2)} \cdot$$ 

$$\ . \tag{20.26c}$$

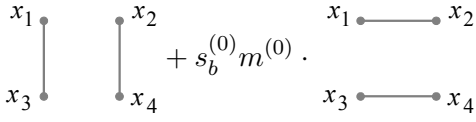
The third-order terms are

$$M^{(3)} \widehat{=} \underbrace{3 \cdot m^{(1)} m_a^{(2)}}_{m^{(3)}} \cdot \quad + \ldots \tag{20.26d}$$

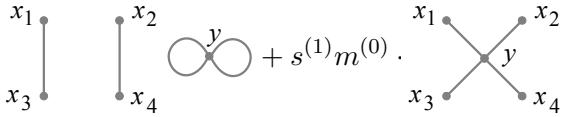


and many other graphs. We don't list the higher order terms of $M^{(n)}$. Instead we compile the first terms of $\widetilde{S}^{(n)}$:

$$\widetilde{S}^{(0)} \widehat{=} s_a^{(0)} m^{(0)} \cdot \quad + s_b^{(0)} m^{(0)} \cdot \tag{20.27a}$$



$$\widetilde{S}^{(1)} \widehat{=} s^{(0)} m^{(1)} \cdot \quad + s^{(1)} m^{(0)} \cdot \tag{20.27b}$$



In (20.16) we had skipped the first diagram of $\widetilde{S}^{(1)}$, because we knew upfront that the four zero-factors of the LSZ-formula would not be compensated by the propagators

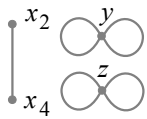
$$s^{(0)} m^{(1)} \cdot G(x_2 - x_1)\, G(x_4 - x_3)\, \underbrace{G(y-y)\, G(y-y)}_{1}$$

of that graph, and therefore that graph would not contribute to the scattering amplitude. But for the completeness of our systematic compilation of graphs, this term is necessary. The two diagrams (20.27b) exhaust[1] the alternatives of graphs, which can be built by contracting the four field-operators $\psi(x_j)$ (related to four incoming or outgoing particles) and the four field-operators $\psi(y)$ (related to a vertex of ψ^4-theory) to propagators.

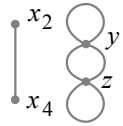
The graphs of second order are

[1] Remember that we are working with renormalized mass-parameter m, and with renormalized field-operators ψ. Therefore graphs similar to the fourth one displayed in (19.44) are suppressed. See the discussion below (19.44).

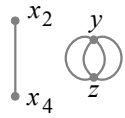
$$\widetilde{S}^{(2)} \widehat{=} s^{(0)} \Big(m_a^{(2)} \cdot$$



$+ m_b^{(2)} \cdot$

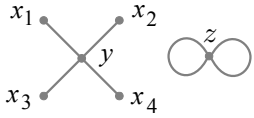


$+ m_c^{(2)} \cdot$



$\Big)$

$+ 2 \cdot s^{(1)} m^{(1)} \cdot$



$$+ s_a^{(2)} m^{(0)} \cdot \ldots + s_b^{(2)} m^{(0)} \cdot \ldots + \ldots \tag{20.27c}$$

Note the additional symmetry factor 2 in the second-last line, caused by exchange of the vertices y and z! We will discuss that important feature immediately.

The graphs with symmetry-factors $s_j^{(2)}$ are not yet known to us, nor the terms $\widetilde{S}^{(n)}$ with $n > 2$. But our overview is sufficient to detect some regularity. For an appropriate description of that regularity, we need the

Definition: A graph is called *connected*, if all of it's structures are connected directly or indirectly to minimum one incoming or outgoing particle. Otherwise the graph is called *unconnected*. (20.28)

For example, the graphs (20.27a) and the second graph of (20.27b) are connected, while the first graph of (20.27b) is unconnected (as the structure ○○ is not connected to any incoming or outgoing particle). All graphs explicitly displayed in (20.27c) are unconnected. Making use of (20.28), we define

$$\overline{S}^{(n)} \equiv \widetilde{S}^{(n)} - (\text{unconnected graphs}) \tag{20.29}$$

as the subset of connected graphs of $\widetilde{S}^{(n)}$.

Now we need to discuss the additional symmetry factor 2 in the second-last line of (20.27c). The general rule is: If a (connected or unconnected)

graph of order j and a (connected or unconnected) graph of order $(n-j)$ are combined to an unconnected graph of order n, then we get — due to permutation of vertices — an additional symmetry factor

$$\binom{n}{j} = \binom{n}{n-j} = \frac{n!}{j!\,(n-j)!}\ . \tag{20.30}$$

We already encountered another example for this in (20.24), with the additional symmetry factor $\binom{3}{2} = 3$. The additional symmetry factors are very important, because in the sequel we want to construct the scattering amplitude of order n in the form

$$\widetilde{S}^{(n)} \;=\; \overline{S}^{(0)} M^{(n)} \;+\; \overline{S}^{(1)} M^{(n-1)} \;+\; \ldots + \overline{S}^{(n)} M^{(0)}\ .$$

Now remember that $\widetilde{S}^{(n)}$ is coming with a factor $1/n!$ in the LSZ-formula (20.3). The additional symmetry factor (20.30) cancels that $1/n!$ and replaces it by the product of the corresponding factors $1/j!$ of $\overline{S}^{(j)}$ and $1/(n-j)!$ of $M^{(n-j)}$.

Thus due to the additional symmetry factors (20.30) we can indeed display (20.27) as

$$\begin{aligned}
\widetilde{S}^{(0)} &= \overline{S}^{(0)} M^{(0)} \\
\widetilde{S}^{(1)} &= \overline{S}^{(0)} M^{(1)} \quad + \overline{S}^{(1)} M^{(0)} \\
\widetilde{S}^{(2)} &= \overline{S}^{(0)} M^{(2)} \quad + \overline{S}^{(1)} M^{(1)} \quad + \overline{S}^{(2)} M^{(0)} \\
&\ \ \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\widetilde{S}^{(n)} &= \overline{S}^{(0)} M^{(n)} \quad + \overline{S}^{(1)} M^{(n-1)} + \overline{S}^{(2)} M^{(n-2)} + \ldots \\
&\qquad\qquad \ldots + \overline{S}^{(n-1)} M^{(1)} + \overline{S}^{(n)} M^{(0)} \\
\widetilde{S}^{(n+1)} &= \overline{S}^{(0)} M^{(n+1)} + \overline{S}^{(1)} M^{(n)} \quad + \overline{S}^{(2)} M^{(n-1)} + \ldots \\
&\qquad\qquad \ldots + \overline{S}^{(n-1)} M^{(2)} + \overline{S}^{(n)} M^{(1)} + \overline{S}^{(n+1)} M^{(0)} \\
&\ \ \vdots
\end{aligned}$$

Summing this infinite system of equations *by columns*, we get

$$\sum_{n=0}^{\infty} \widetilde{S}^{(n)} = \overline{S}^{(0)} \sum_{j=0}^{\infty} M^{(j)} + \overline{S}^{(1)} \sum_{j=0}^{\infty} M^{(j)} + \overline{S}^{(2)} \sum_{j=0}^{\infty} M^{(j)} + \dots$$
$$= M \sum_{n=0}^{\infty} \overline{S}^{(n)} \,. \tag{20.31}$$

In total, the S-matrix thus is

$$S = \sum_{n=0}^{\infty} S^{(n)} \overset{(20.8)}{=} \frac{1}{M} \sum_{n=0}^{\infty} \widetilde{S}^{(n)} \overset{(20.31)}{=} \sum_{n=0}^{\infty} \overline{S}^{(n)} \,. \tag{20.32}$$

We don't need to compute the denominator M nor the unconnected diagrams of the numerator $\widetilde{S}$ of the LSZ-formula, because they mutually cancel! Henceforth we will only compute the connected diagrams $\overline{S}^{(n)} = (20.29)$, because that is sufficient to find the scattering amplitude S.

Note that only the infinite sums (20.32) are equal, while the single terms $\overline{S}^{(j)} \neq \widetilde{S}^{(j)}/M$ are different. When we expand the scattering amplitude with respect to $\overline{S}^{(j)}$, but no more with respect to $\widetilde{S}^{(j)}/M$, then we are changing the algorithm of the approximation. The perturbative computation is leading to the unchanged limit S, but we are approaching that limit at changed step-size.

We continue to evaluate all aspects of perturbation theory, using examples as simple as possible. Therefore we don't investigate next the term $\overline{S}^{(2)}$ of the scattering matrix of ψ^4-interaction, but switch to the scattering matrix of ψ^3-interaction. In zeroth order one finds the same result (i. e. zero) as for ψ^4-interaction, because the interaction-term is

$$\frac{1}{n!}\Big(- i\lambda \hbar^2 c^2 \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 y \, \psi^s(y) \Big)^n \overset{(20.2)}{=} 1 \quad \text{for } n = 0 \tag{20.33}$$

in both cases.

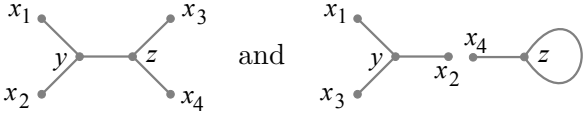
With ψ^3-interaction, $\overline{S}^{(1)}$ is containing four operators $\psi(x_j)$ and three interaction-operators $\psi(y)$, i. e. 7 operators in total. Thus $\overline{S}^{(1)}$ is zero according to theorem (19.40a).

The leading non-vanishing term in the series expansion of the S-matrix in ψ^3-theory is

$$\overline{S}^{(2)} = \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 x_j \, \frac{(-i\lambda\hbar^2 c^2)^2}{2!} \cdot$$
$$\cdot \exp\{-i(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4)\} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 y \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 z \cdot$$
$$\cdot \langle 0 | \, T\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)\, \psi^3(y)\, \psi^3(z) \, | 0 \rangle_c \; . \qquad (20.34)$$

The index c (for "connected") at the matrix element is a reminder, that now all unconnected graphs are to be skipped upfront, because we are computing $\overline{S}^{(2)}$ but not $\widetilde{S}^{(2)}$. To compensate the four zero-factors $\widetilde{G}^{-1}(k_j)$, four Fourier-transformed Greens functions $\widetilde{G}(k_j)$ are needed, which again must be constructed by means of the four exponential functions $\exp\{\pm i(k_j x_j)\}$. A non-vanishing result is possible only, if each operator $\psi(x_j)$ is combined to a propagator with one of the six interaction operators $\psi(y), \psi(z)$, but not with one of the other operators $\psi(x_j)$. In other words: A graph can contribute to the amplitude only, if each external point is directly connected to a vertex, but not to another external point.

Thus from the outset only two types of graphs need to be considered:



(20.35)

But it's not difficult to see, that the second graph gives a vanishing result: If in the term of $\overline{S}^{(2)}$, which corresponds to the second graph, after the substitutions[2] $y_j \equiv x_j - y$ and $z_4 \equiv x_4 - z$ the integrations over y_j and z_4 are performed, then in the result — according to (20.18) — the factors

[2] compare (20.17)

$$\int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4y\, \exp\{-i(k_1+k_2-k_3)y\} \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4z\, \exp\{ik_4z\} =$$
$$\overset{(20.12)}{=} (2\pi\Omega)^2\, \delta(k_1^0+k_2^0-k_3^0)\, \delta_{(\boldsymbol{k}_1\boldsymbol{k}_2),\boldsymbol{k}_3}\, \delta(k_4^0)\, \delta_{\boldsymbol{k}_4,0} \tag{20.36}$$

will turn up. The right graph in (20.35) can add a non-vanishing contribution to the scattering amplitude only, if the outgoing particle, which is described by the operator $\psi(x_4)$, has zero energy and zero momentum. That is impossible, because firstly no Klein-Gordon particle can have zero energy due to $m \neq 0$, and because secondly no particle with zero momentum can leave the interaction range. Due to conservation of energy and momentum, all graphs can be skipped upfront, in which not each incoming or outgoing particle is connected directly or indirectly to minimum one other incoming or outgoing particle.

Thus only two graphs of the left type in (20.35) remain to be computed:

$$\overline{S}^{(2)} \mathrel{\widehat{=}} \underbrace{72 \cdot}_{\overline{S}^{(2)}_{eg}} \underbrace{\begin{array}{ccccc} x_1 & & & & x_3 \\ & \searrow & & \swarrow & \\ & & y - z & & \\ & \swarrow & & \searrow & \\ x_2 & & & & x_4 \end{array}}_{\overline{S}^{(2)}_{eg}} + \underbrace{144 \cdot}_{\overline{S}^{(2)}_{ev}} \underbrace{\begin{array}{ccccc} x_1 & & & & x_2 \\ & \searrow & & \swarrow & \\ & & y - z & & \\ & \swarrow & & \searrow & \\ x_3 & & & & x_4 \end{array}}_{\overline{S}^{(2)}_{ev}} . \tag{20.37}$$

The index eg does mean, that both incoming particles are coupling to the same vertex. The index ev does mean, that the two incoming particles are coupling to different vertices. What is the symmetry factor for $\overline{S}^{(2)}_{eg}$? With tree operators $\psi(y)$ and three operators $\psi(z)$ there are $3 \cdot 3$ alternatives for the construction of the propagator $G(z-y)$. Subsequently there remain 2 alternatives for the construction of the propagators $G(y-x_1)$ and $G(y-x_2)$, and 2 alternatives for the construction of the propagators $G(x_3-z)$ and $G(x_4-z)$. Exchange of the vertices y and z gives another factor 2. Thus the symmetry factor is $9 \cdot 2^2 \cdot 2 = 72$. For $\overline{S}^{(2)}_{ev}$ there is an additional factor 2 due to permutation of x_3 and x_4. We will see, that the two graphs give different results, and therefore cannot be combined to a single graph. Firstly we compute

$$\overline{S}^{(2)}_{eg} \overset{(20.34)}{=} 72 \cdot \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 x_j \, \frac{(-i\lambda\hbar^2 c^2)^2}{2!} \cdot$$

$$\cdot \exp\{-i(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4)\} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 y \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 z \cdot$$

$$\cdot \; G(y - x_1)\, G(y - x_2)\, G(z - y)\, G(x_3 - z)\, G(x_4 - z) \; . \qquad (20.38)$$

In accord with theorem (19.40a), the matrix element in the last line of (20.34) has been replaced by the product of the propagators. The general rule is: For each line in a graph, the corresponding propagator G is inserted as a factor into the scattering amplitude.

Furthermore there is a factor $(-i\lambda\hbar^2 c^2)^2/2!$ in (20.38). The general rule is: Insert a factor $(-i\lambda\hbar^2 c^2)^n/n!$ for a graph with n vertices.

We compile the rules for the construction of matrix elements by means of Feynman-graphs, which we are encountering by and by, into box 20.1 on page 420, because we want to invert the sequence of our actions in future. Until now, we computed the scattering amplitudes algebraically, and then illustrated the results by means of Feynman-graphs. But the essential benefit of the graphs is only realized, if they are used as a mnemonic aid, to draw intuitively all possible alternatives of interaction in each order of perturbative expansion, and then — in the second step — to derive the algebraic formulas from these graphs.

Inserting the Fourier-transformation (20.10) and the definitions

$$y_j \equiv x_j - y \quad \Longrightarrow \quad \mathrm{d}^4 x_j = \mathrm{d}^4 y_j \quad , \quad x_j = y_j + y \qquad (20.39a)$$

$$z_l \equiv x_l - z \quad \Longrightarrow \quad \mathrm{d}^4 x_l = \mathrm{d}^4 z_l \quad , \quad x_l = z_l + z \; , \qquad (20.39b)$$

one gets

$$
\begin{aligned}
\overline{S}^{(2)}_{eg} \overset{(20.38)}{=} 72 \cdot \frac{(-i\lambda\hbar^2c^2)^2}{2!} \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4y \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4z \\
\cdot \prod_{j=1,2} \frac{\widetilde{G}^{-1}(k_j)}{N_j} \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4y_j \, \exp\{-ik_jy\} \\
\cdot \prod_{l=3,4} \frac{\widetilde{G}^{-1}(k_l)}{N_l} \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4z_l \, \exp\{+ik_lz\} \\
\cdot \frac{1}{\Omega 2\pi} \sum_{f_1} \int\limits_{-\infty}^{+\infty} \mathrm{d}f_1^0 \, \widetilde{G}(f_1) \exp\{-i(f_1+k_1)y_1\} \\
\cdot \frac{1}{\Omega 2\pi} \sum_{f_2} \int\limits_{-\infty}^{+\infty} \mathrm{d}f_2^0 \, \widetilde{G}(f_2) \exp\{-i(f_2+k_2)y_2\} \\
\cdot \frac{1}{\Omega 2\pi} \sum_{f} \int\limits_{-\infty}^{+\infty} \mathrm{d}f^0 \, \widetilde{G}(f) \exp\{-if(z-y)\} \\
\cdot \frac{1}{\Omega 2\pi} \sum_{f_3} \int\limits_{-\infty}^{+\infty} \mathrm{d}f_3^0 \, \widetilde{G}(f_3) \exp\{-i(f_3-k_3)z_3\} \\
\cdot \frac{1}{\Omega 2\pi} \sum_{f_4} \int\limits_{-\infty}^{+\infty} \mathrm{d}f_4^0 \, \widetilde{G}(f_4) \exp\{-i(f_4-k_4)z_4\} \; .
\end{aligned}
$$

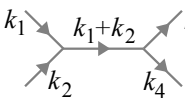
Using the delta function (20.12), this becomes

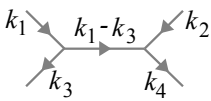
$$\overline{S}^{(2)}_{eg} = 72 \cdot \prod_{j=1}^{4} \frac{\widetilde{G}^{-1}(k_j)}{N_j}\, \widetilde{G}(-k_1)\, \widetilde{G}(-k_2)\, \widetilde{G}(k_3)\, \widetilde{G}(k_4)$$
$$\cdot\, \frac{1}{\Omega 2\pi} \sum_{\boldsymbol{f}} \int_{-\infty}^{+\infty} \mathrm{d}f^0\, \widetilde{G}(f) \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 y\, \exp\{-i(k_1 + k_2 - f)y)\}$$
$$\cdot\, \frac{(-i\lambda\hbar^2 c^2)^2}{2!} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 z\, \exp\{+i(k_3 + k_4 - f)z\}\ . \qquad (20.40)$$

Because of $\widetilde{G}(k_j) \overset{(12.7)}{=} \widetilde{G}(-k_j)$, the four zero-factors can be canceled. The integral over y gives a delta function, which ensures the conservation of energy and momentum at the vertex y, and fixes f accordingly. The integral over z gives a further delta function, which secures the conservation of energy and momentum at the vertex z:

$$\overline{S}^{(2)}_{eg} = 72 \cdot \left(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega N}} \right) \frac{(-i\lambda\hbar^2 c^2)^2}{2!}\, \widetilde{G}(k_1 + k_2) \cdot$$
$$\cdot\, 2\pi\Omega\, \delta(k_1^0 + k_2^0 - k_3^0 - k_4^0)\, \delta_{(\boldsymbol{k}_1 + \boldsymbol{k}_2),(\boldsymbol{k}_3 + \boldsymbol{k}_4)} \qquad (20.41)$$

The particle, which is propagating inbetween the vertices, is transporting the energy and momentum of the outer lines. That can be illustrated due to Feynman-diagrams in wavenumber-space, which often is also called energy-momentum space because of $\hbar c k^0 =$ energy and $\hbar\boldsymbol{k} =$ momentum:

$$\overline{S}^{(2)} \mathrel{\widehat{=}} 72 \cdot$$ 

$$+\, 144 \cdot$$ 

(20.42)

Obviously only k_2 and -k_3 need to be exchanged, to find

$$\overline{S}^{(2)}_{ev} = 144 \cdot \left(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega N}} \right) \frac{(-i\lambda\hbar^2 c^2)^2}{2!}\, \widetilde{G}(k_1 - k_3) \cdot$$
$$\cdot\, 2\pi\Omega\, \delta(k_1^0 + k_2^0 - k_3^0 - k_4^0)\, \delta_{(\boldsymbol{k}_1 + \boldsymbol{k}_2),(\boldsymbol{k}_3 + \boldsymbol{k}_4)}\ . \qquad (20.43)$$

Here we used

$$\delta(k_1^0 - k_3^0 + k_2^0 - k_4^0)\,\delta_{(\boldsymbol{k}_1-\boldsymbol{k}_3),(-\boldsymbol{k}_2+\boldsymbol{k}_4)} = \\ = \delta(k_1^0 + k_2^0 - k_3^0 - k_4^0)\,\delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)}\ .$$

Box 20.1 : **Feynman-rules in time-position-space** for the computation of the component $\overline{S}^{(n)}$ of the scattering amplitude of the uncharged Klein-Gordon field with ψ^s-interaction

A $\overline{S}^{(n)}$ is equal to the sum of all connected 1PI graphs with n vertices.

B The symmetry factor is equal to the number of alternatives for the pairwise combination of the operators ψ, which are constituting the graph, to the propagators of the graph. Only one of the equivalent graphs is inserted as a summand into the scattering amplitude, and multiplied by the symmetry factor.

C s lines are meeting at each vertex. (Here the two ends of a loop are counted as two lines.)

D For the n vertices, a factor

$$\frac{1}{n!}\Big(-i\lambda\hbar^2c^2\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega}\mathrm{d}^4y\Big)^n$$

is inserted. The integrals extend over the complete summand, not only over this factor.

E For each line, the corresponding propagator $G(y-x)$ is inserted as a factor into the summand.

F For each incoming or outgoing particle with wavenumber k_j, a factor

$$\underbrace{\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}}}_{1/N_j}\ \underbrace{\frac{\hbar c}{i}\Big(k_j^2 - \frac{m^2c^2}{\hbar^2} + i\epsilon'\Big)}_{\widetilde{G}^{-1}(k_j)}\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega}\mathrm{d}^4x\,\exp\{\mp ik_jx\}$$

is inserted. The integrals extend over the complete summand, not only over this factor. The sign of the exponent is "$-$" for incoming particles, and "$+$" for outgoing particles.

Note that $\overline{S}^{(2)}_{ev} \neq 2\overline{S}^{(2)}_{eg}$ because of $\widetilde{G}(k_1 - k_3) \overset{(12.7)}{\neq} \widetilde{G}(k_1 + k_2)$.

In future we want to proceed inversely, and derive the algebraic form of the scattering amplitude from Feynman-diagrams in energy-momentum-space. The rules can be formulated in energy-momentum-space much simpler than in time-position-space. They are listed in box 20.2. Obviously the zero-fac-

Box 20.2: **Feynman-rules in energy-momentum-space** for the computation of the component $\overline{S}^{(n)}$ of the scattering amplitude of the uncharged Klein-Gordon field with ψ^s-interaction

A - C as box 20.1 on the facing page

D For the n vertices, insert a factor

$$\frac{1}{n!}(-i\lambda\hbar^2c^2)^n \quad .$$

E For each outer line with wavenumber k_j, a factor

$$\frac{1}{N_j} = \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}}$$

is inserted into the summand.

F For the incoming and outgoing lines with four-wavenumber k_{in} and k_{out}, a factor

$$2\pi\Omega\,\delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}}\Big)\,\delta_{\sum_{\text{in}}\boldsymbol{k}_{\text{in}}\,,\,\sum_{\text{out}}\boldsymbol{k}_{\text{out}}}$$

is inserted into the summand.

G For each inner line with wavenumber k, insert a factor

$$\widetilde{G}(k) = \frac{i}{\hbar c\Big(k^2 - \dfrac{m^2c^2}{\hbar^2} + i\epsilon'\Big)} \quad .$$

H Sum and integrate over the wavenumber k of an inner line,

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi} \quad ,$$

if k is not fixed due to conservation of energy and momentum.

tors of the LSZ-formula are always canceled versus the Fourier-transformed Greens-functions of the incoming and outgoing particles (provided that each external particle is connected to some vertex). Thus one can skip the work to write them down. And always there is a delta function, which secures the conservation of energy and momentum of the incoming and outgoing particles. Therefore one does not need to compute all these integrals anew for each graph, but can write down the results immediately.

The abbreviation 1PI in rule A, box 20.1, stands for one-particle-irreducible. That notion will be explained only around equation (20.82) on page 443.

The wavenumbers of inner lines always must be chosen such, that energy and momentum is conserved at each vertex. We will see that this condition is not sufficient to fix uniquely the wavenumbers of inner lines in diagrams with loops. Rule H is considering this eventuality.

A graph is called tree-graph, if firstly any two vertices are connected by exactly one line (which may run over several intermediate vertices), but never by several parallel lines, and if secondly there is no line, which begins and ends at the same vertex. Diagrams, which don't meet one ore both of these criteria, are called loop-diagrams. We will investigate loop-diagrams in section 20.3.

20.2 The Masses of Virtual Particles

The energies and momenta of the virtual particles, which are represented by the inner lines of ψ^3-theory diagrams like

(20.44)

are uniquely determined due to the delta functions. The invariant wavenumber-square of the virtual particle in the first graph is

$$(k_1^0 + k_2^0)^2 - (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 = 2m^2c^2/\hbar^2 + 2k_1^0 k_2^0 - 2\boldsymbol{k}_1\boldsymbol{k}_2 \ .$$

The wavenumber-square is called invariant, because it is identical in any inertial system. We investigate it in the center-of-mass system $\boldsymbol{k}_2 = -\boldsymbol{k}_1$ of the incoming particles:

$$\begin{aligned} 2m^2c^2/\hbar^2 + 2k_1^0 k_2^0 + 2\boldsymbol{k}_1^2 &= 2m^2c^2/\hbar^2 + 2\sqrt{(\boldsymbol{k}_1^2 + m^2c^2/\hbar^2)^2} + 2\boldsymbol{k}_1^2 \\ &= 4m^2c^2/\hbar^2 + 4\boldsymbol{k}_1^2 > 4m^2c^2/\hbar^2 \ . \end{aligned} \tag{20.45}$$

The invariant wavenumber-square of the virtual particle in the left diagramm (20.44) is at least four times larger than the invariant wavenumber-square of an observed particle. Depending on the momenta of the incoming particles, it may even be much larger.

The second diagram can best be analyzed in the center-of-mass system $\boldsymbol{k}_3 = -\boldsymbol{k}_1$ of the incoming particle k_1 and the outgoing particle k_3. The invariant wavenumber-square of the virtual particle is

$$\begin{aligned} 2m^2c^2/\hbar^2 - 2k_1^0 k_3^0 - 2\boldsymbol{k}_1^2 &= 2m^2c^2/\hbar^2 - 2\sqrt{(\boldsymbol{k}_1^2 + m^2c^2/\hbar^2)^2} - 2\boldsymbol{k}_1^2 \\ &= -4\boldsymbol{k}_1^2 < 0 \ . \end{aligned} \tag{20.46}$$

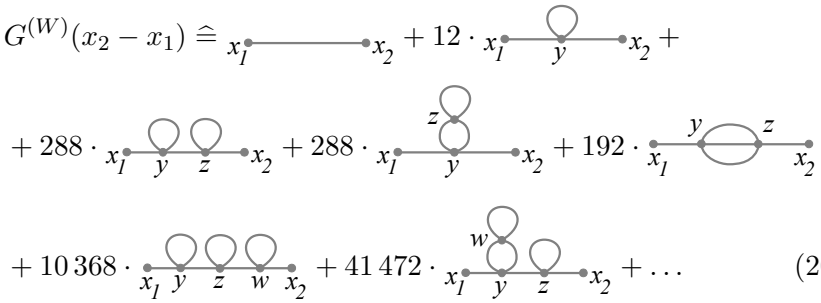
The invariant wavenumber-square of the virtual particle in the right diagram (20.44) always is less than zero. Depending on the momenta of the particles k_1 and k_3, it may be arbitrarily negative. Thus the mass of the virtual particle in this diagram is imaginary.

It's not a specialty of ψ^3-theory, but a general property of all quantum field theories, that virtual particles are not "on mass-shell", i. e. that their mass is differing from the mass of observed particles, and might even be imaginary.

20.3 Loop Graphs

We now will investigate, how the propagator of a particle inbetween the space-time-points x_1 and x_2 is modified in the various orders of perturbation computation of ψ^4-theory:

$$G^{(W)}(x_2 - x_1) \mathrel{\widehat{=}} \quad + 12 \cdot \quad + \quad + 288 \cdot \quad + 288 \cdot \quad + 192 \cdot \quad + 10\,368 \cdot \quad + 41\,472 \cdot \quad + \ldots \tag{20.47}$$



It has been an essential pre-condition of the derivation of the LSZ-formula in section 19.3, that outside of the space-time-range $t_e < t < t_a$, in which interactions may happen, there also exists a space-time-range, in which *no* interactions are taking place. While we eventually shifted t_e and t_a to $\pm\infty$, we have emphasized that this merely meant "far past" and "far future", but not literally "infinity". As self-interactions, like displayed in (20.47), may happen at any time, we have asserted (and will prove that assertion in chapter 22), that such self-interaction-graphs are converted quasi automatically into the simple line of the first summand, if the mass and the field-operator are appropriately renormalized. Later on we will also compute the second diagram in (19.44), and treat it's divergence due to renormalization of the coupling constant.

Accordingly, we used the renormalized quantities in all computations of the previous sections. In contrast, the graphs (20.47) must be computed with the bare (not-renormalized) mass and the bare field-operator, as they would vanish immediately if the renormalized quantities would be inserted. And the second diagram in (19.44) must be computed with the bare coupling constant. We continue to use the notation m for the renormalized mass, while m_0 is our notation for the bare mass. The renormalized coupling constant is λ, the bare coupling constant is λ_0. $\psi(x)$ is the renormalized, $\psi_0(x)$ the bare field-operator.

While the LSZ-formula does not hold for the graphs (20.47), the generic formula[3]

[3] Remember the re-naming (19.21) of indices!

$$\langle 0|\, T\psi_0^{(W)}(x_1)\,\psi_0^{(W)}(x_2)\,|0\rangle \overset{(19.19)}{=} \frac{1}{M}\sum_{n=0}^{\infty}\frac{1}{n!}\cdot$$
$$\cdot\ \langle 0|\, T\psi_0(x_1)\,\psi_0(x_2)\Big(-i\lambda\hbar^2c^2\int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\,\psi_0^4(y)\Big)^n|0\rangle$$
$$\text{with } M = (20.4) \tag{20.48}$$

for matrix elements, which has been derived in section 19.1, is suitable for the computation of propagators with self-interactions. As already done in section 20.1, we will cancel unconnected graphs in the numerator versus the denominator M. Therefore only connected diagrams need to be computed. Consequently only connected diagrams have been indicated in (20.47) right from the outset. As we don't want to change the nomenclature

$$G^{(0)}(x_2-x_1) \equiv G(x_2-x_1) \overset{(15.43)}{=} \langle 0|\, T\psi_0(x_1)\psi_0(x_2)\,|0\rangle$$

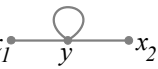
of the Feynman-propagator, we define the notation

$$G^{(W)}(x_2-x_1) = \sum_{n=0}^{\infty} G^{(n)}(x_2-x_1) = \langle 0|\, T\psi_0^{(W)}(x_1)\,\psi_0^{(W)}(x_2)\,|0\rangle$$

for the propagator with self-interaction.

20.3.1 The Tadpole of ψ^4-Theory

The first-order propagator correction is

$12\cdot$ 

$\mathrel{\widehat{=}}\ G^{(1)}(x_2-x_1) =$

$$= \langle 0|\, T\psi_0(x_1)\,\psi_0(x_2)\,(-i\lambda\hbar^2c^2)\int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\,\psi_0^4(y)\,|0\rangle_c$$
$$\overset{(19.40a)}{=} 12\cdot(-i\lambda\hbar^2c^2)\int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\,G(y-x_1)\,G(y-y)\,G(x_2-y)\ .$$

This correction is called tadpole. The naming becomes obvious, if one considers the tadpole of QED, which is displayed at the right. In QED, however, same as in ψ^3-theory, whose tadpole is displayed on the right side of (20.35), the tadpole is a dead end, which is connected only via one propagator to the rest of the world, and thus exists only in second or higher order perturbation theory. In contrast, the tadpole of ψ^4-theory is connected via two propagators to the outside world, and thus showing up already in first order diagrams. For it's computation, we insert the Fourier-transformations (7.13a) of all propagators:

$$\begin{aligned}\frac{1}{\Omega}\sum_{\boldsymbol{f}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}f^0}{2\pi}\,\widetilde{G}^{(1)}(f)\exp\{-if(x_2-x_1)\} = \; -i\,12\lambda\hbar^2c^2\cdot \\ \cdot\int_{\Omega}\int_{-\infty}^{+\infty}\mathrm{d}^4y\,\frac{1}{\Omega^3}\sum_{\boldsymbol{k_1},\boldsymbol{k},\boldsymbol{k_2}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1^0\,\mathrm{d}k^0\,\mathrm{d}k_2^0}{(2\pi)^3}\,\widetilde{G}(k_1)\,\widetilde{G}(k)\,\widetilde{G}(k_2)\cdot \\ \cdot\exp\{-ik_1(y-x_1)-ik(y-y)-ik_2(x_2-y)\} \qquad (20.49)\end{aligned}$$

We identify

$$\int_{\Omega}\int_{-\infty}^{+\infty}\mathrm{d}^4y\,\exp\{+i(k_2-k_1)y\} \overset{(7.16b)}{=} 2\pi\Omega\,\delta(k_2^0-k_1^0)\,\delta_{\boldsymbol{k_2}\boldsymbol{k_1}}\;,$$

compute the integral and the sum over k_2, and rename the integration-variable f on the left side into k_1:

$$\begin{aligned}\frac{1}{\Omega}\sum_{\boldsymbol{k_1}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1^0}{2\pi}\,\widetilde{G}^{(1)}(k_1)\exp\{-ik_1(x_2-x_1)\} = \; -i\,12\lambda\hbar^2c^2\cdot \\ \cdot\,\frac{1}{\Omega^2}\sum_{\boldsymbol{k_1},\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1^0\,\mathrm{d}k^0}{(2\pi)^2}\,\widetilde{G}(k_1)\,\widetilde{G}(k)\,\widetilde{G}(k_1)\,\exp\{-ik_1(x_2-x_1)\}\end{aligned}$$

Consequently the integrands must be equal:

$$12 \cdot \quad \widehat{=} \widetilde{G}^{(1)}(k_1) = \widetilde{G}(k_1) \underbrace{\frac{-i\,12\lambda\hbar^2c^2}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \widetilde{G}(k)}_{\equiv F}\, \widetilde{G}(k_1) \tag{20.50a}$$

$$F \overset{(12.7)}{=} \frac{12\lambda\hbar^2c^2}{\hbar c}\, \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \frac{1}{k^2 - m_0^2c^2/\hbar^2 + i\epsilon'} \tag{20.50b}$$

It's characteristic for loop-graphs, that a wavenumber k is showing up, which is not constrained by conservation of energy and momentum. Furthermore it's a particular feature of the tadpole of ψ^4-theory, that the function F does not depend on the wavenumber k_1 of the propagating particle. Because of

$$\frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \overset{(7.5)}{\approx} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\,, \tag{20.50c}$$

F is diverging for $|k| \to \infty$ like $\mathrm{d}^4k\, k^{-2} \sim k^2$.

The equation of F can alternatively be written in the form

$$F = \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \frac{12\lambda\hbar c}{\Big(k^0 + \frac{\omega_{\boldsymbol{k}}}{c} - i\epsilon\Big)\Big(k^0 - \frac{\omega_{\boldsymbol{k}}}{c} + i\epsilon\Big)}\,, \tag{20.50d}$$

because we have

$$\begin{aligned} k^2 - m_0^2c^2/\hbar^2 = (k^0)^2 - \boldsymbol{k}^2 - m_0^2c^2/\hbar^2 = (k^0)^2 - \frac{\omega_{\boldsymbol{k}}^2}{c^2} = \\ = \Big(k^0 + \frac{\omega_{\boldsymbol{k}}}{c}\Big)\Big(k^0 - \frac{\omega_{\boldsymbol{k}}}{c}\Big) \\ \frac{\omega_{\boldsymbol{k}}}{c} \overset{(7.18)}{=} + \sqrt{\boldsymbol{k}^2 + m_0^2c^2/\hbar^2}\,. \end{aligned} \tag{20.51}$$

Only "on mass-shell"

$$k^2 = \frac{m_0^2c^2}{\hbar^2} \quad \text{and} \quad k^0 = \frac{\omega_{\boldsymbol{k}}}{c}\,.$$

But $\boldsymbol{k}$ and k^0, being the variables of summation and integration, are not fixed to mass-shell. k^0 assumes all values in the continuum $-\infty \ldots +\infty$. $\boldsymbol{k}$ assumes all values, which are compatible with the normalization volume Ω. Therefore the poles of F

$$\begin{aligned} \text{at } k^2 &= m_0^2 c^2/\hbar^2 - i\epsilon' \\ \text{resp. at } k^0 &= \pm\Big(\frac{\omega_{\boldsymbol{k}}}{c} - i\epsilon\Big) \end{aligned}$$

have been shifted from the real k^0-axis into the complex plane:

$$\omega_{\boldsymbol{k}}/c \overset{(12.9)}{\longrightarrow} \omega_{\boldsymbol{k}}/c - i\epsilon \quad \text{with } \epsilon \in \mathbb{R}\ ,\ \epsilon > 0$$

It's important to shift the poles exactly like this, to make sure that the frequency-time-product will always be positive, as discussed at (12.12). Only propagators with positive frequency-time-product are describing particles or antiparticles with positive energy, which are propagating forward through time.

We will encounter integrals of the type (20.50) in all computations of Feynman-graphs with loops. Therefore we compute this integral for future reference in the generalized form

$$F_K \equiv \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{J}{(k^2 - K^2 + i\epsilon')^r} \tag{20.52a}$$

$$= \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{J}{(k^0 + \vartheta_{\boldsymbol{k}} - i\epsilon)^r (k^0 - \vartheta_{\boldsymbol{k}} + i\epsilon)^r} \tag{20.52b}$$

$$1 \le r \in \mathbb{N}\ ,\ 0 \le K^2 \in \mathbb{R}\ ,\ \vartheta_{\boldsymbol{k}} \equiv +\sqrt{\boldsymbol{k}^2 + K^2}$$

$$J \equiv 1 \text{ or } k^2 \text{ or } k^\mu k^\nu \text{ or } (k^2)^2\ .$$

The both poles are indicated in figure 20.3 by red points. With $J = k^\mu$ the integral would be zero, because the integration resp. the summation is symmetrical over $+k^\mu$ and $-k^\mu$. The same would hold for $J = k^\mu k^\nu$ with $\mu \neq \nu$. Therefore these both possibilities can be skipped from further

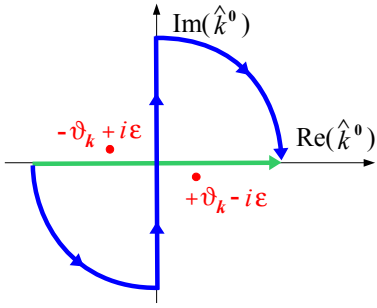


Fig. 20.3 : Integration paths without (green) and with (blue) Wick-Rotation

considerations.

Due to analytical continuation, we extend the definition range of the variable k^0 from the real axis into the complex plane. The complex variable $\hat{k}^0$ is marked by a hat. Actually we did already use that extension, when we shifted the poles from the integration paths into the complex plane. But that have been only infinitesimal deviations from the real axis. Now we consider the complete complex plane as the definition range of the null-component of k, and introduce polar coordinates for the null-component:

$$\hat{k}^0 \equiv |\hat{k}^0| \exp\{i\varphi\} \tag{20.53}$$
$$\mathrm{d}\hat{k}^0 = \exp\{i\varphi\}\mathrm{d}|\hat{k}^0| + i|\hat{k}^0| \exp\{i\varphi\}\mathrm{d}\varphi$$

The polar angle φ is counted from the positive real $\hat{k}^0$-axis counter-clockwise, as usual. Only on the real $\hat{k}^0$-axis $\hat{k}^0 = k^0$.

The integration path of (20.52) is indicated in figure 20.3 as a green arrow. According to Cauchy's integral theorem[4], one gets the same result, if the integration is performed along the path indicated in blue, because there are no singular points in the area enclosed by the two integration paths. In the sequel, we will restrict our analysis to those cases, in which the integral over the two blue circle segments is zero:

[4] A short explanation of this important mathematical tool, tailored to the need of physicists, can be found in [37].

$$\lim_{|\hat{k}^0|\to\infty} \int_{\pi}^{3\pi/2} \int_{\pi/2}^{0} \mathrm{d}\varphi \, \frac{i|\hat{k}^0| \exp\{i\varphi\}\hat{J}}{|\hat{k}^0|^{2r} \exp\{i2r\varphi\}} = 0 \tag{20.54}$$

The hat on $\hat{J}$ is a reminder, that this quantity possibly may be affected by the rotation of the integration path from the real to the imaginary k^0-axis:

$$\hat{J} = \begin{cases} 1 \\ \hat{k}^\mu \hat{k}^\nu = -k^\mu k^\nu \text{ with } \mu = \nu = 0 \\ k^\mu k^\nu \text{ with } \mu = \nu \neq 0 \\ \hat{k}^2 = (ik^0)^2 - \boldsymbol{k}^2 = -(k^0)^2 - \boldsymbol{k}^2 \\ (\hat{k}^2)^2 = [(k^0)^2 + \boldsymbol{k}^2]^2 \end{cases} \tag{20.55}$$

Here we considered, that $\hat{k}^0 \equiv ik^0$ on the imaginary $\hat{k}^0$-axis. Condition (20.54) is fulfilled, if k^0 is showing up in the denominator with higher power than in the numerator. Depending on J, therefore the following restrictions must be fixed for the exponent $r \in \mathbb{N}$:

$$\begin{aligned} J = 1 &\implies r \geq 1 \\ J = k^\mu k^\nu \text{ with } \mu = \nu &\implies \begin{cases} r \geq 2 \text{ if } \mu = 0 \\ r \geq 0 \text{ if } \mu \neq 0 \end{cases} \\ J = k^2 &\implies r \geq 2 \\ J = (k^2)^2 &\implies r \geq 3 \end{aligned} \tag{20.56}$$

If these conditions are met, then the blue integral from $-i\infty$ to $+i\infty$ along the imaginary $\hat{k}^0$-axis is equal to the green integral from $-\infty$ to $+\infty$ along the real $\hat{k}^0$-axis:

$$F_K \overset{(20.52)}{=} \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}\hat{k}^0}{2\pi} \, \frac{\hat{J}}{(\hat{k}^0 + \vartheta_{\boldsymbol{k}})^r (\hat{k}^0 - \vartheta_{\boldsymbol{k}})^r} =$$

$$= \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-i\infty}^{+i\infty} \frac{i\mathrm{d}k^0}{2\pi} \frac{\hat{J}}{\left((ik^0)^2 - \boldsymbol{k}^2 - K^2\right)^r} \tag{20.57}$$

The small terms ϵ could be skipped, because the integration path along the imaginary $\hat{k}^0$-axis is far-off from the poles at $\hat{k}^0 = \pm\vartheta_{\boldsymbol{k}}$, see figure 20.3. The rotation of the integration path in the complex $\hat{k}^0$-plane by the angle $\pi/2$ is a trick, which is often used in quantum field theory. It is called "Wick-rotation".

The Minkowski[5]-metric holds for the wavenumber k:

$$(k)^2 = (k^0, \boldsymbol{k})^2 = (k^0)^2 - \boldsymbol{k}^2$$

We define a four-vector k_E with Euklidean[6] metric, in whose square the squares of all four components are summed-up with the same (in this case positive) sign:

$$\begin{aligned} (k_E)^2 &= (k_E^0, \boldsymbol{k}_E)^2 = (k_E^0)^2 + \boldsymbol{k}_E^2 \\ -(k_E)^2 &= -(k_E^0)^2 - \boldsymbol{k}_E^2 \end{aligned} \tag{20.58}$$

This is to be compared with the variable $\hat{k} = (\hat{k}^0, \boldsymbol{k}) = (ik^0, \boldsymbol{k})$ of summation and integration in (20.57) and in (20.55):

$$(\hat{k})^2 = (ik^0, \boldsymbol{k})^2 = -(k^0)^2 - \boldsymbol{k}^2 \tag{20.59}$$

It's exactly this square, which is showing up in the denominator of (20.57). $\hat{k} = (ik^0, \boldsymbol{k})$ is a vector with negative Euklidean metric. Therefore in (20.57) the replacements

[5] Hermann Minkowski (1864 - 1909)
[6] Euklid of Alexandria (about 360 BCE - about 280 BCE)

$$\hat{k} = (ik^0, \boldsymbol{k}) \longrightarrow (k_E^0, \boldsymbol{k}_E) = \hat{k}_E \tag{20.60a}$$

$$(\hat{k})^2 = (ik^0, \boldsymbol{k})^2 \longrightarrow -(k_E^0, \boldsymbol{k}_E)^2 = -(k_E)^2 \tag{20.60b}$$

$$\hat{k}^\mu \, \hat{k}^\nu \longrightarrow \left\{ \begin{array}{l} -k_E^\mu \, k_E^\nu \,;\, \mu = \nu = 0 \\ +k_E^\mu \, k_E^\nu \,;\, \mu = \nu \neq 0 \end{array} \right\} = -g^{\mu\nu} \frac{(k_E)^2}{4} \tag{20.60c}$$

can be performed, without changing the integral's value:

$$F_K \overset{(20.57)}{=} \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}k_E^0}{2\pi} \frac{J_E}{\Big(-(k_E)^2 - K^2 \Big)^r} \tag{20.61}$$

$$J_E \overset{(20.52)}{=} 1 \text{ or } -(k_E)^2 \text{ or } -g^{\mu\nu}(k_E)^2/4 \text{ or } (k_E)^4$$

The euclidean vector k_E is appearing in the integrand only quadratically. Therefore the change to four-dimensional spherical coordinates is advisable. A n-dimensional sphere with radius R has the volume V_n and the surface S_n. Using the formulae[7]

$$\begin{aligned} V_n &= \frac{2\pi R^2}{n} V_{n-2} & \qquad S_n &= \frac{\mathrm{d}V_n}{\mathrm{d}R} \\ V_2 &= \pi R^2 & \qquad V_3 &= \frac{4}{3}\pi R^3 \end{aligned} \quad , \tag{20.62}$$

the values for arbitrary $n \in \mathbb{N}$ can be computed. The surface of a four-dimensional Euclidean sphere with radius $R = |k_E|$ is $2\pi^2 R^3$. The introduction of spherical coordinates is easier, if the sum over the discrete wavenumbers $\boldsymbol{k}_E$ is replaced by the integral over a continuum of $\boldsymbol{k}_E$-values:

$$\begin{aligned} \frac{1}{\Omega} \sum_{\boldsymbol{k}_E} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}k_E^0}{2\pi} &\overset{(7.5)}{=} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^3 k_E}{(2\pi)^3} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}k_E^0}{2\pi} = i \int_0^{+\infty} \frac{2\pi^2 R^3 \mathrm{d}R}{(2\pi)^4} = \\ &= i \int_0^{+\infty} \mathrm{d}R \, \frac{R^3}{8\pi^2} = i \int_0^{+\infty} \mathrm{d}R^2 \, \frac{R^2}{16\pi^2} \end{aligned} \tag{20.63}$$

[7] A good article "n-sphere" can be found in Wikipedia.

The factor i is showing up, because k_E has one imaginary and three real components, while R is real. Due to these conversions, we have achieved the following result:

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{J}{\left(k^2-K^2+i\epsilon'\right)^r}=\frac{i(-1)^r}{16\pi^2}\int_0^{+\infty}\mathrm{d}R^2\,\frac{R^2J_E}{\left(R^2+K^2\right)^r}$$

$$1\leq r\in\mathbb{N}\ ,\ 0\leq K^2\in\mathbb{R} \tag{20.64}$$

$J=$	1	k^μ	k^2	$k^\mu k^\nu$	$(k^2)^2$
$J_E=$	1	0	$-R^2$	$-g^{\mu\nu}R^2/4$	R^4
$r=$	≥ 1	—	≥ 2	$\begin{cases}\geq 2 \text{ if } \mu=0\\ \geq 0 \text{ if } \mu\neq 0\end{cases}$	≥ 3

For the tadpole of ψ^4-theory, $K^2=m_0^2c^2/\hbar^2$, $r=1$, and $J=1$:

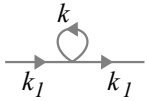
$$\begin{aligned}F &\overset{(20.50b)}{=} \frac{12\lambda\hbar^2c^2}{\hbar c}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{1}{k^2-m_0^2c^2/\hbar^2+i\epsilon'}\\ &\overset{(20.64)}{=} \frac{i(-1)\,12\lambda\hbar c}{16\pi^2}\int_0^{+\infty}\mathrm{d}R^2\,\frac{R^2}{R^2+m_0^2c^2/\hbar^2}\end{aligned} \tag{20.65}$$

This integral is diverging for $R\to\infty$. To regularize it (i. e. to render it computable), we introduce a cutoff-parameter Λ:

$$\begin{aligned}F &= \frac{-i3\lambda\hbar c}{4\pi^2}\lim_{\Lambda\to\infty}\int_0^{\Lambda^2}\mathrm{d}R^2\,\frac{R^2}{R^2+m_0^2c^2/\hbar^2}\\ &= \frac{-i3\lambda\hbar c}{4\pi^2}\lim_{\Lambda\to\infty}\left(R^2-m_0^2c^2/\hbar^2\,\ln(R^2+m_0^2c^2/\hbar^2)\right)\Big|_0^{\Lambda^2}\\ &= \frac{-i3\lambda\hbar c}{4\pi^2}\,m_0^2c^2/\hbar^2\lim_{\Lambda\to\infty}\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}-\ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}+1\right)\right)\end{aligned} \tag{20.66}$$

The term 1 in the logarithm can be skipped because of $\Lambda\to\infty$. Thus

one gets for the propagator with self-interaction in first order perturbative computation (that is for the tadpole) the result

$12 \cdot$  $\widehat{=} \widetilde{G}^{(1)}(k_1) =$

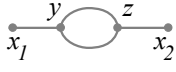
$$\overset{(20.50)}{=} \widetilde{G}(k_1) \underbrace{\frac{-i\,12\lambda\hbar^2c^2}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \widetilde{G}(k)}_{F}\, \widetilde{G}(k_1) \tag{20.67a}$$

$$F = -\frac{i3\lambda\hbar c}{4\pi^2} \Big(\frac{m_0 c}{\hbar}\Big)^2 \lim_{\Lambda\to\infty} \Big(\frac{\Lambda^2\hbar^2}{m_0^2c^2} - \ln\Big(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\Big)\Big)\,. \tag{20.67b}$$

The function F of the tadpole diverges for $\Lambda \to \infty$ quadratically and logarithmically. We will clarify in chapter 22, how this divergence can be cured due to renormalization of the mass m_0 and the field-operator ψ_0.

20.3.2 Propagator-Corrections in ψ^3-Theory

$G^{(1)} = 0$ with ψ^3-interaction. Thus the leading propagator-correction is found in second order perturbation computation:

$$G^{(2)}(x_2 - x_1) \widehat{=} 36 \cdot$$ 

(20.68)

The essential difference to the tadpole is, that there are 2 vertices in this loop, and therefore the loop is composed of two propagator-lines. We will see, that consequently this graph diverges "only" logarithmically, but not quadratically. The computation must be conducted with the bare (not-renormalized) mass m_0.

There are $3^2 \cdot 2^2/2!$ alternatives to build the two internal propagators. Then either x_1 or x_2 can be contracted with y, resulting in another factor 2. Thus the symmetry factor is 36. The graph is computed by means of the generic formula (20.48). As usual, unconnected structures in the numerator are canceled versus the denominator.

$$G^{(2)}(x_2-x_1) \overset{(19.40a)}{=} 36\cdot\frac{(-i\lambda\hbar^2c^2)^2}{2!}\int\limits_\Omega\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\int\limits_\Omega\int\limits_{-\infty}^{+\infty}\mathrm{d}^4z\cdot$$
$$\cdot\, G(y-x_1)\,G(z-y)\,G(z-y)\,G(x_2-z) \qquad (20.69)$$

We insert the Fourier-transformations of the propagators

$$\frac{1}{\Omega}\sum_{\boldsymbol{f}}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}f^0}{2\pi}\,\widetilde{G}^{(2)}(f)\,\exp\{-if(x_2-x_1)\} = 36\cdot\frac{(-i\lambda\hbar^2c^2)^2}{2!}\cdot$$
$$\cdot\int\limits_\Omega\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\int\limits_\Omega\int\limits_{-\infty}^{+\infty}\mathrm{d}^4z\,\frac{1}{\Omega^4}\sum_{\boldsymbol{k}_1,\boldsymbol{k},\boldsymbol{g},\boldsymbol{k}_2}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}k_1^0\,\mathrm{d}k^0\,\mathrm{d}g^0\,\mathrm{d}k_2^0}{(2\pi)^4}\cdot$$
$$\cdot\,\widetilde{G}(k_1)\,\widetilde{G}(k)\,\widetilde{G}(g)\,\widetilde{G}(k_2)\cdot$$
$$\cdot\,\exp\Big\{-i\Big(k_1(y-x_1)+k(z-y)+g(z-y)+k_2(x_2-z)\Big)\Big\}\,,$$

identify the two delta functions

$$\int\limits_\Omega\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\,\exp\{+i(g-k_1+k)y\} \overset{(7.16b)}{=} 2\pi\Omega\,\delta\Big(g^0-(k_1^0-k^0)\Big)\,\delta_{\boldsymbol{g},(\boldsymbol{k}_1-\boldsymbol{k})}$$
$$\int\limits_\Omega\int\limits_{-\infty}^{+\infty}\mathrm{d}^4z\,\exp\{+i(k_2-g-k)y\} \overset{(7.16b)}{=} 2\pi\Omega\,\delta\Big(k_2^0-(g^0+k^0)\Big)\,\delta_{\boldsymbol{k}_2,(\boldsymbol{g}+\boldsymbol{k})}\,,$$

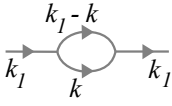
perform the integrations and summations over g and k_2, and rename the integration variable f on the left side into k_1:

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}_1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1^0}{2\pi}\,\widetilde{G}^{(2)}(k_1)\,\exp\{-ik_1(x_2-x_1)\} =$$

$$= 36\cdot\frac{1}{\Omega}\sum_{\boldsymbol{k}_1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_1^0}{2\pi}\,\frac{(-i\lambda\hbar^2c^2)^2}{2!}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}$$

$$\cdot\,\widetilde{G}(k_1)\,\widetilde{G}(k)\,\widetilde{G}(k_1-k)\,\widetilde{G}(k_1)\,\exp\Big\{-ik_1(x_2-x_1)\Big\} \qquad (20.70)$$

The integrands must be equal. Therefore

$36\cdot$ 

$\widehat{=}\ \widetilde{G}^{(2)}(k_1) =$ (20.71a)

$$= \widetilde{G}(k_1)\cdot\underbrace{18(-i\lambda\hbar^2c^2)^2\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\widetilde{G}(k)\,\widetilde{G}(k_1-k)}_{\equiv F(k_1)}\cdot\widetilde{G}(k_1)\ .$$

The conversions, which led to this intermediate result, are quite similar to the conversions, which we encountered before in the tadpole computation. We do not want to write down and compute again and again all those Fourier-transformations and delta functions. Therefore we compile for future reference the general rules for the computation of propagator-corrections in box 20.4 on the next page.

Again — typical for loop-graphs — there is a wavenumber k, which is not determined by a delta function. We compute the function $F(k_1)$, which was defined in the last equation:

$$F(k_1)\overset{(12.7)}{=} 18(-i\lambda\hbar^2c^2)^2\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\Big(\frac{i}{\hbar c}\Big)^2\cdot$$

$$\cdot\,\frac{1}{\Big(k^2-m_0^2\frac{c^2}{\hbar^2}+i\epsilon'\Big)\Big((k_1-k)^2-m_0^2\frac{c^2}{\hbar^2}+i\epsilon'\Big)} \qquad (20.71b)$$

To find $F(k_1)$, we can not simply copy the tadpole computation, because

the undetermined wavenumber does appear in the integrand's denominator not only quadratically (as in case of the tadpole), but also linearly as $-2k_1k$. Therefore first a tricky conversion of the integrand is needed. For arbitrary $A, B \in \mathbb{C}$ and dimension-less $\xi \in \mathbb{R}$

$$\int_0^1 \frac{\mathrm{d}\xi}{\Big(\xi A + (1-\xi)B\Big)^2} = \int_0^1 \frac{\mathrm{d}\xi}{\Big(\xi(A-B)+B\Big)^2} \ . \tag{20.72a}$$

Box 20.4: **Feynman-rules in energy-momentum-space** for the computation of propagator-corrections of the uncharged Klein-Gordon field with ψ^s-interaction

A The propagator-correction corresponds to the sum of all connected graphs with n vertices, which have the structure

$$\widetilde{G}^{(n)} = \widetilde{G}^{(0)} \cdot F^{(n)} \cdot \widetilde{G}^{(0)} \quad \text{with } \widetilde{G}^{(0)} \equiv \widetilde{G}$$

B s lines meet at each vertex. (Two ends of a loop here are counted as 2 lines.)

C The symmetry factor is the number of alternatives for the pairwise combination of the operators ψ of the graph to the graph's propagators. Only one of the equivalent graphs is drawn, and multiplied by the symmetry factor.

D For the n vertices, insert a factor

$$\frac{1}{n!}(-i\lambda\hbar^2c^2)^n \quad .$$

E For each line with wavenumber k, insert a factor

$$\widetilde{G}(k) = \frac{i}{\hbar c\Big(k^2 - \dfrac{m^2c^2}{\hbar^2} + i\epsilon'\Big)} \ .$$

F Summations and integrations are performed over all wavenumbers k, which are not fixed by conservation of energy and momentum:

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}$$

Because of
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\,\frac{1}{\xi(A-B)+B} = \frac{-(A-B)}{\Big(\xi(A-B)+B\Big)^2}$$

we have
$$\int_0^1 \frac{\mathrm{d}\xi}{\Big(\xi A+(1-\xi)B\Big)^2} = \frac{1}{B-A}\cdot\frac{1}{\xi(A-B)+B}\bigg|_0^1 =$$
$$= \frac{1}{B-A}\Big(\frac{B}{AB} - \frac{A}{BA}\Big) = \frac{1}{AB}\ . \tag{20.72b}$$

This can be written in the form
$$\frac{1}{AB} = \int_0^1\int_0^1 \mathrm{d}\xi_1\mathrm{d}\xi_2\,\frac{\delta(\xi_2-1+\xi_1)}{\Big(\xi_1 A+\xi_2 B\Big)^2}\ . \tag{20.72c}$$

For the moment being, this formula is sufficient. For future applications, we state without proof this generalization:

$$n\in\mathbb{N}, m_j\in\mathbb{R}, A_j\in\mathbb{C}, \xi_j\in\mathbb{R}\ :\qquad \frac{1}{A_1^{m_1}\dots A_n^{m_n}} =$$
$$= \int_0^1 \mathrm{d}\xi_1\dots\int_0^1 \mathrm{d}\xi_n\,\frac{\Gamma\Big(\sum_{j=1}^n m_j\Big)\,\delta\Big(\sum_{j=1}^n \xi_j-1\Big)\,\Big(\prod_{j=1}^n \xi_j^{m_j-1}\Big)}{\Big(\prod_{j=1}^n \Gamma(m_j)\Big)\,\Big(\sum_{j=1}^n \xi_j A_j\Big)^{\sum_{j=1}^n m_j}} \tag{20.73a}$$

Here Γ is the gamma-function (the generalization of faculty). If all exponents m_j equal 1, this formula simplifies to

$$\frac{1}{A_1\dots A_n} = \int_0^1 \mathrm{d}\xi_1\dots\int_0^1 \mathrm{d}\xi_n\,\frac{(n-1)!\,\delta\Big(\sum_{j=1}^n \xi_j-1\Big)}{\Big(\sum_{j=1}^n \xi_j A_j\Big)^n}\ , \tag{20.73b}$$

which in case $n=2$ reduces further to (20.72).

The both factors in the denominator of the integrand of $F(k_1)$ can be identified with A and B in (20.72). A further conversion of the denominator leads to

$$\left(\xi\left(k^2 - m_0^2\frac{c^2}{\hbar^2}\right) + (1-\xi)\left((k_1-k)^2 - m_0^2c^2/\hbar^2\right)\right)^2 =$$
$$= \left(\left(k-(1-\xi)k_1\right)^2 + (\xi-\xi^2)k_1^2 - m_0^2c^2/\hbar^2\right)^2 . \qquad (20.74)$$

We resist the temptation to insert $k_1^2 = m_0^2c^2/\hbar^2$, because we want to arrange the computation from the outset so generic, that it is still valid if the bubble (20.71) is the correction of an inner line of a larger graph. For virtual particles $k_1^2 \neq m_0^2c^2/\hbar^2$, as we have seen in section 20.2.

Now we substitute the summation- and integration-variable

$$k \longrightarrow \kappa \equiv k - (1-\xi)k_1 ,$$

and rename subsequently κ into k:

$$F(k_1) \overset{(20.71b)}{=} -\int_0^1 \mathrm{d}\xi\, \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi}\, \frac{2(3\lambda\hbar c)^2}{(k^2 \underbrace{+(\xi-\xi^2)k_1^2 - m_0^2c^2/\hbar^2}_{\equiv -K^2} + i\epsilon')^2}$$

Thus we have arrived at an important intermediate result: Firstly, the undetermined wavenumber k does appear in the integrand only quadratically, same as in the tadpole integrand. Secondly, one can interpret

$$\frac{K\hbar}{c} = \sqrt{\underbrace{-(\xi-\xi^2)}_{-0.25\leq\ldots\leq 0}\, k_1^2\hbar^2/c^2 + m_0^2} \qquad (20.75)$$

as an effective mass, because $F(k_1)$ has poles at $k = \pm K$. In case $k_1^2 \geq 4m_0^2c^2/\hbar^2$, the effective mass becomes imaginary in an interval around $\xi = 0.5$. That's the formal signature of a dissipative process, in which energy disappears from the investigated system, being converted to heat. In this case, the dissipation is caused by the opening of an additional channel, which is a competing alternative to the diagram (20.68): At $k_1^2 \geq 4m_0^2c^2/\hbar^2$, the systems energy is sufficient, to convert the two virtual particles of (20.68) into real particles. Thus these particles can disappear from the probability amplitude (20.68). While for real (i. e. observed) particles $k_1^2 = m_0^2c^2/\hbar^2$,

virtual particles may very well have $k_1^2 > 4m_0^2c^2/\hbar^2$, as we have seen e. g. in the left graph of (20.44).

Applying the generic formula (20.64), we find

$$F(k_1) = -\int_0^1 \mathrm{d}\xi \, \frac{i(3\lambda\hbar c)^2}{8\pi^2} \int_0^{+\infty} \mathrm{d}R^2 \, \frac{R^2}{(R^2+K^2)^2} \tag{20.76a}$$

$$K^2 = (\xi^2 - \xi)k_1^2 + m_0^2c^2/\hbar^2 \ . \tag{20.76b}$$

The integral diverges for $R \to \infty$. To regularize it, we introduce a cut-off parameter Λ:

$$\begin{aligned} F(k_1) &= -\int_0^1 \mathrm{d}\xi \, \frac{i(3\lambda\hbar c)^2}{8\pi^2} \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} \mathrm{d}R^2 \, \frac{R^2}{(R^2+K^2)^2} \\ &= -\int_0^1 \mathrm{d}\xi \, \frac{i(3\lambda\hbar c)^2}{8\pi^2} \lim_{\Lambda\to\infty} \left(\frac{K^2}{R^2+K^2} + \ln(R^2+K^2) \right) \Big|_0^{\Lambda^2} \\ &= -\int_0^1 \mathrm{d}\xi \, \frac{i(3\lambda\hbar c)^2}{8\pi^2} \lim_{\Lambda\to\infty} \left(\frac{K^2}{\Lambda^2+K^2} - 1 + \ln\left(\frac{\Lambda^2+K^2}{K^2} \right) \right) \end{aligned}$$

Because of $\Lambda \to \infty$, this expression simplifies to

$$F(k_1) = -\int_0^1 \mathrm{d}\xi \, \frac{i(3\lambda\hbar c)^2}{8\pi^2} \lim_{\Lambda\to\infty} \ln\left(\frac{\Lambda^2}{K^2} \right) . \tag{20.77}$$

So far we assumed $K^2 \geq 0$. What will happen in case $K^2 < 0$? $K^2 =$ (20.76b) is certainly negative in case $k_{1\,E}^2 < -4\,m_0^2c^2/\hbar^2$ at $\xi = 0.5$. In contrast, K^2 is positive with certainty at $\xi = 0$ and $\xi = 1$. Thus there must exist a value $0 < \eta < 0.5$, $\eta \in \mathbb{R}$, such that K^2 is positive for $\xi < \eta$ and for $\xi > 1 - \eta$, negative for $\eta < \xi < 1 - \eta$, and zero at $\xi = \eta$ and at $\xi = 1 - \eta$. Accordingly we split the integral over ξ into three ranges:

$$\underbrace{\int_0^1 \mathrm{d}\xi \ \ldots}_{F(k_1)} = \underbrace{\int_0^\eta \mathrm{d}\xi \ \ldots}_{F_a(k_1)} + \underbrace{\int_\eta^{1-\eta} \mathrm{d}\xi \ \ldots}_{F_b(k_1)} + \underbrace{\int_{1-\eta}^1 \mathrm{d}\xi \ \ldots}_{F_c(k_1)} \tag{20.78}$$

The two integrals $F_a(k_1)$ and $F_c(k_1)$ differ from (20.77) only by the integration boundaries. $K^2 = 0$ at $\xi = \eta$ and at $\xi = 1 - \eta$. But because of $\lim_{\xi\to 0} \xi \ln(\xi) = 0$, the integrals don't diverge at the integration boundaries.

The integral over the middle range is describing the case, that the virtual particles mutate to real particles. Thus it's contribution to the probability amplitude (20.68) is zero. We can formulate a result, which is generally valid, by introducing a factor V=(20.79c) into the integral:

$$F(k_1) = -\frac{i(3\lambda\hbar c)^2}{8\pi^2} \int_0^1 \mathrm{d}\xi \, V \lim_{\Lambda\to\infty} \ln\left(\frac{\Lambda^2}{K^2}\right) \tag{20.79a}$$

$$K^2 = (\xi^2 - \xi)k_1^2 + m_0^2 c^2/\hbar^2 \tag{20.79b}$$

$$V \equiv \begin{cases} 1 & \text{if } K^2 \geq 0 \\ 0 & \text{else} \end{cases} \tag{20.79c}$$

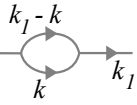
For a free particle, we have $V = 1$ and

$$\int_0^1 \mathrm{d}\xi \, \ln\left(\frac{\Lambda^2}{K^2}\right) = \ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right) - \int_0^1 \mathrm{d}\xi \, \ln\left((\xi^2 - \xi) + 1\right) =$$

$$= \ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right) - 0.81 \approx \ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right) \quad \text{because of } \Lambda \to \infty \ .$$

Thus we arrive at the final result:

$$36 \cdot \quad \widehat{=} \widetilde{G}^{(2)}(k_1) \overset{(20.71)}{=}$$



$$= \underbrace{\widetilde{G}(k_1) \cdot 18(-i\lambda\hbar^2c^2)^2 \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \widetilde{G}(k)\, \widetilde{G}(k_1 - k) \cdot \widetilde{G}(k_1)}_{\equiv F(k_1)} \tag{20.80a}$$

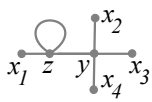
$$F(k_1) = \begin{cases} -\dfrac{i(3\lambda\hbar c)^2}{8\pi^2} \lim\limits_{\Lambda\to\infty} \ln\left(\dfrac{\Lambda^2\hbar^2}{m_0^2c^2}\right) & \text{if } k_1^2 = m_0^2c^2/\hbar^2 \\ (20.79) \quad \text{else} \end{cases} \tag{20.80b}$$

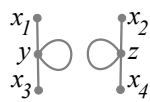
The divergence of this loop is "only" logarithmic, i. e. less catastrophic than in case of the tadpole of ψ^4-theory.

20.3.3 Four-Point Scattering in 2. Order

We still need to investigate two further examples of graphs with loops. Thereby we will find an important extension for the collection of Feynman-rules in the boxes 20.1 and 20.2. In second-order perturbation computation of ψ^4-theory, the scattering amplitude of two incoming and two outgoing particles consists of six connected diagrams:

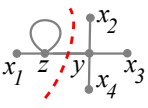
$$\overline{S}^{(2)} \mathrel{\widehat{=}} \underbrace{576 \cdot}_{\overline{S}^{(2)}_{eg}} \quad \underbrace{+1152 \cdot}_{\overline{S}^{(2)}_{ev}} \quad + \quad + 2304 \cdot \quad + 576 \cdot \quad + \ldots \tag{20.81}$$



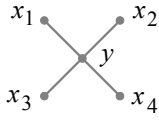


Two diagrams similar to the fourth, one with two tadpoles on one line, and one with a cactus on one line, are not displayed. The third and the fourth (and the two not displayed) diagrams differ only by tadpoles on in- or outgoing lines from two graphs, which we already evaluated before: The fourth graph is a modification of (20.7), and the third graph is a modification

of (20.16). We do not want to consider self-interactions of particles on their way to the proper interaction point as part of the scattering amplitude. Such diagrams can be "amputated" by cutting off the self-interaction, as indicated below by the dashed red line. Then the diagram can be interpreted as the product of the propagator-correction of an in- or outgoing particle, and the scattering in-between the various particles:

$$2304 \cdot$$ 

$$= 2 \cdot 4 \cdot 12 \cdot$$ 

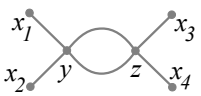
$$\cdot\, 24 \cdot$$ 

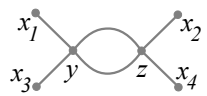
(20.82)

12 and 24 are the symmetry factors of the partial graphs. The self-interaction can be near any x_j ($\rightarrow$ factor 4), and the vertices can be permuted ($\rightarrow$ factor 2). Applying the same method, the self-interactions of the fourth graph in (20.81) can be amputated from the proper scattering event (20.7).

If, due to intersection of one line in a graph, one single particle can be completely cut-off from all other particles, then one says that this graph can be amputated. Graphs, which can be amputated, are also called "one-particle-reducible". A graph, which can not be amputated, is called one-particle-irreducible. 1PI is the abbreviation for the latter notion. As the amputated part is related only to one single particle, it is a self-interaction graph, which mutates due to renormalization "automatically" to the simple propagator-line. The tedious computation of graphs, which can be amputated, therefore would not result into an additional contribution to the scattering amplitude. Graphs, which can be amputated, are therefore skipped from the outset. This rule has been integrated into rule A of the boxes 20.1 and 20.2.

Hence only two graphs contribute to the amplitude of a scattering event with two incoming and two outgoing particles in second order perturbation computation of ψ^4-interaction:

$$\overline{S}^{(2)} \mathrel{\widehat{=}} \underbrace{576 \cdot}_{\overline{S}^{(2)}_{eg}}$$ 

$$\underbrace{+\,1152 \cdot}_{\overline{S}^{(2)}_{ev}}$$ 

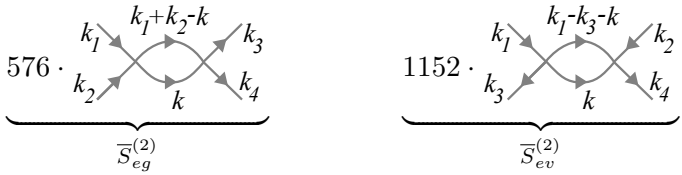
(20.83)

The index eg signifies: The both incoming particles couple to the same vertex. The index ev signifies: The both incoming particles couple to different vertices.

Firstly we check the symmetry factors. In case of $\overline{S}^{(2)}_{eg}$, there are 8 alternatives to couple x_1 to one of the vertices y or z, and 3 alternatives, to couple x_2 to the same vertex. There are 4 alternatives, to couple x_3 to the other vertex, and further 3 alternatives, to couple x_4 to the same vertex as x_3. There remain 2 alternatives for the construction of the two propagators $G(z-y)$. Hence the overall symmetry factor is 576.

In case of $\overline{S}^{(2)}_{ev}$, there is an additional symmetry factor 2, because x_3 may be contracted with the same vertex as x_1, or with the same vertex as x_2. Hence the overall symmetry factor is 1152.

In energy-momentum-space, the both graphs are looking like this:



$\overline{S}^{(2)}_{eg}$ $\overline{S}^{(2)}_{ev}$

As energy and momentum need to be conserved at each vertex, one could in the case $\overline{S}^{(2)}_{eg}$ label the loop's upper branch as well k_3+k_4-k, and in case of $\overline{S}^{(2)}_{ev}$ as well k_4-k_2-k. As for any loop, we find a wavenumber k, which is not fixed by a delta-function, and therefore might cause divergences of the graphs. We compute the graphs with the not-normalized coupling constant λ_0, because we will cure the divergences of these diagrams in chapter 22 due to renormalization of λ_0.

We apply the rules of box 20.2:

$$\overline{S}_{eg}^{(2)} = \Big(\prod_{j=1}^{4}\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega N}}\Big)\cdot$$
$$\cdot\, 2\pi\Omega\,\delta(k_1+k_2-k_3-k_4)\,\delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)}\cdot$$
$$\cdot\, \underbrace{\frac{576\,(-i\lambda_0\hbar^2c^2)^2}{2!\,\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\widetilde{G}(k)\,\widetilde{G}(k_1+k_2-k)}_{\equiv F(k_1+k_2)} \tag{20.84a}$$

$$F(k_1+k_2) \overset{(12.7)}{=} 288\,(-i\lambda_0\hbar^2c^2)^2\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\Big(\frac{i}{\hbar c}\Big)^2\cdot$$
$$\cdot\,\frac{1}{\Big(k^2-m^2c^2/\hbar^2+i\epsilon'\Big)\Big((k_1+k_2-k)^2-m^2c^2/\hbar^2+i\epsilon'\Big)} \tag{20.84b}$$

Aside from a factor 16, $F(k_1+k_2)$ is equivalent to the expression for $F(k_1)$ in (20.71b). Just k_1 is now replaced by k_1+k_2, and this time it is the coupling constant, but not the mass, which is applied not-normalized. Thus we can immediately conclude the result from (20.80):

$$F(k_1+k_2) = -2\,\frac{i(3\lambda_0\hbar c)^2}{\pi^2}\int_0^1 \mathrm{d}\xi\, V\lim_{\Lambda\to\infty}\ln\Big(\frac{\Lambda^2}{K^2}\Big) \tag{20.85a}$$

$$K^2 = \frac{4m^2c^2}{\hbar^2}\Big((\xi^2-\xi)\frac{(k_1+k_2)^2\hbar^2}{4m^2c^2}+\frac{1}{4}\Big) \tag{20.85b}$$

$$V \equiv \begin{cases}1 & \text{if } K^2\geq 0\\ 0 & \text{else}\end{cases} \tag{20.85c}$$

As the four in- and outgoing particles are observed particles with $k_j^2 = m^2c^2/\hbar^2$, we can compute $(k_1+k_2)^2$ in the center-of-mass system $\boldsymbol{k}_2=-\boldsymbol{k}_1$:

$$(k_1+k_2)^2 = \underbrace{(k_1^0)^2-\boldsymbol{k}_1^2}_{m^2c^2/\hbar^2}+\underbrace{(k_2^0)^2-\boldsymbol{k}_2^2}_{m^2c^2/\hbar^2}+\underbrace{2k_1^0k_2^0}_{2(\boldsymbol{k}_1^2+m^2c^2/\hbar^2)}+2\boldsymbol{k}_1^2 =$$
$$= 4m^2c^2/\hbar^2+4\boldsymbol{k}_1^2 > 4m^2c^2/\hbar^2$$

Thus K^2 is always negative at $\xi = 0.5$, always positive at $\xi = 0$ and $\xi = 1$, and zero at $\xi = \eta$ and at $\xi = 1 - \eta$. The second of the three integrals

$$\underbrace{\int_0^1 \mathrm{d}\xi \ldots}_{F(k_1+k_2)} \overset{(20.78)}{=} \underbrace{\int_0^\eta \mathrm{d}\xi \ldots}_{F_a(k_1+k_2)} + \underbrace{\int_\eta^{1-\eta} \mathrm{d}\xi \ldots}_{F_b(k_1+k_2)=0} + \underbrace{\int_{1-\eta}^1 \mathrm{d}\xi \ldots}_{F_c(k_1+k_2)}$$

is set to zero due to the factor $V =$ (20.85c). Hence it does not contribute to the probability amplitude. In the high-relativistic case $\boldsymbol{k}_1^2\hbar^2/(mc)^2 \gg 1$ and $\boldsymbol{k}_2^2\hbar^2/(mc)^2 \gg 1$, the contributions of the two integrals F_a and F_c are marginal. Therefore $\overline{S}^{(2)}_{eg}$ is very small in the high-relativistic case, and actually negligible versus $\overline{S}^{(2)}_{ev}$, as we will see immediately. In the sequel of (20.75) we introduced the notion of a "competing graph". The competing graph, which is extracting probability amplitude out of the left graph in (20.88), interestingly in this case is the first-order graph (20.16).

As there is an additional symmetry factor 2 in $\overline{S}^{(2)}_{ev}$ as compared to $\overline{S}^{(2)}_{eg}$,

$$F(k_1 - k_3) = -4\,\frac{i(3\lambda_0\hbar c)^2}{\pi^2} \int_0^1 \mathrm{d}\xi \lim_{\Lambda\to\infty} \ln\Big(\frac{\Lambda^2}{L^2}\Big) \tag{20.86a}$$

$$L^2 = (k_1 - k_3)^2\Big((\xi^2 - \xi) + \frac{m^2c^2}{(k_1 - k_3)^2\hbar^2}\Big)\,. \tag{20.86b}$$

We don't need to consider different cases of L^2, because in the center-of-mass system $\boldsymbol{k}_3 = -\boldsymbol{k}_1$

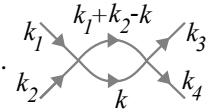
$$(k_1 - k_3)^2 = \underbrace{(k_1^0)^2 - \boldsymbol{k}_1^2}_{m^2c^2/\hbar^2} + \underbrace{(k_3^0)^2 - \boldsymbol{k}_3^2}_{m^2c^2/\hbar^2} - \underbrace{2k_1^0k_3^0}_{2(\boldsymbol{k}_1^2+m^2c^2/\hbar^2)} - 2\boldsymbol{k}_1^2 = -4\boldsymbol{k}_1^2 < 0\,.$$

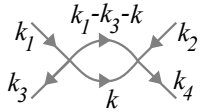
Therefore L^2 can never be negative.

In the highly relativistic case $\boldsymbol{k}_1^2\hbar^2/(mc)^2 \gg 1$ and $\boldsymbol{k}_3^2\hbar^2/(mc)^2 \gg 1$, the integration over ξ results into

$$\int_0^1 \mathrm{d}\xi \, \ln\Big(\frac{\Lambda^2}{L^2}\Big) = \ln\Big(\frac{-\Lambda^2}{(k_1-k_3)^2}\Big) - \int_0^1 \mathrm{d}\xi \, \ln(\xi-\xi^2) \, = \\ = \ln\Big(\frac{-\Lambda^2}{(k_1-k_3)^2}\Big) + 2 \approx \ln\Big(\frac{+\Lambda^2}{|(k_1-k_3)^2|}\Big) \, . \tag{20.87}$$

Thus the total result for both diagrams is:

$$576 \cdot$$ 

$$+ 1152 \cdot$$ 

$$\widehat{=} \, \overline{S}^{(2)}$$

$$\overline{S}^{(2)} = \Big(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega N}} \Big) \cdot \Big(F(k_1+k_2) + F(k_1-k_3) \Big) \cdot \\ \cdot \, 2\pi\Omega \, \delta(k_1+k_2-k_3-k_4) \, \delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)} \tag{20.88a}$$

$$F(k_1+k_2) + F(k_1-k_3) = (20.85) + (20.86) \, \approx \tag{20.88b}$$

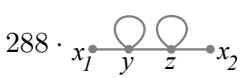
$$\approx -4 \, \frac{i(3\lambda_0\hbar c)^2}{\pi^2} \lim_{\Lambda\to\infty} \ln\Big(\frac{\Lambda^2}{|(k_1-k_3)^2|}\Big) \quad \text{if} \begin{cases} \boldsymbol{k}_1^2\hbar^2/(mc)^2 \gg 1 \\ \boldsymbol{k}_3^2\hbar^2/(mc)^2 \gg 1 \end{cases}$$

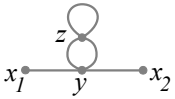
The loop (20.88) of ψ^4-theory is diverging logarithmically. We will clarify in chapter 22, how this divergence can be cured due to renormalization of the coupling constant.

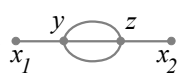
20.3.4 The Double-Loop of ψ^4-Theory

We still need to consider the second-order correction of the propagator of ψ^4-theory. It is consisting of three graphs:

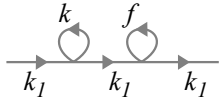
$$G^{(2)}(x_2-x_1) \overset{(20.47)}{\widehat{=}} \tag{20.89}$$

$$\underbrace{288 \cdot}_{G^{(2a)}(x_2-x_1)}$$ 

$$\underbrace{+ 288 \cdot}_{G^{(2b)}(x_2-x_1)}$$ 

$$\underbrace{+ 192 \cdot}_{G^{(2c)}(x_2-x_1)}$$ 

To compute theses graphs, we apply the rules of box 20.4. Thereby we use the bare, not renormalized mass m_0 and the not renormalized field-operator $\psi_0(x)$.

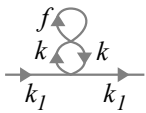
$288 \cdot$ 

$\widehat{=} \widetilde{G}^{(2a)}(k_1) \; =$

$$= \frac{2}{2!}\,\widetilde{G}(k_1)\,\underbrace{\frac{12(-i\lambda\hbar^2c^2)}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\widetilde{G}(k)}_{F\,,\text{see (20.67)}}\cdot$$

$$\cdot\,\widetilde{G}(k_1)\,\underbrace{\frac{12(-i\lambda\hbar^2c^2)}{\Omega}\sum_{\boldsymbol{f}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}f^0}{2\pi}\,\widetilde{G}(f)}_{F\,,\text{see (20.67)}}\,\widetilde{G}(k_1) \qquad (20.90)$$

For the cactus-diagramm $G^{(2b)}(x_2 - x_1)$ one gets, using the same rules:

$288 \cdot$ 

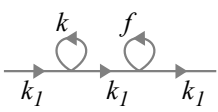
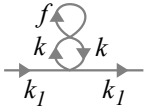
$\widehat{=} \widetilde{G}^{(2b)}(k_1) =$

$$= \frac{2}{2!}\,\widetilde{G}(k_1)\,\underbrace{\frac{12(-i\lambda\hbar^2c^2)}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\widetilde{G}(k)}_{F\,,\text{see (20.67)}}\cdot$$

$$\cdot\,\underbrace{\frac{12(-i\lambda\hbar^2c^2)}{\Omega}\sum_{\boldsymbol{f}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}f^0}{2\pi}\,\widetilde{G}(f)}_{F\,,\text{see (20.67)}}\,\widetilde{G}(k_1) \qquad (20.91)$$

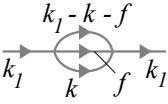
As a line 

may be amputated at any time, with same justification that line may be implanted at any time into a diagram. That means, that a factor $\widetilde{G}(k_1)$ may be inserted into $\widetilde{G}^{(2b)}(k_1)$. Thereby this graph becomes identical to $\widetilde{G}^{(2a)}(k_1)$, and we find the result:

$288 \cdot$  $+ 288 \cdot$  $\widehat{=}$

$$\widehat{=} \widetilde{G}^{(2a+2b)}(k_1) = \widetilde{G}(k_1)\, F\, \widetilde{G}(k_1)\, F\, \widetilde{G}(k_1) \quad \text{with } F = (20.67) \tag{20.92}$$

The structure of the third term in $\widetilde{G}^{(2)}(k_1)$ is different:

$192 \cdot$  $\widehat{=} \widetilde{G}^{(2c)}(k_1) = \widetilde{G}(k_1)\, J(k_1)\, \widetilde{G}(k_1)$ (20.93a)

$$J(k_1) \equiv \frac{192(-i\,\lambda\hbar^2c^2)^2}{2!\,\Omega^2} \sum_{\boldsymbol{k},\boldsymbol{f}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0\,\mathrm{d}f^0}{(2\pi)^2}\, \widetilde{G}(k)\, \widetilde{G}(f)\, \widetilde{G}(k_1-k-f)$$

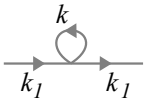
$$= \frac{192(-i\,\lambda\hbar^2c^2)^2}{2!\,\Omega^2} \sum_{\boldsymbol{k},\boldsymbol{f}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{\mathrm{d}f^0}{2\pi}\, \frac{(i/\hbar c)^3}{\left(k^2 - m^2c^2/\hbar^2 + i\epsilon'\right)} \cdot$$

$$\cdot\, \frac{1}{\left(f^2 - m^2c^2/\hbar^2 + i\epsilon'\right)\left((k_1-k-f)^2 - m^2c^2/\hbar^2 + i\epsilon'\right)} \tag{20.93b}$$

We will not explicitly calculate $J(k_1)$, but merely clarify due to comparison with the diagrams computed before, whether — and if, how — this graph diverges. Using

$$\frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4}\ ,$$

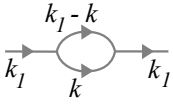
we consider the structure of the diverging terms of the graphs, which we computed before:



structure: $\dfrac{\mathrm{d}^4k}{k^2}$

$$F \overset{(20.50)}{\sim} \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{1}{k^2 - m_0^2c^2/\hbar^2 + i\epsilon'}$$

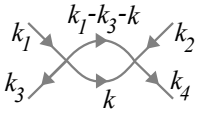
$$\overset{(20.67b)}{\sim} \lim_{\Lambda\to\infty} \left(\frac{\Lambda^2\hbar^2}{m_0^2c^2} - \ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right)\right) \tag{20.94a}$$



structure: $\dfrac{\mathrm{d}^4k}{k^4}$

$$F(k_1) \overset{(20.71b)}{\sim} \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{1}{\left(k^2 - m_0^2\frac{c^2}{\hbar^2} + i\epsilon'\right)\left((k_1-k)^2 - m_0^2\frac{c^2}{\hbar^2} + i\epsilon'\right)}$$

$$\overset{(20.80b)}{\sim} \lim_{\Lambda\to\infty} \ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right) \tag{20.94b}$$

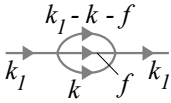


structure: $\dfrac{\mathrm{d}^4k}{k^4}$

$$F(k_1 - k_3) \overset{(20.84b)}{\sim} \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{1}{\left(k^2 - m^2c^2/\hbar^2 + i\epsilon'\right)} \cdot$$

$$\cdot \frac{1}{\left((k_1 - k_3 - k)^2 - m^2c^2/\hbar^2 + i\epsilon'\right)}$$

$$\overset{(20.88)}{\sim} \lim_{\Lambda\to\infty} \ln\left(\frac{\Lambda^2}{|(k_1-k_3)^2|}\right) \tag{20.94c}$$



$$\text{structure}: \frac{\mathrm{d}^4k}{k^4}\frac{\mathrm{d}^4f}{f^4}$$

$$J(k_1) \overset{(20.93b)}{\sim} \frac{1}{\Omega^2}\sum_{\boldsymbol{k},\boldsymbol{f}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\frac{\mathrm{d}f^0}{2\pi}\frac{1}{\left(k^2 - m^2c^2/\hbar^2 + i\epsilon'\right)}\cdot$$
$$\cdot\frac{1}{\left(f^2 - m^2c^2/\hbar^2 + i\epsilon'\right)\left((k_1 - k - f)^2 - m^2c^2/\hbar^2 + i\epsilon'\right)} \tag{20.94d}$$

Comparing this to the graphs above, it is plausible to assume a logarithmic divergence of $J(k_1)$:

$$J(k_1) = C\left((k_1)^2\right)\cdot\lim_{\Lambda\to\infty}\ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right) \tag{20.94e}$$

C is a finite function of $(k_1)^2$. We found the form of this result not by calculation, but took it from the literature. Note the important difference inbetween the tadpole and the three other loops: Different from $F(k_1) =$ (20.94b), $F(k_1 - k_3) =$ (20.94c), and $J(k_1) =$ (20.94d), the tadpole's diverging function $F =$ (20.94a) is independent of the propagating particle's wavenumber k_1.

From (20.94) obviously this general rule can be concluded:

$$\text{structure}: \frac{\mathrm{d}^4k}{k^n} \implies \begin{cases} n = 2 : & \text{quadratic divergence} \\ n = 4 : & \text{logarithmic divergence} \\ n \geq 6 : & \text{no divergence} \end{cases} \tag{20.95}$$

Each vertex in a loop adds a factor k^{-2}. Therefore no loop with three or more vertices can diverge. We have already seen all loops with less than three vertices, i. e. one in ψ^3-theory and three in ψ^4-theory. No further loops with less than three vertices exist. (Clearly these loops will appear again as partial diagrams inside graphs of higher order.) All other diagrams of ψ^3- and ψ^4-theory, which do not contain these four types of loops, are convergent. Once a method is found, by which the four diverging diagrams can be converted to convergent diagrams, the total ψ^3- and ψ^4-theory is convergent. No new types of divergences will show up in any higher order of

perturbation theory. In the two following chapters we will clarify, how the convergence of the four loop-graphs can be enforced due to renormalization.

21 Renormalization: Physical Interpretation

Divergences are the unpleasant hallmark of relativistic quantum field theory. In this chapter we will discuss the treatment of the divergences, which appear in loop-diagram-computations of interacting quantum fields. We have already computed all diverging loop-diagrams of ψ^3-theory and ψ^4-theory in the previous chapter. And we will compute all diverging loop-diagrams of quantum-electrodynamics in chapter 26. All of them can be cured by a method of renormalization, which in principle is always the same.

After it's detection by end of the forties, this renormalization-procedure remained a mystery for almost a quarter of a century. It was a baffling mathematical trick. It worked, and provided correct results with a precision, which had been unknown in physics before. But nobody understood it's secret of success. In the second article linked under [24] a short, worth reading review can be found about the intricate paths and meanders, which were followed by theorists in those years.

Light into the darkness came due to the detection of the renormalization group, which will be presented in the next section. Thereafter, we will explicitly perform the renormalization of a simple minimal-model of quantum field theory. In this chapter's last section, we will append a short discussion about effective theories, and the possible existence of a fundamental length.

21.1 The Renormalization-Group

This section — and in some sense the complete chapter — is concerned with the question, why and under which conditions physical processes on different scales of length will (or will not) de-couple. Wilson [26] considered as an example the water of an ocean. The scale of length of it's size is 10^7 m.

This is as well the scale of length of the tides and of it's large-scale ocean currents. In contrast, the scale of length of the water's surface waves is about 10^{-3} m up to 10^{+2} m. For a description of these waves, it does not matter whether they are waves in the pacific ocean or in the black sea. And also the movements of single water molecules (scale of length 10^{-10} m), of which the water is composed, are of no relevance for a description of waves. That is the normal case: Processes on quite different scales of length in most cases are well decoupled from another, and therefore can be investigated independently.

The same holds true for the continuum physics of liquids and solids, in which the crystal grid constant G resp. the distance of nearest neighbor-molecules is defining a scale of length, which is setting a limit to the continuum-description. Applying mean-field theory, macroscopic properties like hardness, magnetization, viscosity, speed of sound, etc. can in most cases be described appropriately on scales of length $L \gg G$, without the need to make explicit reference to their atomic structure or to the length G. Only if the continuum-description is extended down to the order of magnitude of the length G, nonsensical results are to be expected.

Good decoupling of phenomena on a macroscopic scale of length L from phenomena on the scale G of distances between neighbor molecules is the normal case in classical continuum theory. As to any good rule, there exist exceptions to this rule: In phase-transitions, a very strong coupling is observed between quite different scales of length. The paradigmatic example is the magnetization of a ferromagnetic solid. The magnetization is caused by the alignment of the magnetic dipole moments of electrons. Their detailed description must happen on the scale of length of the solid's crystal grid constant of typically $G \approx 10^{-10}$ m. In contrast, the macroscopic magnetization M is observed on length scales of typically $L > 10^{-6}$ m.

A ferromagnetic solid is tending to a state of macroscopic magnetization, because it's energy decreases if neighbor spins are aligned parallel. At the same time, the thermal energy of the spins is countervailing the alignment. Below the Curie-temperature T_c, which also is called the critical temperature, the interaction of spins is dominating. Above T_c there is no macroscopic observable magnetization, because the thermal energy of the spins is larger than the possible gain of energy due to parallel spin alignment. In this

range of high temperature, the orientation of spins is correlated only over ranges of order of magnitude G. This is to say, that the orientation of a spin is correlated only to the orientation of it's nearest neighbors, and possible to the orientation of it's over-next nearest neighbors, but not to spins in larger distance.

That changes drastically, if the temperature is lowered down to T_c. Then the correlation length increases to the diameter of the complete solid. This means in theory, that the correlation length diverges. The orientation of one spin, which before only acted onto neighbor spins on the length scale of the grid constant G, now is affecting other spins in arbitrary distance.

The antagonistic tendencies of magnetic interaction and thermal energy of the spins can be modeled by a parameter λ, which is called coupling strength. In the most simple case, it is inversely proportional to temperature: $\lambda \sim 1/T$. The coupling strength is indicating the probability, that neighbor spins will be aligned. This probability is $1/2$ at $T = \infty$ resp. $\lambda = 0$, and it is 1 at $T = 0$ resp. $\lambda = \infty$. In realistic models, λ is a complicated function, which does depend on the range and as well on the strength of the spin-spin-coupling. We choose a simplifying description, in which λ is a number (multiplied by a physical unit).

Once the coupling strength is known as a function of temperature, it is relatively simple to derive from the theory the macroscopic magnetization. In contrast, the computation of the coupling strength is quite difficult. The computation is possible using the mean-field theory, but only far-off from the Curie-temperature. Slightly below of T_c, the mean-field theory is predicting the magnetization $M \sim (T_c - T)^{1/2}$ — independent of the details on the atomic length-scale G, i. e. independent of the crystal structure of the ferromagnet. This prediction turned out to be qualitatively correct, but quantitatively wrong. The magnetization at T_c indeed is "universal", i. e. independent of the details on the scale of length G. But it's value is $M \sim (T_c - T)^{1/3}$.

The universality of the phase transition encouraged the search for an effective theory, which should describe the macroscopic magnetization correctly on the length-scale L, independent of details on the atomic scale G, in particular at temperatures near T_c. The mean-field theory fails near T_c, because it is based on the assumption that the correlation-length, which

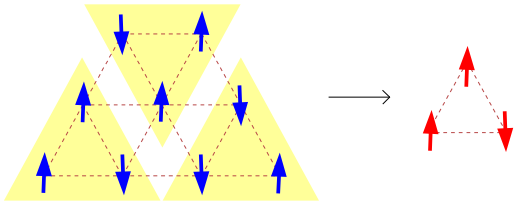


Fig. 21.1 : Spins j_n (left) and spins j_{n+1} (right)

characterizes the range of significant mutual interaction of spins, does not depend appreciably on temperature. But actually the correlation-length is changing drastically near the critical temperature.

A better method for the computation of the coupling strength between the spins in a magnet at temperatures near T_c has been found in form of the block-spin-method of Kadanoff[1] and the theory of the renormalization group due to Wilson[2]. To explicate the blockspin-method, we consider the two-dimensional Ising-model, which is depicted left in figure 21.1 . Nine spins, which can assume only the values $j_0 = +1$ or $j_0 = -1$, are arranged in a hexagonal grid with grid-constant G. The coupling strength between nearest neighbors is λ_0, while the coupling strength between spins, which are not nearest neighbors, is by definition zero in this model. The center spin has 6 nearest neighbors, the spins in the bottom corners have 2 nearest neighbors, and the remaining spins have 3 or 4 nearest neighbors. As each spin can assume two directions, there are in total $2^9 = 512$ different possible arrangements of these spins.

The coupling strength λ_0 determines the probability P_0 of parallel resp. the probability $1 - P_0$ of antiparallel orientation of neighbor spins. Thus the probability of each of the 512 possible configurations can be computed. Furthermore in each configuration the spins j_0 are combined to groups of 3, which in figure 21.1 are marked by yellow triangles. According to the majority principle, these 3 spins define a blockspin j_1. The blockspin is $j_1 = +1$, if minimum two of the spins j_0 in this block have the value $+1$.

[1] Leo Philip Kadanoff (∗1937)
[2] Kenneth Geddes Wilson (1936 - 2013)

Else the blockspin is $j_1 = -1$. The blockspins are indicated red in the right part of the figure.

There are only $2^3 = 8$ different possible configurations of the blockspins j_1, while there are 512 different configurations of the spins j_0. The sum of the probabilities of all different configurations of the spins j_0, which result into the identical configuration of the blockspins j_1, is the probability of this blockspin-configuration. From the probabilities of the different blockspin-configurations one can compute back to the probabilities P_1 of parallel resp. the probability $1-P_1$ of antiparallel orientation of neighbored blockspins. In general $P_1 \neq P_0$. (In [26, figure on page 166] this is explicitly demonstrated for a simple example.) From P_1 again the renormalized coupling strength λ_1 can be computed.

With the renormalized coupling strength λ_1, the next round of the blockspin-procedure is started. One again computes the probabilities of all 512 configurations of the spins j_1 and of the 8 configurations of the blockspins j_2. That results into the renormalized coupling constant λ_2. And so on.

Only the limited capability of the computers causes us to do the computation like this. Rather we would like to compute a larger model of 3^N spins, with N being a *very* large natural number. Then in the first step the probabilities of all 2^{3^N} possible configurations of the spins j_0 are computed, and thus the probabilities of the $2^{3^{N-1}}$ possible configurations of the blockspins j_1. From that again the value of the renormalized coupling constant λ_1 follows. The blockspin-grid is shrunk — as indicated in figure 21.1 — to the same grid-constant G as the original grid. But it has only a third of the size of the original grid. All structures (i. e. domains with parallel spin orientation), which were smaller than $3\,G$ at the start of the procedure, have disappeared from the model.

In the next step, the probabilities of all $2^{3^{N-1}}$ possible configurations of the spins j_1 are computed with the coupling constant λ_1. This gives at the same time the probabilities of the $2^{3^{N-2}}$ possible configurations of the blockspins j_2. From this again the renormalized coupling constant λ_2 is computed. After the n^{th} run of the blockspin-procedure, the number of spins in the grid is reduced to 3^{N-n}, which still shall be a very large number. All structures of size $< 3^n G$, which existed in the grid at the start of the procedure, have disappeared from the model. The stepwise coarsening of

the model is accompanied by the stepwise renormalization of the coupling constant. By this method one finds the sequence of paired values

$$\begin{aligned}&(\lambda_0, G) \to (\lambda_1, 3G) \to (\lambda_2, 3^2G) \to \dots \\ &\dots \to (\lambda_s, 3^sG) \to (\lambda_t, 3^tG) \to (\lambda_u, 3^uG) \to \dots ,\end{aligned} \tag{21.1}$$

which are called the "flow" of the renormalization group. Why is this a group? We define the transformation

$$\lambda_s \xrightarrow{T_{st}} T_{st}(\lambda_s) = \lambda_t , \tag{21.2}$$

and the concatenation of two transformations

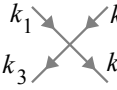
$$T_{su}(\lambda_s) = T_{tu}T_{st}(\lambda_s) = T_{tu}(T_{st}\lambda_s) = T_{tu}(\lambda_t) = \lambda_u . \tag{21.3}$$

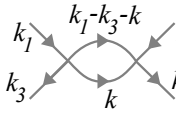
There also exists a unit element: T_{ss}. The inverse elements, however, are missing, because the stepwise computations of blockspins can not be inversed. The 3 spin orientations of the previous generation can not be reconstructed from the spin of a block. Therefore the renormalization-group $\{T\}$ of the model fig. 21.1 is not a group, but only a half-group. But the renormalization group of QFT, with which we will occupy ourselves in the sequel, is a full-fledged group.

21.2 A minimal QFT-Model

We want to compare the renormalization procedure of quantum field theory with the renormalization procedure of solid state theory, which has been illustrated in the previous section by the example of an Ising-model. For that purpose we construct a most simple, minimal model of QFT, which is concentrating onto the essential topics. As a guideline we use the example of the scattering event with two incoming and two outgoing particles in ψ^4-theory. If the energies of the scattering particles are highly relativistic, the probability amplitude for this event is

$$\overline{S} = \overline{S}^{(1)} + \overline{S}^{(2)} + \mathcal{O}(\lambda_0^3) \mathrel{\widehat{=}}$$

$$\mathrel{\widehat{=}} 24 \cdot$$ 

$$+ 1152 \cdot$$ 

$$+ \mathcal{O}(\lambda_0^3)$$

$$\overline{S} \overset{(20.19),(20.88)}{=} \Big(\prod_{\text{in}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{in}}}\Omega N}}\Big)\Big(\prod_{\text{out}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{out}}}\Omega N}}\Big) \cdot$$

$$\cdot\ 2\pi\Omega\,\delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}}\Big)\,\delta_{\sum_{\text{in}}\boldsymbol{k}_{\text{in}}\,,\,\sum_{\text{out}}\boldsymbol{k}_{\text{out}}} \cdot$$

$$\cdot\ \Big(\lambda_0(-24i\hbar^2c^2) + \lambda_0^2\,\frac{(-36i\hbar^2c^2)}{\pi^2}\lim_{\Lambda\to\infty}\ln\Big(\frac{\Lambda^2}{|(k_3-k_1)^2|}\Big)\Big) \tag{21.4}$$

Instead of $\overline{S} =$ (21.4), we define for the minimal model the simpler, but in it's structure essentially similar function

$$J(k) \equiv \lambda_0 f^{(1)}(k) + \lambda_0^2 f^{(2)}(k) + \lambda_0^3 f^{(3)}(k) + \dots\ . \tag{21.5a}$$

The coefficient

$$f^{(1)}(k) \equiv f_1 = \text{constant} \neq 0 \tag{21.5b}$$

is according to (21.4) by definition a constant. With regard to (20.84b), (20.71b), and (20.76), we choose for $f^{(2)}(k)$ the definition

$$f^{(2)}(k) \equiv u \int_0^{+\infty} \mathrm{d}R\ \frac{1}{R+k}\ . \tag{21.5c}$$

R and k are wavenumbers. The wavenumber k — same as the wavenumber q, which will be defined immediately — shall be interpreted in this model as

$$k \equiv \sqrt{\boldsymbol{k}^2}\ \neq\ \sqrt{(k^0) - \boldsymbol{k}^2} = \frac{mc}{\hbar}\ . \tag{21.6}$$

Different from the root of the square of a Minkowski-four-wavenumber, k thus is not a constant. Instead it's value is high for high-energy particles,

and low for low-energy particles. In u those factors are combined, which are independent of the integration variable R (which is a wavenumber). To design the model as simple as possible, no mass parameter is used, and only one incoming particle with wavenumber k is considered. $J(k)$ may e. g. be visualized as the scattering matrix of a massless and high-relativistic particle, which is incoming with wavenumber k, and is scattered by a very heavy nucleus. Like the term $\sim \lambda_0^2$ in (21.4), $f^{(2)}$ diverges logarithmically at $R \to \infty$. We assume, that the coefficients $f^{(n)}(k)$ of higher order are diverging as well, at least some of them.

These divergences shall be eliminated due to the renormalization of the coupling constant. That is to say: We expand the constant λ_0 with respect to another, unknown constant λ. The coefficients of the series expansions are called l_j:

$$\lambda_0 = l_1\lambda + l_2\lambda^2 + l_3\lambda^3 + \dots \tag{21.7}$$

This is a completely undetermined formula, because at the outset neither the constant λ nor the coefficients l_j are known. The series expansion is inserted into (21.5a):

$$\begin{aligned} J(k) &= (l_1\lambda + l_2\lambda^2 + l_3\lambda^3)f_1 + (l_1\lambda + l_2\lambda^2 + l_3\lambda^3)^2 f^{(2)}(k) + \\ &\quad + (l_1\lambda + l_2\lambda^2 + l_3\lambda^3)^3 f^{(3)}(k) + \dots \\ &= \lambda\Big(l_1 f_1\Big) + \lambda^2\Big(l_2 f_1 + l_1^2 f^{(2)}(k)\Big) + \\ &\quad + \lambda^3\Big(l_3 f_1 + 2l_1 l_2 f^{(2)}(k) + l_1^3 f^{(3)}(k)\Big) + \mathcal{O}(\lambda^4) \end{aligned} \tag{21.8}$$

Already at this point the formal trick of the renormalization procedure is becoming visible. In (21.5a) each factor λ_0^n is multiplied by exactly one coefficient $f^{(n)}$, with some of the $f^{(n)}$ being divergent. In contrast, in (21.8) each λ^n is multiplied by the sum of all $f^{(1)}, f^{(2)}, \dots, f^{(n)}$. Thus diverging $f^{(j)}$ can compensate mutually, provided that the coefficients l_j are chosen appropriately. $J(k) =$ (21.5a) and $J(k) =$ (21.8) are equal. But the algorithm of the series expansion in (21.8) is skillfully modified, such that each single expansion coefficient is finite — provided it meets some certain conditions, which will be specified in the sequel.

Now the coefficients l_j are determined successively in a perturbative computation according to the orders of λ. The value of J is determined experimentally at only one single wavenumber of the incoming particle, which we call q. It is a particular property of the renormalization procedure, that exactly one measured value is needed per parameter, which shall be renormalized (in this example λ_0). In first order of λ, one finds with that measured value the result

$$\begin{aligned} J_{\text{exp}}(q) &\overset{(21.8)}{=} \lambda l_1 f_1 + \mathcal{O}(\lambda^2) = \lambda f_1 + \mathcal{O}(\lambda^2) \\ \Longrightarrow \quad & l_1 = 1 \ . \end{aligned} \tag{21.9}$$

The choice $l_1 = 1$ is natural, but not compulsory. Another choice would merely lead to an unnecessary re-scaling of λ. As f_1 is by definition of our model finite and different from zero, and as the experimentally found value $J_{\text{exp}}(q)$ is finite as well, one gets in first order a finite, well-defined value of λ. The renormalization obviously was not really necessary for the first-order computation. We could have sticked as well to λ_0. In first order $J(k)$ does not depend on k. Thus the result is for arbitrary $k \neq q$

$$J(k) = \lambda f_1 + \mathcal{O}(\lambda^2) \ . \tag{21.10}$$

In second order of the coupling constant we compute the value of $J(k)$ to be

$$J(k) \overset{(21.8)}{=} \lambda f_1 + \lambda^2 \Big(l_2 f_1 + f^{(2)}(k) \Big) + \mathcal{O}(\lambda^3) \ . \tag{21.11}$$

Here we are facing the problem, that $f^{(2)}$ is diverging. To make a computation possible, $f^{(2)}$ is regularized due to the replacement of the upper integration limit $+\infty$ by a very large, but finite wavenumber Λ:

$$f^{(2)}_\Lambda(k) \equiv u \int_0^\Lambda \mathrm{d}R \, \frac{1}{R+k} = u \ln\Big(\frac{\Lambda + k}{k} \Big) \tag{21.12a}$$

$$f^{(2)}(k) = \lim_{\Lambda \to \infty} f^{(2)}_\Lambda(k) \tag{21.12b}$$

If J or a coefficient f resp. l is computed with the cut-off parameter Λ, then we mark it by the index $_\Lambda$. Thus one gets for J in second order

$$J_\Lambda(k) = \lambda f_1 + \lambda^2\Big(l_{2,\Lambda} f_1 + f_\Lambda^{(2)}(k)\Big) + \mathcal{O}(\lambda^3) \tag{21.13a}$$

$$J(k) = \lim_{\Lambda\to\infty} J_\Lambda(k)\ . \tag{21.13b}$$

The coefficient l_2 is determined by means of the experimentally found value:

$$J_{\exp}(q) = \underbrace{\lambda f_1}_{J_{\exp}(q)\text{, see (21.9)}} + \lambda^2\Big(l_2 f_1 + f^{(2)}(q)\Big) + \mathcal{O}(\lambda^3) \tag{21.14}$$

If λ is considered to be undetermined, then this equation is depending on two variables (namely λ and l_2), and therefore has no unique solution. But there is no reason why we should not stick to the value of λ as fixed in (21.9). Then

$$l_2 = \lim_{\Lambda\to\infty} l_{2,\Lambda} = -\frac{f^{(2)}(q)}{f_1} = -\lim_{\Lambda\to\infty}\frac{f_\Lambda^{(2)}(q)}{f_1} \tag{21.15}$$

must hold. f_1 is finite and constant, and $f_\Lambda^{(2)}$ is diverging logarithmically at $\Lambda\to\infty$. Consequently $l_{2,\Lambda}$ must diverge logarithmically at $\Lambda\to\infty$ as well. Thus

$$J_\Lambda(k) = \lambda f_1 + \lambda^2\Big(f_\Lambda^{(2)}(k) - f_\Lambda^{(2)}(q)\Big) + \mathcal{O}(\lambda^3)$$

for arbitrary wavenumbers k. This relation shall again be finite in the limit $\Lambda\to\infty$. It is finite if and only if

$$\lim_{\Lambda\to\infty}\Big(f_\Lambda^{(2)}(k) - f_\Lambda^{(2)}(q)\Big) = \text{finite}\ . \tag{21.16}$$

With our definition of $f^{(2)}$ this is true, because

$$\lim_{\Lambda\to\infty}\Big(f_\Lambda^{(2)}(k) - f_\Lambda^{(2)}(q)\Big) \overset{(21.12)}{=} \lim_{\Lambda\to\infty} u\ln\Big(\frac{(\Lambda+k)q}{k(\Lambda+q)}\Big) = u\ln\Big(\frac{q}{k}\Big) \tag{21.17}$$

is independent of Λ. Thus one gets for arbitrary k

$$J(k) = \lambda f_1 + \lambda^2 \underbrace{\Big(f^{(2)}(k) - f^{(2)}(q)\Big)}_{u \ln(q/k)} + \mathcal{O}(\lambda^3) \ . \tag{21.18a}$$

Note firstly, that the condition (21.16) imposes a very strong restriction onto the possible form of $f^{(2)}$. Analogous restrictions will arise for all further $f^{(n)}$ with $n > 2$. Note secondly, that due to the renormalization large and small wavenumbers have decoupled: In (21.5a) the small wavenumber k in the argument of $J(k)$ is coupled in second order of λ_0 due to the coefficient $f^{(2)}(k)$ very strong (even diverging) to the (infinitely) large wavenumber Λ. In (21.18a) the small wavenumber k in the argument of $J(k)$ is coupled in second order of λ only to the wavenumber q, which is of same order of magnitude as k. The large wavenumber Λ does not show up at all in (21.18a).

We have seen at the Ising-model, that the values of the coupling constant correlate to the fineness, with which the model is evaluated. (The more spins j_0 are combined to a single blockspin j_n, the coarser the becomes the model, resulting into an accordingly modified coupling strength λ_n.) Therefore it is no surprise, that in QFT as well the value of the renormalized coupling constant $\lambda(q)$ does depend on the fineness $\approx q^{-1}$, which is applied in the measurement of $J_{\text{exp}}(q)$.

If the experimental determination of J is performed with a different wavenumber q' of the incoming particle, then one gets instead of (21.18a) the equation

$$J(k) = \lambda' f_1 + \lambda'^2 \Big(f^{(2)}(k) - f^{(2)}(q')\Big) + \mathcal{O}(\lambda'^3) \ . \tag{21.18b}$$

We subtract (21.18b) from (21.18a), and insert (21.17):

$$0 = (\lambda - \lambda') f_1 + \lambda^2 u \ln\Big(\frac{q}{k}\Big) - \lambda'^2 u \ln\Big(\frac{q'}{k}\Big) + \mathcal{O}(\lambda^3) - \mathcal{O}(\lambda'^3)$$

Using $\lambda(q) \equiv \lambda$ and $\lambda(q') \equiv \lambda'$ and the series expansion

$$\lambda(q') = \lambda(q) + a_2\lambda^2(q) + \mathcal{O}\Big(\lambda^3(q)\Big) \ ,$$

one gets the result

$$\begin{aligned}\lambda(q') &= \lambda(q) + \lambda^2(q)\frac{u}{f_1}\ln\Big(\frac{q}{q'}\Big) + \mathcal{O}\Big(\lambda^3(q)\Big) \\ &= \lambda(q) + \frac{q'-q}{q}\underbrace{q\frac{\mathrm{d}\,\lambda(q)}{\mathrm{d}\,q}}_{\beta(\lambda,q)} + \mathcal{O}\Big(\lambda^3(q)\Big) \ . \end{aligned} \tag{21.19}$$

The second line is the start of a Taylor-expansion, and it contains the definition of the beta-function. If $\beta > 0$, the the coupling becomes stronger at increasing q' resp. decreasing distance of the interacting particles. In chapter 26 we will see, that quantum electrodynamics is an example for this case. Charges are in QED surrounded by clouds of virtual charges, which screen them dielectrically from the outer world. If a second particle penetrates with high momentum into this cloud of charges and comes near to the first particle, then it feels a larger charge and consequently a stronger coupling than in larger distance. The value of the coupling constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \begin{cases} 1/137 \text{ at } k \approx 0 \\ 1/128 \text{ at } k \approx 100\,\mathrm{GeV}/(\hbar c) \end{cases} \tag{21.20a}$$

of QED is $\alpha \approx 1/137$ for large distance. At high wavenumbers of about $100\,\mathrm{GeV}/(\hbar c)$, as are needed for the creation of Z-bosons, the results of the measurements could be explained only by the assumption of a coupling constant $\alpha \approx 1/128$.

In case of $\beta < 0$, the coupling becomes weaker for increasing q resp. decreasing distance of the interacting particles. The is the case for the coupling constant of the strong interaction. The dependence of the strong coupling constant α_s on the momentum transfer $\hbar k$ is more pronounced than in the electromagnetic interaction. The values

$$\alpha_s \approx \begin{cases} 0.35 \text{ at } k \approx 2\,\mathrm{GeV}/(\hbar c) \\ 0.18 \text{ at } k \approx 10\,\mathrm{GeV}/(\hbar c) \\ 0.11 \text{ at } k \approx 120\,\mathrm{GeV}/(\hbar c) \end{cases} \tag{21.20b}$$

have been extracted from measurements. Quarks are at large wavenumbers (i. e. at small distances) "asymptotic free". But the bonding energy between them becomes infinitely large if one tries to extract a single quark out of a baryon.

As the β-function of QED is positive, while the β-function of QCD is negative, one may speculate that there should be a wavenumber K with $\alpha(K) = \alpha_s(K)$. Some estimations are predicting $K \approx 10^{15}\mathrm{GeV}/(\hbar c)$. Obviously an extrapolation extending thus far must be considered with much reservation.

The notion "coupling constant" is so deeply ingrained in the vocabulary of physicists, that it is still being used for the renormalized parameters, even though they are no constants. If this fact shall be emphasized, the notion "running coupling constants" is used. The experimentally confirmed running of the coupling constants of quantum fields is strong evidence for the assumption, that the conclusion by analogy from the renormalization procedure of the classical continuum-theory of solids and liquids to the renormalization procedure of QFT is correct. In both cases, the renormalization of coupling strength correlates with the coarsening of the models, and at the same time with the decoupling of different length-scales.

In perturbation theory of third order in λ one gets

$$\begin{aligned} J(k) &\overset{(21.8)}{=} \lambda l_1 f_1 + \lambda^2\Big(l_2 f_1 + l_1^2 f^{(2)}(k)\Big) + \\ &\quad + \lambda^3\Big(l_3 f_1 + 2 l_1 l_2 f^{(2)}(k) + l_1^3 f^{(3)}(k)\Big) + \mathcal{O}(\lambda^4) \\ &\overset{(21.15),(21.18a)}{=} \lambda f_1 + \lambda^2\Big(f^{(2)}(k) - f^{(2)}(q)\Big) + \\ &\quad + \lambda^3\Big(l_3 f_1 - \frac{2}{f_1} f^{(2)}(q) f^{(2)}(k) + f^{(3)}(k)\Big) + \mathcal{O}(\lambda^4)\ . \end{aligned}$$

If the measured value $J_{\mathrm{exp}}(q)$ is inserted into this equation, and if we request that the value of λ as fixed in (21.9) shall be retained, then

$$l_3 = \lim_{\Lambda\to\infty} l_{3,\Lambda} = \frac{1}{f_1}\Big(\frac{2}{f_1} f^{(2)}(q) f^{(2)}(q) - f^{(3)}(q)\Big) = \\ = \lim_{\Lambda\to\infty} \frac{1}{f_1}\Big(\frac{2}{f_1} f_\Lambda^{(2)}(q) f_\Lambda^{(2)}(q) - f_\Lambda^{(3)}(q)\Big) \tag{21.21}$$

must hold. We did not fix $f^{(3)}$ in our simple model. Therefore this expression may be finite or infinite. If $\lim_{\Lambda\to\infty} l_{3,\Lambda} = \pm\infty$ is accepted, then this equation has in any case a solution, because we are completely free in the choice of $l_{3,\Lambda}$.

Obviously the structure

$$l_n = \lim_{\Lambda\to\infty} l_{n,\Lambda} = \text{function}\Big(f^{(1)}(q), f^{(2)}(q), \ldots, f^{(n)}(q)\Big)$$

is recurring in all orders of perturbation theory. As l_n can be chosen arbitrarily, this equation has a solution in any order n, provided that $\lim_{\Lambda\to\infty} l_{n,\Lambda} = \infty$ is accepted.

For arbitrary k one gets in third order of perturbation theory

$$J(k) = \lambda f_1 + \lambda^2\Big(f^{(2)}(k) - f^{(2)}(q)\Big) + \lambda^3\Big(-\frac{2}{f_1} f^{(2)}(q)\, f^{(2)}(k) + \\ + f^{(3)}(k) + \frac{2}{f_1} f^{(2)}(q) f^{(2)}(q) - f^{(3)}(q)\Big) + \mathcal{O}(\lambda^4)\ . \tag{21.22}$$

This equation is only reasonable, if firstly

$$\Big(-\frac{2}{f_1} f^{(2)}(q)\, f^{(2)}(k) + f^{(3)}(k) + \frac{2}{f_1} f^{(2)}(q) f^{(2)}(q) - f^{(3)}(q)\Big) = \\ = \lim_{\Lambda\to\infty}\Big(-\frac{2}{f_1} f_\Lambda^{(2)}(q)\, f_\Lambda^{(2)}(k) + f_\Lambda^{(3)}(k) + \frac{2}{f_1} f_\Lambda^{(2)}(q) f_\Lambda^{(2)}(q) - \\ - f_\Lambda^{(3)}(q)\Big) = \text{finite}\ , \tag{21.23}$$

and if secondly (21.16) continues to be valid.

For the computation of the quantity $J(k)$, we have renormalized the coupling constant, using a measured value

$$J_{\exp}(q) \overset{(21.9)}{=} \lambda f_1 + \mathcal{O}(\lambda^2)$$

which was found at the wavenumber q of the incoming particle. In (21.18) we have derived the concept of the running coupling constant due to the requirement, that the value of the observable $J(k)$ must not depend on the wavenumber q, which is applied for the experimental determination of $J_{\exp}(q)$:

$$J_{q,\lambda}(k) = J_{q',\lambda'}(k) = J_{q'',\lambda''}(k) = \dots \tag{21.24}$$

If this is compared with the

Renormalization-flow of the Ising-Model:

$$\begin{aligned}&(\lambda_0, G) \;\to\; (\lambda_1, 3G) \;\to\; (\lambda_2, 3^2G) \;\to\; \dots \\ &\dots \;\to\; (\lambda_s, 3^sG) \;\to\; (\lambda_t, 3^tG) \;\to\; (\lambda_u, 3^uG) \;\to\; \dots\ ,\end{aligned} \tag{21.25}$$

then the

Renormalization-flow of QFT:

$$(\lambda_0, \infty) \leftrightarrow \dots \leftrightarrow (\lambda, q) \leftrightarrow (\lambda', q') \leftrightarrow (\lambda'', q'') \leftrightarrow \dots\ ,$$

becomes visible, which is correlating the renormalized coupling constant with the length-scale q^{-1}, at which the experimental determination of $J_{\exp}(q)$ has been performed. As a (quasi-)continuum of wavenumbers is experimentally accessible in field-theory, the flow of the renormalization group is described as a continuum of infinitely many infinitesimal small steps:

$$\boxed{\lambda(q) \;\to\; \lambda(q + \mathrm{d}q) \overset{(21.19)}{=} \lambda(q) + \frac{\mathrm{d}q}{q}\, \underbrace{q\, \frac{\mathrm{d}\lambda(q)}{\mathrm{d}q}}_{\beta(\lambda,q)}} \tag{21.26a}$$

Finite steps can be described by integrals:

$$\boxed{\lambda(q_r) \;\to\; \lambda(q_s) = \lambda(q_r) + \int_{q_r}^{q_s} \frac{\mathrm{d}k}{k}\, \beta(\lambda, k)} \tag{21.26b}$$

The renormalization group of QFT is a full-fledged group. Different from the half-group of blockspin-transformations (21.2), the transformations (21.26) can be performed in direction to larger or to smaller wavenumbers q, i. e. for each transformation there exists as well the inverse transformation.

In spite of the obvious analogy between renormalization in solid-state physics and renormalization in QFT, a fundamental difference must not be overlooked: The renormalization-flow of the Ising-model is starting with the pair of values (λ_0, G), while the renormalization-flow of QFT is starting with (λ_0, ∞). Quantum field theory is constructed, as if it's formalism would be valid without changes up to infinitely large wavenumbers, i. e. down to infinitesimal small distances. This is the reason for the divergences, which are encountered in QFT. Even in the most "extreme" pair of values (λ_0, G) in the renormalization-flow of the Ising-model, λ_0 and G both are finite. In constrast, the coupling constant

$$\lambda_0 \overset{(21.7)}{=} l_1\lambda + l_2\lambda^2 + l_3\lambda^3 + \ldots$$

of not renormalized QFT is infinitely large, and does hold only at infinitely small distance, because the coefficient $l_2 =$ (21.15) diverges, and in general further l_n are diverging as well.

Seen from a formal point of view, one is working in QFT at a critical point. The strong coupling in-between quite different scales of length, which must be compensated due to renormalization, does show up in the classical continuum-theories of liquids and solids only near critical temperatures. Appreciable technical skills of the experimentalists are required, to adjust a system with sufficient precision to T_c and allow for measurements of the parameters. In contrast, temperature is not considered at all in quantum field theory. Quantum fields are "automatically" at any time adjusted to a critical state. The critical state is characterized by a correlation-length which is very large in comparison to the evaluated scale of length. If — as in not renormalized QFT — the evaluated scale of length is assumed to

be zero, then of course any finite correlation-length seems to be infinitely large in comparison. Thus the formal criterion of a critical state is in not renormalized QFT always fulfilled.

21.3 The fundamental Length

For any good theory (in other theories we are anyway not interested) there exists some certain range of applicability, within which it can be reasonably used with correct results. The range of applicability of Newton's mechanics for example is characterized by the fact, that expressions $\mathcal{O}(v^2/c^2)$ are negligibly small. Here v is the typical velocity of objects relative to the observer, and c is the speed of light in vacuum. If a mechanical process does not meet this condition, then it must be described by relativistic mechanics, from which Newton's mechanics follow in the limit $(v^2/c^2) \to 0$.

A quantum-theoretical description is required for processes, in which terms $\mathcal{O}(\hbar/S)$ are significant. Here S is the action of the observed system. If terms $\mathcal{O}(\hbar/S)$ are negligible, then that process belongs to the range of applicability of the classical (Newtonian or relativistic) theory.

In an article [47], which is worth reading still by today, Heisenberg in 1938 speculated that the divergences of quantum field theory should be considered as an indication, that QFT is the low-energy limit of a yet unknown more general theory. Like Newton's mechanics turned out to be on the one hand the limit of relativistic mechanics, which depend on the appropriate handling of the constant of nature c, and on the other hand as the limit of quantum mechanics, which depend on the appropriate handling of the constant of nature $\hbar$, the deficient quantum field theory would according to Heisenberg's assumption sooner or later turn out to be the limit of a theory, which depends on the appropriate handling of a still unknown constant of nature r. The dimension of r should be length, and it's order of magnitude should be 10^{-15}m.

By today we know, that the value 10^{-15}m was a much to high guess. Since those days, quantum electrodynamics has been evaluated experimentally down to about 10^{-18}m, without a new fundamental length coming to the fore of the experimentalists. In the younger discussion it is sometimes guessed, that the inverse $K^{-1} \approx 10^{-30}$m of the wavenumber, at which possibly the

coupling constants of strong, weak, and electromagnetic interactions are of same order of magnitude, might define a fundamental length.

Certainly the Planck-length and Planck-time

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \cdot 10^{-35}\,\mathrm{m}\ ,\ t_P = \frac{l_P}{c} = 5.4 \cdot 10^{-44}\,\mathrm{s}\ , \tag{21.27}$$

which are constructed by combinations of the quantum of action, the gravitational constant, and the speed of light, set a lower limit to the reasonable use of our notions of space intervals and time intervals. This assumption can be confirmed by the following consideration: If we want to speak reasonably about some certain minimum length r, then we must have — at least in principle — a probe which can measure that length. According to Heisenberg's indeterminacy relations, a particle cannot be localized with an accuracy better than half it's reduced Compton wavelength, i. e. better than half it's inverted invariant wave-number κ. This localization must not be less than two times the Schwarzschild radius r_S, because otherwise the particle would collapse to a black hole:

$$r \approx \frac{1}{2\kappa} \geq 2r_S = \frac{4G}{c^2} \cdot \frac{\kappa\hbar}{c}$$

$$\kappa \leq \sqrt{\frac{c^3}{8G\hbar}} = \frac{1}{l_{\mathrm{Planck}}\sqrt{8}} = 2.2 \cdot 10^{34}\mathrm{m}^{-1} \tag{21.28a}$$

$$r \approx \frac{1}{2\kappa} \approx 2 \cdot 10^{-35}\mathrm{m} \overset{(21.27)}{\approx} l_P \tag{21.28b}$$

Instead of venture on speculations, we confine ourselves to the cautious formulation: If there should exist a fundamental length r, which marks the limit of the range of applicability of the quantum field theories as known by today, then it is smaller (probably much smaller) than the scale of length which has been explored experimentally until today, i. e. smaller (probably much smaller) than $10^{-18}\mathrm{m}$, and greater or equal to the Planck length:

$$10^{-18}\mathrm{m} > r \geq l_P \approx 2 \cdot 10^{-35}\mathrm{m} \tag{21.29a}$$

For the maximum wavenumber κ we conclude:

$$\frac{2\pi}{10^{-18}\mathrm{m}} \approx 6 \cdot 10^{18}\mathrm{m}^{-1} < \kappa \leq \frac{1}{l_{\mathrm{Planck}}\sqrt{8}} \approx 2 \cdot 10^{34}\mathrm{m}^{-1} \ . \qquad (21.29b)$$

In the same way as the classical continuum-theory of solids is a useful effective theory, which produces reasonable results on scales of length $L \gg G$, QFT may be considered a useful effective continuum-theory of quantum fields, which produces reasonable results on scales of length $k^{-1} \gg r$. The

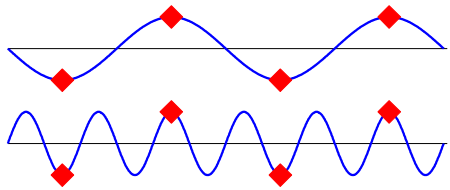


Fig. 21.2 : Phonon with wavelength $2\,G$ (top) and $2\,G/3$ (bottom)

existence of a fundamental length due to the grid constant G in solid-state physics implies, that an integral over the wavenumbers of phonons must be cut-off at $k \approx \pi/G$ by the latest. In figure 21.2 two transversal phonons of a one-dimensional solid, consisting of red atoms, are sketched. Even though to the bottom phonon a three times larger wavenumber is ascribed theoretically than to the top phonon, actually both phonons are identical. Therefore the integration over wavenumbers $k > \pi/G$ would be pointless and give unreasonable results.

In total, the assumption of the length r results into two changes only: Firstly the limits $\Lambda \to \infty$ are omitted. Secondly all divergences disappear, because $\Lambda_{\mathrm{max}} = \kappa$ is finite. But the quantum field theories still need renormalization, because we do not know the values of r and κ, and because the coupling constant anyway must be adapted by (21.26) to the wavenumbers, at which the theory is being tested experimentally.

22 Renormalization of ψ^s-Theory

We have identified in section 20.3 the four diverging loop-diagrams occurring in ψ^3- and ψ^4-theory, and listed them in (20.94). In this chapter we will clarify, how those divergences can be removed due to the renormalization of mass, coupling constant, and field-operator.

We will use the following notations: m_0 is the not renormalized mass, λ_0 is the not renormalized coupling constant, $\psi_0(x)$ is the not renormalized field-operator. m without zero is denoting the renormalized mass, λ without zero the renormalized coupling constant, $\psi(x)$ without zero the renormalized field-operator. As defined in (19.21), operators and propagators with the index (W) are denoting the complete quantities of the interacting systems, while the same quantities without that index are operators and propagators in the interaction picture.

There are two types of corrections for the propagator

$$G^{(W)}(x_2 - x_1) = \langle 0| T\psi_{(W)}(x_1)\, \psi_{(W)}(x_2) |0\rangle = \sum_{n=0}^{\infty} G^{(n)}(x_2 - x_1)$$

of ψ^4-theory, namely tadpole- and cactus-diagrams like (20.92), and double-loop diagrams like (20.93a). In zeroth order of perturbation theory, $G^{(W)}$ is identical to the Feynman-propagator:

$$G^{(0)}(x_2 - x_1) \equiv G(x_2 - x_1) \overset{(15.43)}{=} \langle 0| T\psi_0(x_1)\psi_0(x_2) |0\rangle \tag{22.1}$$

In first order, there is only the tadpole. In second order, there are the double-loop, the cactus consisting of two tadpoles, and two tadpoles in series. In higher orders of perturbation theory there are combinations of these diagrams, e. g. double-loops with tadpoles on their branches, or serial replications of diagrams which are already known from lower orders. Serial

replications are one-particle-reducible (definition below (20.82)), because they can be split into two complete diagrams by cutting one single line. The graph with one tadpole each on the three branches of the double-loop is a one-particle-irreducible graph (abbreviation: 1PI) of order $\mathcal{O}(\lambda^5)$. We define

$$F_{1\mathrm{PI}} = F + J + \ldots + \ldots$$

as the sum of all one-particle-irreducible graphs GFG, GJG, ..., which are showing up in any order of perturbation theory. From (20.67) and (20.92) it becomes visible, that these adders to $G^{(W)}$ are the elements of a geometrical series:

$$\begin{aligned}
\widetilde{G}^{(W)} &= \widetilde{G} + \widetilde{G}F_{1\mathrm{PI}}\widetilde{G} + \widetilde{G}F_{1\mathrm{PI}}\widetilde{G}F_{1\mathrm{PI}}\widetilde{G} + \\
&\quad + \widetilde{G}F_{1\mathrm{PI}}\widetilde{G}F_{1\mathrm{PI}}\widetilde{G}F_{1\mathrm{PI}}\widetilde{G} + \ldots = \frac{\widetilde{G}}{1 - F_{1\mathrm{PI}}\widetilde{G}} = \\
&\overset{(12.7)}{=} \frac{i}{\hbar c\Big(k_1^2 - m_0^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)\Big(1 - \dfrac{iF_{1\mathrm{PI}}}{\hbar c(k_1^2 - m_0^2\frac{c^2}{\hbar^2}) + i\epsilon'}\Big)} \\
&= \frac{i}{\hbar c\Big(k_1^2 - m_0^2\frac{c^2}{\hbar^2} - iF_{1\mathrm{PI}}\frac{1}{\hbar c} + i\epsilon'\Big)}
\end{aligned} \tag{22.2}$$

This series will certainly converge, because in each term in $F_{1\mathrm{PI}}$ there is the coupling constant λ, for which we assumed $\lambda^n \to 0$ in the limit $n \to \infty$. In $F_{1\mathrm{PI}}$ there are terms, which are independent of k_1 (namely the tadpoles and the cacti), and terms, which depend on k_1^2 (namely all diagrams with double-loops). We expand $F_{1\mathrm{PI}}\frac{1}{\hbar c}$ in a Taylor-series around $k_1^2 = m_0^2c^2/\hbar^2$:

$$\begin{aligned}
F_{1\mathrm{PI}}\tfrac{1}{\hbar c} &= A + (k_1^2 - m_0^2\tfrac{c^2}{\hbar^2})B + \underbrace{\sum_{n=2}^{\infty}(k_1^2 - m_0^2\tfrac{c^2}{\hbar^2})^n C_n}_{\mathcal{O}(\lambda^4)} \\
A &\equiv F_{1\mathrm{PI}}\tfrac{1}{\hbar c}\Big|_{k_1^2 = m_0^2\frac{c^2}{\hbar^2}} \quad , \quad B \equiv \frac{1}{\hbar c}\frac{\partial F_{1\mathrm{PI}}}{\partial k_1^2}\Big|_{k_1^2 = m_0^2\frac{c^2}{\hbar^2}}
\end{aligned} \tag{22.3}$$

Contributions to C_n with $n \geq 2$ can come only from diagrams with minimum two double-loops, i. e. which are minimum $\mathcal{O}(\lambda^4)$. In A there is one term $\mathcal{O}(\lambda)$, namely the tadpole, and infinitely many terms of higher order:

$$\begin{aligned} A &= F_{1\mathrm{PI}} \tfrac{1}{\hbar c}\Big|_{k_1^2 = m_0^2 \frac{c^2}{\hbar^2}} \overset{(20.67b)}{=} \\ &= -\frac{i3\lambda}{4\pi^2}\Big(\frac{m_0 c}{\hbar}\Big)^2 \lim_{\Lambda\to\infty}\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2} - \ln\Big(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\Big)\right) + \mathcal{O}(\lambda^2) \end{aligned} \tag{22.4}$$

In B there is one term $\mathcal{O}(\lambda^2)$, namely the double-loop, and infinitely many terms of higher order:

$$\begin{aligned} B &= \frac{1}{\hbar c}\frac{\partial F_{1\mathrm{PI}}}{\partial k_1^2}\Big|_{k_1^2 = m_0^2 \frac{c^2}{\hbar^2}} \overset{(20.94e)}{=} \\ &= \frac{1}{\hbar c}\lim_{\Lambda\to\infty}\ln\Big(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\Big)\cdot \underbrace{\frac{\partial\, C\Big((k_1)^2\Big)}{\partial k_1^2}\Big|_{k_1^2 = m_0^2\frac{c^2}{\hbar^2}}}_{\neq 0 \text{ and finite}} + \mathcal{O}(\lambda^3) \end{aligned} \tag{22.5}$$

In the sequel we content ourselves with results, which are correct up to and including $\mathcal{O}(\lambda^2)$. Under this condition we have

$$\begin{aligned} &k_1^2 - m_0^2\tfrac{c^2}{\hbar^2} - iF_{1\mathrm{PI}} = k_1^2 - m_0^2\tfrac{c^2}{\hbar^2} - iA - i(k_1^2 - m_0^2\tfrac{c^2}{\hbar^2})B = \\ &= \frac{-iA + (k_1^2 - m_0^2\frac{c^2}{\hbar^2})(1-iB)(1+iB)}{(1+iB)} = \frac{-iA + (k_1^2 - m_0^2\frac{c^2}{\hbar^2})}{(1+iB)}\,. \end{aligned}$$

Terms AB and BB have been neglected. This result is inserted into the formula (22.2) of the propagator with self-interaction:

$$\widetilde{G}^{(W)} = \frac{i(1+iB)}{\hbar c\Big(k_1^2 - m_0^2\frac{c^2}{\hbar^2} - iA + i\epsilon'\Big)} \tag{22.6}$$

Both A and B are constants, which do not depend on k_1. Because of $\Lambda \to \infty$, both are infinitely large. We now are going to eliminate the divergences due

to renormalization:

It is impossible to determine the "bare mass" m_0 of a particle experimentally. Only the mass, which is causing the pole of the propagator, i. e. the

$$\boxed{\text{physical mass} = m \equiv +\sqrt{m_0^2 + iA\tfrac{\hbar^2}{c^2}}} \tag{22.7}$$

can be measured. The physical mass is finite, as proved by experiments. The computed quantity A diverges. Thus also the not measurable bare mass m_0 diverges exactly such, that the sum of the both diverging quantities gives the finite value (22.7).

The field-operators $\psi_0(x)$ are implicitly contained in the propagator (22.6) by the relation

$$\widetilde{G} \sim G \sim \langle 0|\, T\psi_0(x_1)\, \psi_0(x_2)\, |0\rangle \ .$$

The field-operators are renormalized due to

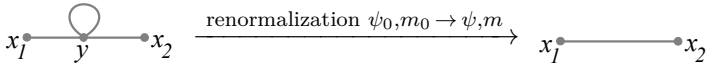
$$\boxed{\psi(x) \equiv \psi_0(x)\, \sqrt{1 + iB}} \ . \tag{22.8}$$

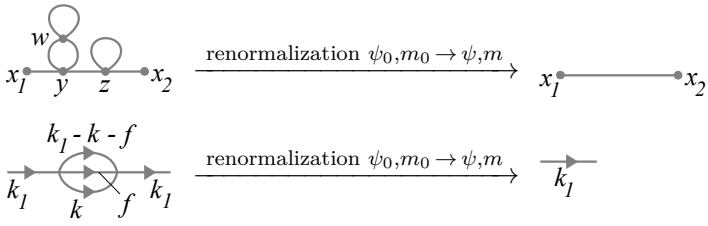
B diverges. Thus $\psi_0(x)$ must diverge as well exactly such, that the product of the diverging quantities gives the finite value $\psi(x)$.

Once the mass and the field-operators are renormalized according to (22.7) and (22.8), the propagator gets the simple structure

$$\widetilde{G}^{(W)} = \widetilde{G} = \frac{i}{\hbar c\Big(k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)} \ . \tag{22.9}$$

At the same time the tadpoles, cacti, and double-loop structures disappear from all diagrams:



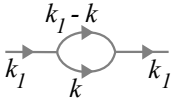


etc. ... (22.10)

Wherever in a diagram tadpoles, cacti, or double-loops show up, they are replaced by the simple propagator-line. It is not necessary to take care of the symmetry factors, which are indicated in (20.47), because the renormalized diagram is identical to a simpler diagram which has been computed already in lower order of perturbation theory. As it shall not be considered a second time, it is simply skipped, no matter what may be it's symmetry factor. Here we encounter again the argument, by which in section 20.3 also the cancellation of all graphs was justified, which can be amputated.

By the same method the diverging diagram

$$36 \cdot \quad \stackrel{(20.68)}{\widehat{=}} \quad \widetilde{G}^{(2)}(k_1) =$$

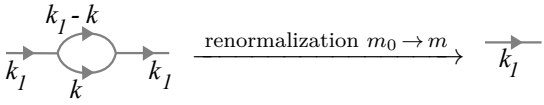


$$= \widetilde{G}(k_1) \cdot \underbrace{\left(\frac{3\lambda\hbar c}{2\pi}\right)^2 \left(- \frac{1}{2} \lim_{\Lambda\to\infty} \ln\left(\frac{\Lambda^2\hbar^2}{m_0^2c^2}\right) + C(k_1)\right)}_{F(k_1)} \cdot \widetilde{G}(k_1)$$

of ψ^3-theory, which again has been computed with the bare mass m_0, can be renormalized. The diverging terms of all order of perturbation theory are summed up as a geometrical series (22.2). Then the

$$\text{physical mass} = m \equiv + \sqrt{m_0^2 + iF(k_1)\,\frac{\hbar}{c^3}} \tag{22.11}$$

is fixed. That means, that not only in propagator-corrections, but in all diagrams of ψ^3-theory, in which the diverging loop is showing up, the replacement



(22.12)

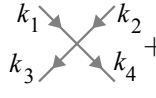
shall be performed. As the simplified diagram thus created has been considered already in lower order of perturbation theory, it is to be canceled, no matter what may be it's symmetry factor. This completes the renormalization of ψ^3-theory, because this is the only type of divergence in this theory.

We still need to clarify the handling of the diverging four-point diagram (20.94c). It is treated due to renormalization of the coupling constant. The bare coupling constant λ_0 can not be measured. Only the probability of a scattering event with two incoming particles with wavenumbers k_1 and k_2 and two outgoing particles with wavenumbers k_3 and k_4 can be measured. The probability amplitude of this scattering event is

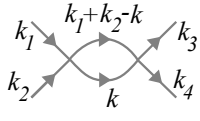
$$\overline{S} \overset{(24.10)}{\equiv} \mathcal{M} \cdot \Big(\prod_{\text{in}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{in}}}\Omega N}} \Big) \Big(\prod_{\text{out}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{out}}}\Omega N}} \Big) \cdot$$
$$\cdot \; 2\pi\Omega\, \delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}} \Big)\, \delta_{\sum_{\text{in}} \boldsymbol{k}_{\text{in}}\,,\, \sum_{\text{out}} \boldsymbol{k}_{\text{out}}} \; \cdot$$

The value of the scattering matrix $\mathcal{M}$, which here has been defined, is for this scattering event:

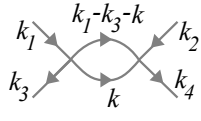
$$\mathcal{M} = \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{O}(\lambda_0^3) \mathrel{\widehat{=}} 24 \cdot$$



$+$



$+ 576 \cdot$



$+ 1152 \cdot + \mathcal{O}(\lambda_0^3)$

$$\overset{(20.19),(20.88)}{\widehat{=}} 24(-i\lambda_0\hbar^2c^2) - 4\,\frac{i(3\lambda_0\hbar c)^2}{\pi^2} \lim_{\Lambda\to\infty} \ln\Big(\frac{\Lambda^2}{|(k_3 - k_1)^2|} \Big)$$
$$\text{if } \boldsymbol{k}_1^2\hbar^2/(mc)^2 \gg 1 \;,\; \boldsymbol{k}_3^2\hbar^2/(mc)^2 \gg 1 \qquad (22.13)$$

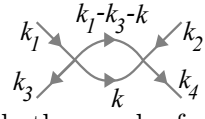
Just to simplify the notation, we here restrict ourselves to the case of high-

relativistic particles, in which the contribution of the second of the three diagrams is negligible. The following renormalization could very well be done for scattering events at low energy. For that case we would need to insert the complete formulas (20.85) and (20.86).

We compute (22.13) with the renormalized mass m and with the renormalized field-operator $\psi(x)$, but with the bare coupling constant λ_0. The expression diverges for $\Lambda \to \infty$. The renormalized coupling constant λ is defined due to

$$\boxed{\lambda = \lambda_0 + \frac{3\lambda_0^2}{2\pi^2} \lim_{\Lambda\to\infty} \ln\Big(\frac{\Lambda^2}{|(k_3-k_1)^2|}\Big)} \,. \tag{22.14}$$

The experimentally observed probability for the scattering $k_1, k_2 \to k_3, k_4$ is finite. Thus λ is finite as well. The logarithm diverges for $\Lambda \to \infty$. Thus λ_0 diverges as well, and exactly such that the sum (22.14) is finite.



Different from propagator-corrections, the graph does *not* become invisible due to the renormalization. While the graph of second order

$$24(-i\lambda_0\hbar^2c^2) - 4\,\frac{i(3\lambda_0\hbar c)^2}{\pi^2} \lim_{\Lambda\to\infty} \ln\Big(\frac{\Lambda^2}{|(k_3-k_1)^2|}\Big) = 24(-i\lambda\hbar^2c^2) \,,$$

formally disappears from the theory, this does hold only for the wave numbers, with which the renormalization (22.14) has been done. For particles with different wavenumbers q_j one gets in second order of perturbation theory:

$$\begin{aligned}
\mathcal{M} &= 24(-i\lambda_0\hbar^2c^2) - 4\,\frac{i(3\lambda_0\hbar c)^2}{\pi^2} \lim_{\Lambda\to\infty} \cdot \\
&\quad \cdot \Big(\ln\Big(\frac{\Lambda^2}{|(q_3-q_1)^2|}\Big) - \ln\Big(\frac{\Lambda^2}{|(k_3-k_1)^2|}\Big)\Big) + \mathcal{O}(\lambda^3) \\
&= 24(-i\lambda\hbar^2c^2) - 4\,\frac{i(3\lambda\hbar c)^2}{\pi^2} \ln\Big(\frac{|(k_3-k_1)^2|}{|(q_3-q_1)^2|}\Big) + \mathcal{O}(\lambda^3)
\end{aligned}$$

In the last line we were not completely accurate when changing to the renormalized coupling constant. But that inaccuracy is $\mathcal{O}(\lambda^3)$ and thus acceptable. Therefore it is not necessary to perform new renormalizations, if the scattering of particles with different wavenumbers shall be computed. This is a general rule: For the renormalization of one parameter (in this case the renormalization $\lambda_0 \to \lambda$), one single measured value is sufficient.

The cutoff-parameters Λ have disappeared from all results. Only those wavenumbers k_j, q_j have remained in the formulas, which can be observed experimentally, i. e. which are very small as compared to the inverse r^{-1} of the fundamental length (see section 21.3). This means: Due to renormalization, the processes on the scale of the small wavenumbers k_j, q_j have been decoupled from the (infinitely) large wavenumbers Λ. Due to renormalization the ψ^3-theory and the ψ^4-theory became effective theories, which give correct, finite results for processes on the scale of the small wavenumbers k_j, q_j.

23 Yukawa Theory

It is well known since Rutherford's[1] experiments in 1910, that most of the mass of an atom is concentrated in a nucleus, whose diameter is by several orders of magnitude smaller than the grid constant of crystalline solids. As the constituents of the nucleus have positive electric charge or are neutral (the existence of neutrons has been established in the early thirties), it is obvious that there must be an attractive force acting between the nucleons, which at small distances is much stronger than their electric repulsion. At large distances, however, this force obviously is much weaker the the electromagnetic force. In 1935 Yukawa[2] published the first reasonable theory of nuclear forces. Before describing his theory, we will shortly sketch the line of thought which guided him to that theory.

$$\varphi_E(r) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r} \tag{23.1}$$

is the electrostatic potential which is created by a point-charge q at distant r. The range of the electrostatic interaction is infinite. To model an interaction with finite range, Yukawa introduced a cutoff-parameter R:

$$\varphi_K(r) = \frac{g}{r}\,\exp\{-r/R\} \tag{23.2}$$

$q/4\pi\epsilon_0$ is replaced by the coupling constant g. Due to the additional exponential function, the nuclear force becomes unmeasurable small as soon as the distance r from the potential's source becomes significantly larger than R. φ_E and φ_K are solutions of the potential equations

[1] Ernest Rutherford (1871-1937)
[2] Yukawa Hideki (1907-1981)

$$\Delta\varphi_E(r) = 0 \qquad \text{resp.} \qquad \Big(\Delta - \frac{1}{R^2}\Big)\varphi_K(r) = 0 \ , \tag{23.3}$$

as can be checked by insertion. (The Laplace-operator in spherical coordinates can be found in appendix A.20.) If the Laplace-operator is expanded to the d'Alembert-operator, then φ_E furthermore becomes a solution of the wave equation

$$\Big(-\frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \Delta\Big)\varphi_E(\boldsymbol{r}, t) = 0 \ . \tag{23.4}$$

Based on (23.3), one then may speculate that φ_K should be a solution of the wave equation

$$\Big(-\frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \Delta - \frac{1}{R^2}\Big)\varphi_K(\boldsymbol{r}, t) = 0 \ . \tag{23.5}$$

Using the ansatz $\varphi_K(\boldsymbol{r}, t) = \varphi_{K0}\exp\{-i(\omega t - \boldsymbol{k}\boldsymbol{r})\}$, one can derive from that wave equation

$$\frac{\omega^2}{c^2} - \boldsymbol{k}^2 = \frac{1}{R^2} \overset{(7.18)}{=} \frac{M^2c^2}{\hbar^2} \ . \tag{23.6}$$

This equation is constituting a relation between the mass M of the field φ_K and the range R of the force. Inserting the experimentally determined range R of the nuclear force of about one femtometer, Yukawa could estimate the field's rest energy

$$Mc^2 \approx \frac{\hbar c}{R} \approx \frac{\hbar c}{10^{-15}\mathrm{m}} \approx 200\,\mathrm{MeV} \ . \tag{23.7}$$

The pions thus invented by Yukawa could only by 1947 be first time directly observed in the cosmic radiation.

Yukawa's theory is describing nucleons and antinucleons as fermions with spin 1/2 (i. e. as Dirac-fields), and the force inbetween them as an uncharged Klein-Gordon field. The Lagrangian of this model of nuclear forces is

$$\mathcal{L} = \overline{\psi}_{(W)}\Big(i\hbar c\gamma^\nu \mathrm{d}_\nu - g(\hbar c)^{3/2}\phi_{(W)} - mc^2\Big)\psi_{(W)} + \\ + \frac{1}{2}\Big(\hbar^2 c^2 \mathrm{d}_\mu \mathrm{d}^\mu - M^2c^4\Big)\phi^2_{(W)} \ . \tag{23.8}$$

It is consisting of the Lagrangian of the Dirac field (without interaction with a gauge field), the Lagrangian of the uncharged Klein-Gordon field, and the interaction term with coupling constant g. As declared in (19.21), $\psi_{(W)}, \overline{\psi}_{(W)}, \phi_{(W)}$ are the field-operators of the interacting fields. Operators without index are operators in the interaction-picture. The comparison with the Lagrangian

$$\mathcal{L} \overset{(8.21)}{=} \overline{\psi}_{(W)}\Big(i\hbar c\gamma^\nu(\mathrm{d}_\nu + \frac{i}{\hbar}qA_{(W)\nu}) - mc^2\Big)\psi_{(W)} - \frac{1}{4\mu_0}F_{\sigma\tau}F^{\sigma\tau}$$

of Dirac-field plus gauge-field is instructive: The Klein-Gordon-field $\phi_{(W)}(x)$, which is conveying the Yukawa-interaction, has — different from the gauge-field $A_{(W)}(x)$ — a finite rest-mass M. It's Lagrangian does not have the structure of rotation terms like $F_{\sigma\tau}F^{\sigma\tau}$. Furthermore $\phi_{(W)}$ is not multiplied by the gamma-matrices, and thus is not as closely as the gauge-field interwoven with the spinor formalism of the Dirac-field. And $\phi_{(W)}$ is — again different from the gauge-field — no vector field, but a scalar field.

As the Dirac-field's dimension is volume$^{-1/2}$, and the Klein-Gordon-field's dimension is energy$^{-1/2}$ · volume$^{-1/2}$ — as can be clearly seen again in equation (23.14) below — , the coupling constant is dimensionless:

$$[g] \cdot \frac{(\text{energy}\cdot\text{length})^{3/2}}{\text{volume}\cdot\text{energy}^{1/2}\cdot\text{volume}^{1/2}} = [\mathcal{L}] = \frac{\text{energy}}{\text{volume}} \\ \Longrightarrow \quad [g] = 1 \tag{23.9}$$

23.1 Scattering Processes

We already encountered the Feynman-propagators of the Dirac field

$$S(x-y) \overset{(16.31)}{=} \langle 0|\, T\psi(x)\overline{\psi}(y)\,|0\rangle \tag{23.10a}$$

and of the Klein-Gordon field

$$G(x-y) \overset{(15.43)}{=} \langle 0|\, T\phi(x)\phi^\dagger(y)\,|0\rangle\ . \tag{23.10b}$$

Recall, that the time-order operator $T =$ (15.44) brings about a factor of (-1) at each permutation of two fermion operators. Furthermore in Yukawa-theory $\phi^\dagger = \phi$, because this Klein-Gordon field is not charged. When we describe scattering events with two incoming and two outgoing nucleons, we will encounter matrix elements of the type

$$\langle 0|\, T\psi_{(W)}(x_3)\,\psi_{(W)}(x_4)\,\overline{\psi}_{(W)}(x_1)\,\overline{\psi}_{(W)}(x_2)\,|0\rangle\ .$$

Such matrix elements can be computed by means of the formula

$$\begin{aligned}
&\langle 0|\, T\psi_{(W)}(x_r)\dots\overline{\psi}_{(W)}(x_1)\,|0\rangle \overset{(19.19)}{=} \\
&= \frac{\langle 0|\, T\psi(x_r)\dots\overline{\psi}(x_1)\sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^n|0\rangle}{\langle 0|\, T\sum_{m=0}^{\infty}\frac{1}{m!}\Big(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^m|0\rangle}
\end{aligned} \tag{23.11}$$

as matrix elements of field-operators in the interaction picture, because we nowhere made reference to the particular properties of the uncharged Klein-Gordon field when we derived this formula in section 19.1. It does hold as well for the Dirac field ψ and it's antifield $\overline{\psi}$. The Hamilton operator $H(\tau)$, which must be inserted into this formula, can be derived from the Lagrangian (23.8):

$$H_{(W)} = H_{\text{Dirac}} + H_{\text{Klein-Gordon}} + \underbrace{g(\hbar c)^{3/2}\int_\Omega \mathrm{d}^3x\,\overline{\psi}(x)\,\phi(x)\,\psi(x)}_{H(\tau)}$$

Same as we did in (20.20)ff, we will as well in (23.11) cancel the unconnected diagrams of the numerator versus the denominator. Thus only connected

diagrams need to be computed. This is indicated by the index $_c$ at the matrix element:

$$\langle 0|\, T\psi_{(W)}(x_r)\dots\overline{\psi}_{(W)}(x_1)\,|0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}(-ig\sqrt{\hbar c})^n\cdot$$
$$\cdot\ \langle 0|\, T\psi(x_r)\dots\overline{\psi}(x_1)\Big(\int\limits_{\Omega}\int\limits_{-\infty}^{+\infty}\mathrm{d}^4y\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\Big)^n\,|0\rangle_c \qquad (23.12)$$

Wick's theorem is useful for the computation of matrix elements of field-operators in the interaction picture. Is has been stated in (19.40) for both uncharged and charged Klein-Gordon fields. The extension to combinations of Dirac-field operators and Klein-Gordon-field operators is given in the following

Theorem: The vacuum-expectation-value of a time-ordered product of n field-operators $\psi(x)$ and m field-operators $\overline{\psi}(x)$ of the Dirac field and j field-operators $\phi(x)$ of an uncharged boson field in the interaction picture is

∗ zero, if j is uneven and/or if $n \neq m$.

∗ in case of even j and $n = m$ equal to the sum of the

$$\Big((2n-1)\cdot(2n-3)\cdot(2n-5)\cdot\ \dots\ \cdot 1\Big)\cdot$$
$$\cdot\Big((j-1)\cdot(j-3)\cdot(j-5)\cdot\ \dots\ \cdot 1\Big)$$

different products of the n Feynman-propagators of the Dirac-field and the $j/2$ Feynman-propagators of the boson field, to which the $n+m+j$ operators can be contracted. (23.13)

Note that the the permutation of any two fermion operators, which is needed when contracting the operators to propagators, gives a factor of (-1).

The LSZ-reduction-formula, which has been described in section 19.3 for uncharged boson fields, will be most useful for the computation of scattering amplitudes. Comparing the field-operators, the required replacements become visible. The field-operators of the Dirac-field in the interaction picture are

$$\psi(x) \overset{(16.1a)}{=} \sum_{\boldsymbol{k},r} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\} \Big)$$

$$\overline{\psi}(x) = \sum_{\boldsymbol{k},r} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^{r}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}\bar{u}^{\boldsymbol{k}} \exp\{+ikx\} + {}^{r}b_{\boldsymbol{k}}\, {}^{r}\bar{v}^{\boldsymbol{k}} \exp\{-ikx\} \Big) \ . \quad (23.14a)$$

The summation over the spin-variable r is to be performed only, if the spin-state ($r = 1$ or $r = 2$) of the fields is not observed. If the spin-state is controlled, the respective value r must be inserted instead of the sum over the spin-states. As we don't want to write so many summation symbols, we assume the latter case in the sequel. The field operators of the uncharged Klein-Gordon field in the interaction picture are

$$\phi(x) \overset{(19.11)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} + a^{\dagger}_{\boldsymbol{k}} \exp\{+ikx\} \Big) \ . \quad (23.14b)$$

The normalization factors $\hbar\omega_{\boldsymbol{k}}$ are different due to the different rest masses of the Klein-Gordon field and the Dirac field. Furthermore the Fourier-operators ${}^{r}a_{\boldsymbol{k}}$ of the Dirac field differ from the Fourier-operators $a_{\boldsymbol{k}}$ of the Klein-Gordon field. But from a purely formal point of view, the field-operators differ only by the spinor functions $u, v, \bar{u}, \bar{v}$. The dimension of these spinor functions is $\sqrt{\text{energy}}$.

The relations (A.187) in the derivation of the LSZ-formula are for nucleons to be replaced by the relations

$${}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} = +ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int_{\Omega} \mathrm{d}^3x \exp\{+ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \psi(x) \quad (23.15a)$$

$${}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} = -ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int_{\Omega} \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \psi(x) \quad (23.15b)$$

$${}^{r}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}\bar{u}^{\boldsymbol{k}} = -ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int_{\Omega} \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \overline{\psi}(x) \quad (23.15c)$$

$$ {}^{r}b_{\boldsymbol{k}}\,{}^{r}\overline{v}^{\boldsymbol{k}} = +ic\sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}}\int\limits_{\Omega}\mathrm{d}^3x\,\exp\{+ikx\}\overset{\leftrightarrow}{\mathrm{d}_0}\,\overline{\psi}(x)\ . \tag{23.15d} $$

The third and the fourth relation are the adjoints of the first and the second. As done in (A.187), these relations are checked due to insertion of the field-operator. Then we multiply (23.15a) from left by $\sum_s {}^{s}\overline{u}^{\boldsymbol{k}}$, multiply (23.15b) from left by $\sum_s {}^{s}\overline{v}^{\boldsymbol{k}}$, multiply (23.15c) from right by $\sum_s {}^{s}u^{\boldsymbol{k}}$, and multiply (23.15d) from right by $\sum_s {}^{s}v^{\boldsymbol{k}}$. Using

$$ \begin{aligned} {}^{r}\overline{u}^{\boldsymbol{k}}\,{}^{s}u^{\boldsymbol{k}} &\overset{(8.72a)}{=} 2mc^2\,\delta_{rs} \\ {}^{r}\overline{v}^{\boldsymbol{k}}\,{}^{s}v^{\boldsymbol{k}} &\overset{(8.72b)}{=} -2mc^2\,\delta_{rs}\ , \end{aligned} $$

we then get

$$ {}^{r}a_{\boldsymbol{k}} = +ic\sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}}\int\limits_{\Omega}\mathrm{d}^3x\,\exp\{+ikx\}\overset{\leftrightarrow}{\mathrm{d}_0}\,\frac{{}^{r}\overline{u}^{\boldsymbol{k}}\psi(x)}{2mc^2} \tag{23.16a} $$

$$ {}^{r}b^{\dagger}_{\boldsymbol{k}} = +ic\sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}}\int\limits_{\Omega}\mathrm{d}^3x\,\exp\{-ikx\}\overset{\leftrightarrow}{\mathrm{d}_0}\,\frac{{}^{r}\overline{v}^{\boldsymbol{k}}\psi(x)}{2mc^2} \tag{23.16b} $$

$$ {}^{r}a^{\dagger}_{\boldsymbol{k}} = -ic\sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}}\int\limits_{\Omega}\mathrm{d}^3x\,\exp\{-ikx\}\overset{\leftrightarrow}{\mathrm{d}_0}\,\frac{\overline{\psi}(x)\,{}^{r}u^{\boldsymbol{k}}}{2mc^2} \tag{23.16c} $$

$$ {}^{r}b_{\boldsymbol{k}} = -ic\sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}}\int\limits_{\Omega}\mathrm{d}^3x\,\exp\{+ikx\}\overset{\leftrightarrow}{\mathrm{d}_0}\,\frac{\overline{\psi}(x)\,{}^{r}v^{\boldsymbol{k}}}{2mc^2}\ . \tag{23.16d} $$

All four Fourier-operators are dimension-less spinor-scalars. Comparing these relations with (A.187), it becomes clear that in the following steps of the derivation of the LSZ-formula, which are corresponding to (A.187)ff, the field-operators of the uncharged Klein-Gordon bosons are to be replaced as follows:

$$ \text{incoming or outgoing boson}: \quad \phi(x) = \phi^{\dagger}(x) $$

$$\text{incoming} \begin{cases} \text{nucleon}: & (2mc^2)^{-1}\,\overline{\psi}(x)\,{}^r u^{\boldsymbol{k}} \\ \text{antinucleon}: & -(2mc^2)^{-1}\,{}^r\overline{v}^{\boldsymbol{k}}\,\psi(x) \end{cases}$$
$$\text{outgoing} \begin{cases} \text{nucleon}: & (2mc^2)^{-1}\,{}^r\overline{u}^{\boldsymbol{k}}\,\psi(x) \\ \text{antinucleon}: & -(2mc^2)^{-1}\,\overline{\psi}(x)\,{}^r v^{\boldsymbol{k}} \end{cases} \tag{23.17}$$

Then we introduced in (A.200) the multipoint-Greensfunction of the Klein-Gordon field

$$G(y,x) \equiv \langle 0|\, T\psi_{(W)}(y)\,\psi_{(W)}(x)\,|0\rangle \ ,$$

which because of (23.17) must be replaced for nucleons by

$$G(y,x) \ \longrightarrow \ S(y,x)\cdot \frac{{}^{s_1}\overline{u}^{\boldsymbol{q}_1}\ldots(-1)^{s_n}v^{\boldsymbol{q}_n}\,{}^{r_1}u^{\boldsymbol{k}_1}\ldots(-1)^{r_n}\overline{v}^{\boldsymbol{k}_m}}{(2mc^2)^{n+m}} \ .$$

Eventually we inserted in (A.201)

$$\Big(k^2 - m^2\frac{c^2}{\hbar^2}\Big)\,G(y,x) = \frac{i\,G(y,x)}{\hbar c\,\widetilde{G}(k)} \ . \tag{23.18}$$

This must be replaced for nucleons by

$$\Big(k^2 - m^2\frac{c^2}{\hbar^2}\Big)\frac{S(y,x)}{2mc^2} = \frac{(k^2 - m^2\frac{c^2}{\hbar^2})\,S(y,x)}{\hbar c(\gamma^\nu k_\nu + m\frac{c}{\hbar})} = \frac{i\,S(y,x)}{\hbar c\,\widetilde{S}(k)}$$
$$\text{with } \widetilde{S}(k) \overset{(12.24)}{=} \frac{i(\gamma^\nu k_\nu + m\frac{c}{\hbar})}{k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'} \ . \tag{23.19}$$

It does not seem implausible that this replacement is correct, because the derivation of the LSZ-formula is done almost (but not exactly!) on mass-shell. And on mass-shell

$$\hbar^2c^2k^2 = m^2c^4 \implies \hbar c(\gamma^\nu k_\nu + mc/\hbar) \approx 2mc^2 \ .$$

Still it is obvious, that we have not really proved (23.19), but at best have made it plausible. With an additional small complication, because of which

an additional operator $\mathcal{U}_{\mathrm{S}}$ (which will be defined immediately) is inserted, one then gets — like in (19.49) — the LSZ-formula for nucleons:

$$
\begin{aligned}
S_{\boldsymbol{q}_1\dots\boldsymbol{q}_n\boldsymbol{k}_1\dots\boldsymbol{k}_m} &= \langle \boldsymbol{q}_1 \dots \boldsymbol{q}_n |\, S \,| \boldsymbol{k}_1 \dots \boldsymbol{k}_m \rangle \;= \qquad (23.20)\\
&= \mathcal{U}_{\mathrm{S}} \prod_{j=1}^{n} \frac{{}^{r_j}\bar{u}^{\boldsymbol{q}_j}\, \widetilde{S}^{-1}(q_j)}{\sqrt{2\hbar\omega_{\boldsymbol{q}_j}\Omega}} \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4 y_j \, \exp\{+iq_j y_j\} \cdot \\
&\quad \cdot \prod_{l=1}^{m} \frac{\widetilde{S}^{-1}(k_l)\, {}^{r_l}u^{\boldsymbol{k}_l}}{\sqrt{2\hbar\omega_{\boldsymbol{k}_l}\Omega}} \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4 x_l \, \exp\{-ik_l x_l\} \sum_{j=0}^{\infty} \frac{1}{j!} \cdot \\
&\quad \cdot \frac{\langle 0|\, T\psi(y_1)\dots\psi(y_n)\,\overline{\psi}(x_1)\dots\overline{\psi}(x_m)\Big(-\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^{j}\,|0\rangle}{\langle 0|\, T\sum\limits_{r=0}^{\infty}\frac{1}{r!}\Big(-\frac{i}{\hbar}\int\limits_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^{r}\,|0\rangle}
\end{aligned}
$$

Here it is assumed that only nucleons (but no antinucleons) are going in and out. If an antinucleon is coming in, then ${}^{r_l}u^{\boldsymbol{k}_l}$ must be replaced by $-{}^{r_j}\bar{v}^{\boldsymbol{k}_l}$ and $\overline{\psi}(x_l)$ must be replaced by $\psi(x_l)$. If an antinucleon is going out, then ${}^{r_j}\bar{u}^{\boldsymbol{q}_j}$ must be replaced by $-{}^{r_j}v^{\boldsymbol{q}_j}$ and $\psi(y_j)$ must be replaced by $\overline{\psi}(y_j)$.

To check the consistency of our considerations, we analyze the dimensions: Each incoming or outgoing particle is adding to (23.20) the dimension

$$
\left[\frac{\bar{u}\,\widetilde{S}^{-1}}{\sqrt{\hbar\omega\Omega}}\,\mathrm{d}^4 x\,\psi(x)\right] = \frac{\sqrt{\text{energy}}\cdot\text{length}^{-1}}{\sqrt{\text{energy}\cdot\text{volume}}}\cdot\frac{\text{length}^4}{\sqrt{\text{volume}}} = 1\ .
$$

Thus the probability-amplitude is dimension-less, as should be.

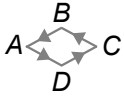
The announced small complication is resulting from the fact, that the spinor-functions of Dirac-theory — in contrast to the scalar functions of the Klein-Gordon-field — are not commuting in general. We save ourselves the lengthly investigation of the correct sequence of factors in the LSZ-formula, and extract from the literature the following simple result: The rearrangement-operator for spinor-functions $\mathcal{U}_{\mathrm{S}}$ must be inserted into the LSZ-formula. This is the definition of that operator:

Definition : The rearrangement-operator $\mathcal{U}_{\mathrm{S}}$ arranges the spinor-functions $\mathsf{A}, \mathsf{B}, \ldots, \mathsf{Z}$, which are related to a continuous fermion line (which may extend over one ore several vertices) to a product, in which those factors, to which the arrows on the line are pointing, are arranged left of those factors, from which the arrows are originating:

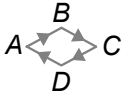
$\Longrightarrow \mathsf{DCBA}$

$\Longrightarrow \mathsf{ABCD}$

(23.21)



$\Longrightarrow \mathsf{C}_{ab}\mathsf{D}_{bc}\mathsf{A}_{cd}\mathsf{B}_{da}$
$= \mathrm{tr}\{\mathsf{CDAB}\}$



$\Longrightarrow \mathsf{B}_{ab}\mathsf{A}_{bc}\mathsf{D}_{cd}\mathsf{C}_{da}$
$= \mathrm{tr}\{\mathsf{BADC}\}$

It doesn't matter, where the starting point of a closed fermion loop is chosen, because the spinor factors are cyclically combined to the trace.

Due to this rule, there will always be a row-spinor at the left end of a product and a column-spinor at the right end of a product, making the product in total a spinor-scalar. If the fermion-line does not have a beginning or an end, because it is closed to a loop of fermion lines, then this rule results in the trace of a product of spinor matrices, which again is a spinor-scalar. We will encounter examples for both cases, and the functionality of the rearrangement-operator will become clear as soon as we will consider those examples.

23.1.1 Nucleon-Nucleon Scattering

First we investigate a scattering event with two incoming and two outgoing nucleons (not antinucleons). k_1 and k_2 are the wavenumbers of the incoming nucleons. k_3 and k_4 are the wavenumbers of the outgoing nucleons. Using the abbreviation

$$N_j \equiv \sqrt{2\hbar\omega_{\boldsymbol{k}_j}\Omega}\ ,$$

the probability amplitude

$$S_{\boldsymbol{k}_4\boldsymbol{k}_3\boldsymbol{k}_2\boldsymbol{k}_1} \overset{(20.1)}{=} \langle\boldsymbol{k}_3\boldsymbol{k}_4|\, S\, |\boldsymbol{k}_1\boldsymbol{k}_2\rangle \overset{(20.32)}{=} \sum_{n=0}^{\infty} \overline{S}^{(n)} \tag{23.22}$$

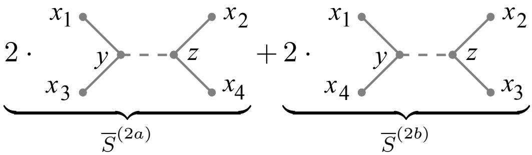
according to the LSZ-formula is:

$$\begin{aligned}\overline{S}^{(n)} \overset{(23.20)}{=} \mathcal{U}_{\text{S}} \prod_{j=1}^{4} \frac{\widetilde{S}^{-1}(k_j)}{N_j} \int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4x_j\, \frac{(-ig\sqrt{\hbar c})^n}{n!}\,\cdot \\ \cdot\, {}^{r_4}\overline{u}^{\boldsymbol{k}_4}\, {}^{r_3}\overline{u}^{\boldsymbol{k}_3}\, {}^{r_2}u^{\boldsymbol{k}_2}\, {}^{r_1}u^{\boldsymbol{k}_1} \exp\{-i(k_1x_1 + k_2x_2 - k_3x_3 - k_4x_4\} \\ \cdot\, \langle 0|\, T\psi(x_3)\,\psi(x_4)\,\overline{\psi}(x_1)\,\overline{\psi}(x_2)\Big(\int\limits_{-\infty}^{+\infty}\int\limits_{\Omega} \mathrm{d}^4y\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\Big)^n\, |0\rangle_c\end{aligned}$$

$\overline{S}^{(0)}$ is zero, because — same as in (20.14) — the two propagators $S(x_3 - x_1)$ and $S(x_4 - x_2)$, to which the matrix element can be contracted in zeroth order, are not sufficient to compensate the four zero-factors $\widetilde{S}^{-1}$ of the LSZ-formula. In first order there is one operator $\phi(y)$ in the matrix element, and it is again zero because of (23.13). The same holds true for all uneven orders $n = 1, 3, 5, \ldots$ of perturbation theory. Only even orders $n = 2, 4, \ldots$ can add to the probability amplitude.

The leading non-zero term is found in second-order perturbation theory:

$\overline{S}^{(2)} \mathrel{\widehat{=}} 2\cdot$ [diagram (2a)] $+\,2\cdot$ [diagram (2b)] (23.23)



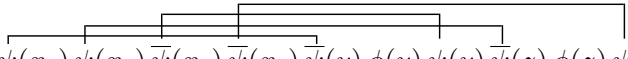
Boson lines are drawn dashed, to contrast them versus the continuous fermion lines.

$$\overline{S}^{(2)} = \mathcal{U}_{\mathrm{S}} \prod_{j=1}^{4} \frac{\widetilde{S}^{-1}(k_j)}{N_j} \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4x_j \frac{(-ig\sqrt{\hbar c})^2}{2!} \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4y \int_{-\infty}^{+\infty}\int_{\Omega} \mathrm{d}^4z \cdot$$
$$\cdot {}^{r_4}\overline{u}^{\boldsymbol{k}_4}\, {}^{r_3}\overline{u}^{\boldsymbol{k}_3}\, {}^{r_2}u^{\boldsymbol{k}_2}\, {}^{r_1}u^{\boldsymbol{k}_1} \exp\{-i(k_1x_1 + k_2x_2 - k_3x_3 - k_4x_4\}$$
$$\cdot \langle 0|\, T\psi(x_3)\, \psi(x_4)\, \overline{\psi}(x_1)\, \overline{\psi}(x_2)\, \overline{\psi}(y)\, \phi(y)\, \psi(y)\, \overline{\psi}(z)\, \phi(z)\, \psi(z)\, |0\rangle_c \quad (23.24)$$

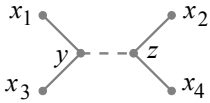
There is only one possibility, to combine the two operators $\phi(y)$ and $\phi(z)$ to a propagator. The field-operators of the incoming and outgoing nucleons must not be combined mutually, but must be contracted with the operators of the interaction terms, to make sure that there is a sufficient number of propagators to compensate the four zero-factors $\widetilde{S}^{-1}(k_j)$. The options for combinations are furthermore restricted by the fact, that each nucleon-propagator must be built from one operator ψ and one operator $\overline{\psi}$.

$\overline{\psi}(x_1)$ can be contracted with $\psi(y)$ or $\psi(z)$ (2 alternatives). Subsequently only one option remains for the contraction of $\overline{\psi}(x_2)$. $\psi(x_3)$ can be contracted with $\overline{\psi}(y)$ or $\overline{\psi}(z)$ (2 alternative), subsequently the contraction of $\psi(x_4)$ is uniquely fixed. In total there are 4 different non-vanishing alternatives of contraction. Two of them each are combined to one graph because their numerical values obviously are equal (they only differ by the permutation of the vertices y and z). This is characteristic for the Yukawa-theory (and — as we will see — for quantum electrodynamics as well): As the three operators $\overline{\psi}\,\phi\,\psi$, which belong to any vertex, can not replace themselves mutually, the symmetry factor of any graph with n vertices is $n!$, because the permutations of the vertices are the only modifications which don't change the graph's topology. The symmetry factor just compensates in the Yukawa-theory (and in QED) in all diagrams the factor $1/n!$, which is resulting from the series expansion of the interaction term.

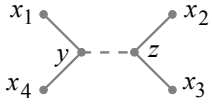
When contracting the field-operators to propagators, a factor (-1) must be inserted for each permutation of two fermion operators. In the matrix element



$$\langle 0|\, T\psi(x_3)\, \psi(x_4)\, \overline{\psi}(x_1)\, \overline{\psi}(x_2)\, \overline{\psi}(y)\, \phi(y)\, \psi(y)\, \overline{\psi}(z)\, \phi(z)\, \psi(z)\, |0\rangle_c \quad (23.25a)$$



the contractions of the graph are marked by the horizontal brackets, which we introduced in section 19.2.1. The brackets are crossing at 7 positions. Furthermore $\overline{\psi}(x_1)$ must be permuted with $\psi(y)$ and $\overline{\psi}(x_2)$ with $\psi(z)$, to make sure that in all propagators the annihilation-operators are left of the creation-operators. Thus in total 9 permutations of fermion operators are required for the contraction of the matrix element to the propagators of the diagram, resulting into a factor (-1).

For the diagram 

one permutation less is required:

$$\langle 0|\, T\psi(x_3)\,\psi(x_4)\,\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c \qquad (23.25b)$$

Therefore $\overline{S}^{(2a)}$ gets a factor (-1), while $\overline{S}^{(2b)}$ does not.

Remark: If we had not started from the matrix element in (23.24), but for example from

$$\langle 0|\, T\psi(x_4)\,\psi(x_3)\,\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c\ ,$$

then we would have found for $\overline{S}^{(2b)}$ a factor (-1), but not for $\overline{S}^{(2a)}$. This is just a particular case of the general indeterminacy of matrix elements in quantum theory, which are fixed only up to an arbitrary phase-factor. Only the square-moduli of the matrix elements, i. e. in this case $|\overline{S}^{(2a)} + \overline{S}^{(2b)}|^2$, can be proved experimentally. For the square modulus it doesn't matter which term has the positive sign and which term has the negative sign. But it is important, and has experimentally verifiable consequences, that the signs of $\overline{S}^{(2a)}$ and $\overline{S}^{(2b)}$ are different.

By application of the re-arrangement operator one eventually arrives at the following result:

$$\overline{S}^{(2)} = \prod_{j=1}^{4} \frac{1}{N_j} \int_{-\infty}^{+\infty} \int_{\Omega} \mathrm{d}^4 x_j \, \frac{(-ig\sqrt{\hbar c})^2}{2!} \cdot$$

$$\cdot \exp\{-i(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4\} \cdot 2\, G(z-y) \cdot \Big($$

$$- \Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3} \widetilde{S}^{-1}(k_3) S(x_3 - y)\, S(y - x_1) \widetilde{S}^{-1}(k_1)^{r_1} u^{\boldsymbol{k}_1}\Big) \cdot$$

$$\cdot \Big({}^{r_4}\bar{u}^{\boldsymbol{k}_4} \widetilde{S}^{-1}(k_4) S(x_4 - z)\, S(z - x_2) \widetilde{S}^{-1}(k_2)^{r_2} u^{\boldsymbol{k}_2}\Big) +$$

$$+ \Big({}^{r_4}\bar{u}^{\boldsymbol{k}_4} \widetilde{S}^{-1}(k_4) S(x_4 - y)\, S(y - x_1) \widetilde{S}^{-1}(k_1)^{r_1} u^{\boldsymbol{k}_1}\Big) \cdot$$

$$\cdot \Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3} \widetilde{S}^{-1}(k_3) S(x_3 - z)\, S(z - x_2) \widetilde{S}^{-1}(k_2)^{r_2} u^{\boldsymbol{k}_2}\Big)\Big) \qquad (23.26)$$

The spinor-functions of the four last lines each belong to a continuous fermion-line of a diagram. The factors are arranged according to the definition (23.21) and result into a spinor-scalar each, because S and $\widetilde{S}$ are 4×4 spinor-matrices — see (12.24) and (12.27) — , while $\bar{u}$ is a row-spinor and u is a column-spinor.

As we did when we computed the structurally similar diagrams (20.42), one now must substitute appropriate variables, transform the Greensfunctions S into wavenumber-space and cancel them against the zero-factors $\widetilde{S}^{-1}$, thus getting several delta-functions. We do not want to go through all these exercises step by step again. In box 20.2 on page 421 we listed the rules, according to which the scattering amplitudes of uncharged Klein-Gordon fields can be formulated immediately in energy-momentum-space. The transformations of these rules to the Yukawa-interaction can be found in box 23.1 on the next page. The new rule K will be justified in section 23.1.4.

We apply the rules of box 23.1 to the scattering of two nucleons (not antinucleons) which are coming in with wavenumbers k_1 and k_2:

Box 23.1: **Feynman-rules in energy-momentum-space** for the computation of the component $\overline{S}^{(n)}$ of the scattering amplitude in Yukawa-theory

A , **B** See box 20.1 on page 420

B′ Insert a factor (-1) for each permutation of two fermion operators when combining them to propagators.

C The structure of any vertex is  .

D For n vertices insert a factor

$$\frac{1}{n!}(-ig\sqrt{\hbar c})^n \quad .$$

E For incoming and outgoing lines with four-dimensional wavenumbers k_{in} and k_{out} insert a factor

$$2\pi\Omega\,\delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}}\Big)\,\delta_{\sum_{\text{in}}\boldsymbol{k}_{\text{in}},\,\sum_{\text{out}}\boldsymbol{k}_{\text{out}}} \ .$$

F For each outer line with wavenumber k_j insert a factor

$$\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}}\cdot\begin{cases} {}^{r}u^{\boldsymbol{k}} \text{ for incoming nucleon} \\ {}^{r}\bar{u}^{\boldsymbol{k}} \text{ for outgoing nucleon} \\ (-{}^{r}\bar{v}^{\boldsymbol{k}}) \text{ for incoming antinucleon} \\ (-{}^{r}v^{\boldsymbol{k}}) \text{ for outgoing antinucleon} \\ 1 \text{ for incoming or outgoing boson} \ . \end{cases}$$

G For each inner boson-line with wavenumber k insert a factor

$$\widetilde{G}(k) = \frac{i}{\hbar c(k^2 - M^2\frac{c^2}{\hbar^2} + i\epsilon')} \ .$$

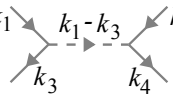
For each inner fermion-line with wavenumber k insert a factor

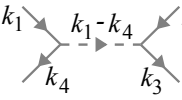
$$\widetilde{S}(k) \overset{(12.24)}{=} \frac{i(\gamma^\nu k_\nu + m\frac{c}{\hbar})}{k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'} \ .$$

H See box 20.2 on page 421

J Insert the rearrangement-operator $\mathcal{U}_{\text{S}} = (23.21)$.

K Inner fermion-lines must *always* be interpreted as particles (but not as antiparticles).

$$2 \cdot$$ 

$$+\, 2 \cdot$$ 

$$\widehat{=}\, \overline{S}^{(2)} = \frac{(-ig\sqrt{\hbar c})^2}{2!} \cdot$$

$$\cdot \left(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}} \right) 2\pi\Omega\, \delta\Big(k_1^0 + k_2^0 - k_3^0 - k_4^0\Big)\, \delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)} \cdot$$

$$\cdot \Bigg(-2\, ({}^{r_3}\bar{u}^{\boldsymbol{k}_3\, r_1}u^{\boldsymbol{k}_1})({}^{r_4}\bar{u}^{\boldsymbol{k}_4\, r_2}u^{\boldsymbol{k}_2}) \frac{i}{\hbar c\Big((k_1-k_3)^2 - M^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)} +$$

$$+\, 2\, ({}^{r_4}\bar{u}^{\boldsymbol{k}_4\, r_1}u^{\boldsymbol{k}_1})({}^{r_3}\bar{u}^{\boldsymbol{k}_3\, r_2}u^{\boldsymbol{k}_2}) \frac{i}{\hbar c\Big((k_1-k_4)^2 - M^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)} \Bigg) \tag{23.27}$$

The rearrangement-operator (23.21) has already been applied here, and therefore is not displayed again explicitly.

The computation of the four spinor-products ${}^{r_\alpha}\bar{u}^{\boldsymbol{k}_\alpha\, r_\beta}u^{\boldsymbol{k}_\beta}$ is demonstrated in appendix A.21. It is shown there, that the following approximations hold in the non-relativistic limit:

$${}^{r}\bar{u}^{\boldsymbol{f}\ s}u^{\boldsymbol{k}} \approx 2mc^2\, \delta_{rs} \tag{23.28a}$$

$${}^{r}\bar{v}^{\boldsymbol{f}\ s}v^{\boldsymbol{k}} \approx -2mc^2\, \delta_{rs} \tag{23.28b}$$

$${}^{r}\bar{u}^{\boldsymbol{f}\ s}v^{\boldsymbol{k}} \approx 0 \tag{23.28c}$$

$${}^{r}\bar{v}^{\boldsymbol{f}\ s}u^{\boldsymbol{k}} \approx 0 \tag{23.28d}$$

$$\text{if } |c\hbar\boldsymbol{f}| \ll mc^2 \text{ and } |c\hbar\boldsymbol{k}| \ll mc^2.$$

In the non-relativistic limit, the spin of the nucleons is conserved at each vertex. If the nucleons are coming in with different spins ($r_1 \neq r_2$), then the second term in the scattering amplitude will vanish. This term is based on the assumption, that the outgoing particle k_3 in (23.26) can be described either by the propagator $S(x_3 - y)$, or — with equal right — by the propagator $S(x_3 - z)$. But if we define to name that one of the two outgoing particles k_3, which has the spin r_1, then the second diagram in (23.27) must be discarded. Then the scattering amplitude simplifies to

$$2 \cdot \quad \widehat{=} \; \overline{S}^{(2)} \approx \frac{ig^2\, 4m^2c^4\, \delta_{r_3 r_1}\, \delta_{r_4 r_2}}{(k_1 - k_3)^2 - M^2 \frac{c^2}{\hbar^2} + i\epsilon'} \cdot$$

$$\cdot \left(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}} \right) 2\pi\Omega\, \delta\left(k_1^0 + k_2^0 - k_3^0 - k_4^0\right) \delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)}$$

$$\text{if } |c\hbar\boldsymbol{k}_j| \ll mc^2 \text{ for } j = 1, 2 \text{ and } r_1 \neq r_2 \; . \tag{23.29}$$

23.1.2 Antinucleon-Antinucleon Scattering

In all energy-momentum-space diagrams considered so far, the directions of the triangular arrows and the directions of the particle momenta have been identical. But in case of antiparticles it is common practice to draw the directions of the triangular arrows in the diagrams in opposite direction to the particle momenta.

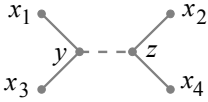
Consequently the diagram allows for different interpretations:

- A nucleon is arriving from left at the vertex. There it interacts with a boson, and then continues to the right.
- An antinucleon is arriving from right at the vertex. There it interacts with a boson, and then continues to the left.
- A nucleon is coming in from left, and an antinucleon is coming in from right. At the vertex they meet and annihilate into a boson, which is going out upwards.
- A boson is coming in from top. At the vertex it decays due to pair production into a nucleon, which is going out to the right, and into an antinucleon, which is going out to the left.

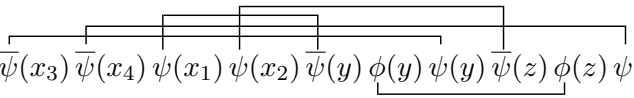
To ensure unique interpretations, the directions of the antinucleon momenta can be indicated by additional arrows at the outer lines (not at the inner lines, as will be explained below) of the diagrams.

According to rule C of box 23.1, there is *always* one fermion-arrow directed towards the vertex and one fermion-arrow directed away from the vertex. Never both fermion-arrows are pointing towards the vertex or away from the vertex. This rule of Yukawa-theory is reflecting the observation, that atomic nuclei, which are composed of several nucleons, do not annihilate into bosons.

While the Feynman-graphs in time-position-space are looking identical for nucleon-nucleon-scattering and for antinucleon-antinucleon-scattering, the matrix elements of antinucleon-antinucleon-scattering are different from (23.25), and different contractions are required.

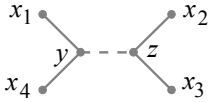
To build the graph 

for antinucleon-antinucleon-scattering from the matrix element

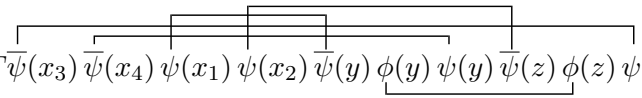


$$\langle 0|\, T\overline{\psi}(x_3)\,\overline{\psi}(x_4)\,\psi(x_1)\,\psi(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c\ , \tag{23.30a}$$

11 permutations of fermion operators are needed, resulting in a factor (-1).

For the diagram 

one permutation less is required:



$$\langle 0|\, T\overline{\psi}(x_3)\,\overline{\psi}(x_4)\,\psi(x_1)\,\psi(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c \tag{23.30b}$$

Thus one gets a factor (+1). Consequently the only difference inbetween

$$2\cdot \quad + 2\cdot \quad \widehat{=}\ \overline{S}^{(2)} = \frac{(-ig\sqrt{\hbar c})^2}{2!}\cdot$$

$$\cdot\left(\prod_{j=1}^{4}\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}}\right) 2\pi\Omega\,\delta\Big(k_1^0+k_2^0-k_3^0-k_4^0\Big)\,\delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)}\cdot$$

$$\cdot\left(-2\,({}^{r_3}\bar{v}^{\boldsymbol{k}_3}\,{}^{r_1}v^{\boldsymbol{k}_1})({}^{r_4}\bar{v}^{\boldsymbol{k}_4}\,{}^{r_2}v^{\boldsymbol{k}_2})\,\frac{i}{\hbar c\Big((k_1-k_3)^2-M^2\frac{c^2}{\hbar^2}+i\epsilon'\Big)}+\right.$$

$$\left.+\,2\,({}^{r_4}\bar{v}^{\boldsymbol{k}_4}\,{}^{r_1}v^{\boldsymbol{k}_1})({}^{r_3}\bar{v}^{\boldsymbol{k}_3}\,{}^{r_2}v^{\boldsymbol{k}_2})\,\frac{i}{\hbar c\Big((k_1-k_4)^2-M^2\frac{c^2}{\hbar^2}+i\epsilon'\Big)}\right) \tag{23.31}$$

and (23.27) is the replacement of all spinors $\bar{u}$ and u by $\bar{v}$ and v. In the non-relativistic approximation the spin of the antinucleons is conserved at each

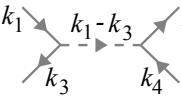
vertex because of (23.28). If the antinucleons are coming in with different spin, then the second term will vanish for the reason named after (23.28). As the both negative signs mutually compensate in the product

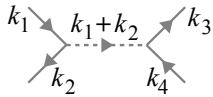
$$({}^{r_3}\overline{v}^{\boldsymbol{k}_3\ r_1}v^{\boldsymbol{k}_1})({}^{r_4}\overline{v}^{\boldsymbol{k}_4\ r_2}v^{\boldsymbol{k}_2}) \overset{(23.28)}{\approx} (-2mc^2)^2\,\delta_{r_3r_1}\,\delta_{r_4r_2} \qquad (23.32)$$
$$\text{if } |c\hbar\boldsymbol{k}_j| \ll mc^2 \text{ for } j = 1,2,3,4\ ,$$

one gets for the elastic scattering of antinucleons by antinucleons in the non-relativistic approximation exactly the same amplitude (23.29) as for the elastic scattering of nucleons by nucleons.

23.1.3 Nucleon-Antinucleon-Scattering

As a third variant we consider the scattering of a nucleon, which is coming in with wavenumber k_1 and going out with wavenumber k_3 by an antinucleon, which is coming in with wavenumber k_2 and going out with wavenumber k_4. This process can be described by two graphs:

$$\overline{S}^{(2)} \mathrel{\widehat{=}} 2 \cdot$$


$$+\,2 \cdot$$


(23.33)

Note, that these two graphs differ only by the labeling (and the accompanying text), but not by the lines and arrows. In the first graph, time is proceeding from top to bottom. A nucleon is coming in with wavenumber k_1, emits a boson, and then goes out with wavenumber k_3. An antinucleon comes in with wavenumber k_2, absorbs a boson, and then goes out with wavenumber k_4. In the second diagram, time is proceeding from left to right. A nucleon with wavenumber k_1 and an antinucleon with wavenumber k_2 annihilate into a boson, which then decays due to pair creation into a nucleon with wavenumber k_3 and an antinucleon with wavenumber k_4.

The nucleon, which is incoming with wavenumber k_1, can dock at either vertex (2 alternatives), after that the remaining parts of the two graphs are uniquely fixed. According to the rules of box 23.1 the probability amplitude is

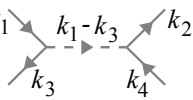
$$\overline{S}^{(2)} = 2 \cdot \frac{(-ig\sqrt{\hbar c})^2}{2!} \Big(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}} \Big) \cdot$$
$$\cdot\ 2\pi\Omega\,\delta\Big(k_1^0 + k_2^0 - k_3^0 - k_4^0\Big)\,\delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)} \cdot$$
$$\cdot \Bigg(({}^{r_3}\overline{u}^{\boldsymbol{k}_3\,r_1}u^{\boldsymbol{k}_1})({}^{r_2}\overline{v}^{\boldsymbol{k}_2\,r_4}v^{\boldsymbol{k}_4}) \frac{i}{\hbar c\Big((k_1-k_3)^2 - M^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)} -$$
$$- ({}^{r_2}\overline{v}^{\boldsymbol{k}_2\,r_1}u^{\boldsymbol{k}_1})({}^{r_3}\overline{u}^{\boldsymbol{k}_3\,r_4}v^{\boldsymbol{k}_4}) \frac{i}{\hbar c\Big((k_1+k_2)^2 - M^2\frac{c^2}{\hbar^2} + i\epsilon'\Big)} \Bigg) . \qquad (23.34)$$

To check the signs, we start from the matrix element

$$\langle 0|\, T\psi(x_3)\,\overline{\psi}(x_4)\,\overline{\psi}(x_1)\,\psi(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c\ ,$$

which has been constructed with regard to (23.17). One gets the two graphs due to the contractions

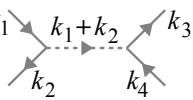
$$\langle 0|\, T\psi(x_3)\,\overline{\psi}(x_4)\,\overline{\psi}(x_1)\,\psi(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c$$
$$\Longrightarrow\ 2 \cdot$$



(23.35a)

and

$$\langle 0|\, T\psi(x_3)\,\overline{\psi}(x_4)\,\overline{\psi}(x_1)\,\psi(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle_c$$
$$\Longrightarrow\ 2 \cdot$$



(23.35b)

In the first matrix element there are 4 crossings of the contraction brackets. Furthermore $\overline{\psi}(x_1)$ must be permuted with $\psi(y)$, and $\psi(x_2)$ with $\overline{\psi}(y)$, to achieve the propagators of the graph. The 6 permutations of fermion operators result into the factor $(+1)$.

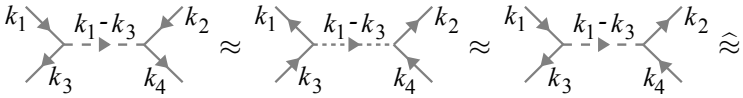
There is 1 crossing of the contraction brackets in the second matrix element. Furthermore $\overline{\psi}(x_1)$ must be exchanged with $\psi(y)$ and $\psi(x_2)$ with $\overline{\psi}(y)$, to get the propagators of the graph. The 3 permutations of fermion operators result into a factor (-1).

In the low-energy limit, the spinor products of the second term are zero according to (23.28). This seems surprising in the first moment. We know, that a proton and an antiproton, which are colliding with arbitrarily low energy, will mutually annihilate. But the product of that annihilation is not a Yukawa-boson, but a photon. This means that for a complete description of the interactions between nucleons we must include quantum electrodynamics into our considerations. There exists, however, in Yukawa-theory a process of nucleon-antinucleon-annihilation into *two* bosons. We will consider that process later.

The sign of the first term, which is describing the elastic scattering of a nucleon and an antinucleon, differs from the sign of the scattering events considered before, because of

$$({}^{r_3}\overline{u}^{\boldsymbol{k}_3\ r_1}u^{\boldsymbol{k}_1})({}^{r_2}\overline{v}^{\boldsymbol{k}_2\ r_4}v^{\boldsymbol{k}_4}) \overset{(23.28)}{\approx} -(2mc^2)^2\,\delta_{r_3r_1}\,\delta_{r_2r_4} \qquad (23.36)$$
$$\text{if } |c\hbar\boldsymbol{k}_j| \ll mc^2 \text{ for } j = 1,2,3,4\ .$$

This is compensated by the fact, that this term gets a positive sign due to the even number of permutations of fermion operators, while the analogous term for the scattering of nucleons by nucleons or for the scattering of antinucleons by antinucleons required an odd number of operator permutations and thus got a negative sign. Therefore in total also the scattering of nucleons by antinucleons is described in the low-energy approximation by equation (23.29). According to Yukawa-theory, for all three alternatives of (anti-)nucleon scattering in non-relativistic approximation



$$\approx \frac{ig^2\, 2m^2c^4\, \delta_{r_3r_1}\, \delta_{r_4r_2}}{(k_1-k_3)^2 - M^2\frac{c^2}{\hbar^2} + i\epsilon'} \left(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}} \right) \cdot$$
$$\cdot\; 2\pi\Omega\, \delta\Big(k_1^0 + k_2^0 - k_3^0 - k_4^0\Big)\, \delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)}$$
$$\text{if } |c\hbar\boldsymbol{k}_j| \ll mc^2 \text{ for } j = 1, 2 \quad . \tag{23.37}$$

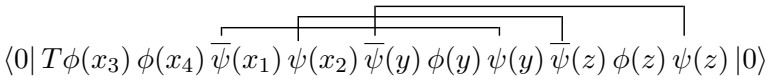
The value and the sign are identical for all three diagrams. From that equation it is not obvious, however, whether the interaction is attractive or repulsive in all three cases. Of course we might simply thrust that Yukawa, who wanted to construct an attractive interaction, did the job right. We postpone the proof, that Yukawa-interaction indeed is attractive, to our discussion of quantum electrodynamics. There we will confirm the attractive nature of Yukawa-interaction by comparing (23.37) with (24.73).

23.1.4 Inner Fermion-Lines

Until now we computed diagrams, in which the inner line was a boson-line. Ambiguities turn up as soon as there is an inner fermion-line in a diagram. A simple example of a scattering diagram with an inner fermion line is the annihilation of a nucleon (coming in with wavenumber k_1) and an antinucleon (coming in with wavenumber k_2) into two bosons (going out with wavenumbers k_3 and k_4), see the left diagram:

(23.38)

This diagram allows for two different interpretations: Either a nucleon moves to y, emits a boson, then moves to z, and annihilates there with an antinucleon into the boson with wavenumber k_4. Or an antinucleon emits at z a boson, then moves to y, and annihilates there with a nucleon into the boson with wavenumber k_3. For two reasons, the both interpretations give different results. Firstly there is the matrix element



$$\langle 0|\, T\phi(x_3)\,\phi(x_4)\,\overline{\psi}(x_1)\,\psi(x_2)\,\overline{\psi}(y)\,\phi(y)\,\psi(y)\,\overline{\psi}(z)\,\phi(z)\,\psi(z)\,|0\rangle$$

in the probability amplitude of the scattering event. The fermion operators must definitely be contracted as indicated. If the inner line is interpreted as an antiparticle moving from z to y, then the sequence of the operators $\overline{\psi}(y)$ and $\psi(z)$ is already correct, because the creation operator must be on the right side. But if the inner line is interpreted as a particle moving from y to z, then the operators $\overline{\psi}(y)$ and $\psi(z)$ must be permuted, resulting into a change of sign.

Secondly according to rule G of box 23.1 the propagator

$$\widetilde{S}(k) \overset{(12.24)}{=} \frac{i(\gamma^\nu k_\nu + m\frac{c}{\hbar})}{k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'} \,,$$

which is linear in k, must be inserted for the inner line. For an antinucleon moving from z to y, the momentum's sign is opposite to that of a nucleon moving from y to z. This difference of sign is not compensated by the sign change due to permutation of the operators in the matrix element, because the sign of $mc/\hbar$ does not change.

Another example: If a perturbative computation is done for the propagator of the Yukawa-boson, similar to the perturbative computation done in (20.47) for the boson-propagator of ψ^4-theory, then in 2. order the right diagram in (23.38) is encountered. This diagram again allows for different interpretations. First interpretation: The boson is moving with wavenumber k_1 from left to y, decays due to pair-creation into a fermion and an anti-fermion, which then at z mutually annihilate into a boson, which is going out with wavenumber k_1 to the right. Second interpretation: A boson is moving from left to y, while at the same time a fermion is moving from z to y. At y the fermion absorbs the boson, moves on to z, and there emits a boson, which is going out to the right with wavenumber k_1. Third interpretation: A boson moves from left to y, while at the same time an anti-fermion is moving from z to y. At y the anti-fermion absorbs the boson, then moves to z, and there emits a boson, which moves out to the right with wavenumber k_1.

To compute the diagram, a matrix element must be contracted to propagators. The contractions are incompatible for the different interpretations:

$$\langle 0|\, T\phi(x_1)\, \phi(x_2)\, \overline{\psi}(y)\, \phi(y)\, \psi(y)\, \overline{\psi}(z)\, \phi(z)\, \psi(z)\, |0\rangle$$

The contraction brackets of the fermions are identical for all three interpretations. There are no crossings of fermion brackets. With the first interpretation (a nucleon and an antinucleon are moving from y to z) the operators of both fermion-pairs must be permuted, because the creation operators must be right of the annihilation operators. The 2 permutations in total give a factor (+1). The second interpretation (one nucleon moves from z to y, and one nucleon moves from y to z) requires only the permutation of $\overline{\psi}(y)$ and $\psi(z)$, resulting into a factor (-1). The third interpretation (one antinucleon moves from z to y, and one antinucleon moves from y to z) requires only the permutation of $\overline{\psi}(z)$ and $\psi(y)$, again giving the factor (-1).

The graphs are not painted arbitrarily, but they are abstract representations of algebraic formulas, which we are translating back to algebraic formulas following the rules of box 23.1. If there are ambiguities, then we must go a step backwards, and clarify the correct application and interpretation of the graphs due to algebraic computations ab initio. We avoid that tedious task, and take without proof from the literature the simple result:

> **Theorem:** The computations of graphs of Dirac fields are consistent with algebraic computations if and only if inner fermion lines are interpreted with no exceptions as particles but not as antiparticles. (23.39)

This rule has been inserted as K in box 23.1. A general result can be concluded from this rule: If in a graph there is a loop consisting of n fermion-lines and zero boson-lines, then in the matrix element there is a factor

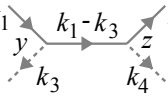
$$\langle 0|\, \ldots\, \overline{\psi}(y_1)\, \psi(y_1)\, \overline{\psi}(y_2) \ldots\, \psi(y_{n-1})\, \overline{\psi}(y_n)\, \psi(y_n) \ldots\, |0\rangle\ .$$

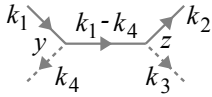
The contraction of the $2 \times n$ fermion operators to a loop of n particles (and zero antiparticles) always adds a factor (-1) to the overall result, because

always one operator pair $\overline{\psi}\,\psi$ must be permuted to close the loop.

23.1.5 Annihilation into Bosons

The annihilation of a nucleon (which is coming in with wavenumber k_1) and an antinucleon (which is coming in with wavenumber k_2) into two bosons (which are going out with wavenumbers k_3 and k_4) has been mentioned already in the previous section as an example for a graph with an inner fermion line. Now we are going to investigate this process in detail.

$$\overline{S}^{(2)} \mathrel{\widehat{=}} 2 \cdot$$ 

$$+\, 2 \cdot$$ 

(23.40)

The momentum $\hbar(\boldsymbol{k}_1 - \boldsymbol{k}_3)$ resp. $\hbar(\boldsymbol{k}_1 - \boldsymbol{k}_4)$ tagged to the inner line is interpreted according to rule K of box 23.1 parallel (not antiparallel) to the triangular arrow, because the inner line must be interpreted as a particle (not as an antiparticle).

The particle coming in with k_1 may dock to vertex y or z:

$$\langle 0|\, T\phi(x_3)\, \phi(x_4)\, \overline{\psi}(x_1)\, \psi(x_2)\, \overline{\psi}(y)\, \phi(y)\, \psi(y)\, \overline{\psi}(z)\, \phi(z)\, \psi(z)\, |0\rangle_c$$

$$\langle 0|\, T\phi(x_3)\, \phi(x_4)\, \overline{\psi}(x_1)\, \psi(x_2)\, \overline{\psi}(y)\, \phi(y)\, \psi(y)\, \overline{\psi}(z)\, \phi(z)\, \psi(z)\, |0\rangle_c$$

In the first variant of the possible contractions of the fermion operators there are 3 crossings of the contraction brackets. In addition, all 3 operator-pairs must be permuted to make sure that the creation operator is always on the right side. Thus one gets in total the factor $(+1)$. In the second variant there are no crossings of the contraction brackets. Only 2 operator-pairs need to be permuted, because the sequence of the pair $\psi(y)\,\overline{\psi}(z)$ is already correct (the virtual inner fermion is moving in this case from z to y). Thus we again get a factor $(+1)$ in this case, and both alternatives can be combined to a graph with symmetry factor 2. As in all diagrams of Yukawa-theory (and of QED) the symmetry factor (in this case 2!) is caused exclusively by the permutations of the vertices, and will be compensated

in the following by the factor 1/2! caused by the Taylor expansion of the interaction term. This comes due to the fact, that — different from ψ^s-interaction — the three operators $\overline{\psi}(y)\,\phi(y)\,\psi(y)$ of an interaction term can not mutually replace themselves.

The outgoing boson k_3 can be constructed by contraction of $\phi(x_3)$ with $\phi(y)$ or with $\phi(z)$. The contraction of the second, outgoing boson then is uniquely fixed. There is a factor (-1) due to rule F of box (23.1) for the incoming antinucleon. Thus one gets in second order of perturbation theory the following probability amplitude for the annihilation of a nucleon and an antinucleon into two bosons:

$$\begin{aligned}\overline{S}^{(2)} &\overset{(23.40)}{=} -2\cdot\frac{(-ig\sqrt{\hbar c})^2}{2!}\\ &\cdot\left(\prod_{j=1}^{4}\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}}\right)2\pi\Omega\,\delta\Big(k_1^0+k_2^0-k_3^0-k_4^0\Big)\,\delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)}\\ &\cdot\,{}^{r_2}\overline{v}^{\boldsymbol{k}_2}\left(\frac{i\Big(\gamma^\nu(k_1-k_3)_\nu+m\frac{c}{\hbar}\Big)}{(k_1-k_3)^2-m^2\frac{c^2}{\hbar^2}+i\epsilon'}+\frac{i\Big(\gamma^\nu(k_1-k_4)_\nu+m\frac{c}{\hbar}\Big)}{(k_1-k_4)^2-m^2\frac{c^2}{\hbar^2}+i\epsilon'}\right){}^{r_1}u^{\boldsymbol{k}_1}\end{aligned}\tag{23.41}$$

24 Quantum Electrodynamics

Quantum electrodynamics is the description of the interaction of the electrically charged elementary fields with their gauge field. The

$$\text{photon} = \gamma$$

is the quantum of the gauge field. These electrically charged elementary fields are known:

The three leptons

$$\text{electron} = e = e^- \quad \text{myon} = \mu = \mu^- \quad \text{tauon} = \tau = \tau^-$$

and their three antiparticles

$$\text{positron} = e^+ \quad \text{anti-myon} = \mu^+ \quad \text{anti-tauon} = \tau^+$$

as well as the six quarks

$$\text{up} = u \quad \text{down} = d \quad \text{strange} = s$$

$$\text{charm} = c \quad \text{bottom} = b \quad \text{top} = t$$

and their six antiparticles

$$\text{anti-up} = \bar{u} \quad \text{anti-down} = \bar{d} \quad \text{anti-strange} = \bar{s}$$

$$\text{anti-charm} = \bar{c} \quad \text{anti-bottom} = \bar{b} \quad \text{anti-top} = \bar{t}$$

and the two vector bosons

$$W^+ \qquad W^- \ .$$

The photon, the W^+, and the W^- have no antiparticles.[1] The photon

[1] Some people say: The photon is it's own antiparticle. That's a question of definition. If the notion particle/antiparticle is *defined* such that particle and antiparticle are related due to a C transformation (see section 11.2.3), then the photon indeed is the antiparticle of the photon, and the W^+ is the antiparticle of the W^-. But if the definition is, that a

carries no electric charge. The charge of the three leptons and the W^- is

$$-e = -1,6 \cdot 10^{-19}\text{C} \ ,$$

while the charge of the three antileptons and the W^+ is

$$+e = +1,6 \cdot 10^{-19}\text{C} \ .$$

The electrical charge of the quarks u, c, t is $+2e/3$, the charge of the quarks d, s, b is $-e/3$. Their antiparticles carry the reversed charges, respectively.

Attention: Different from us, about 50 % of the authors define $+e = -1,6 \cdot 10^{-19}\text{C}$.

The three leptons have same charges and same spins, but differ by their masses and electric and magnetic moments. The masses of the six quarks are different from another as well. The photon (the gauge boson of electromagnetic interaction) has rest mass zero, while the gauge bosons W^+ and W^- of weak interaction have finite rest mass.

Leptons and quarks are the quanta of Dirac fields, i. e. their spin is $\hbar/2$. The photon and the W^+ and the W^- have spin $\hbar$.

All listed particles are assumed to be elementary particles with no substructure.

24.1 Feynman-Rules

In the previous chapter we listed in box 23.1 the rules for the computation of S-matrix elements of Yukawa-theory. To transfer these rules to quantum-electrodynamics, all formulas which are referring to the pion-field of Yukawa-theory must be translated into the corresponding formulas of the photon field. That has been done in box 24.1 on page 510. We now will check the translation of those rules step by step:

particle and it's antiparticle can be created pairwise due to decay of a photon, and that a particle and it's antiparticle can annihilate pairwise into a photon, then the notion of particles and their antiparticles is applicable only to fermions. Neither photons, nor W^+/W^- bosons nor Z^0 bosons nor gluons nor Higgs bosons are created in pairs due to photon decay, nor can they pairwise annihilate into photons. In this book, the latter definition for particles/antiparticles is preferred.

Rule C: The lines of massless bosons are drawn in Feynman-graphs as wavy lines. According to experience, the flavor of Fermions is conserved at any vertex in QED. That means: If e. g. an e^- is coming in at a vertex, then the second fermion line at this vertex must be either an outgoing electron or an incoming positron. If a $\bar{c}$ is coming in, then the second fermion line at this vertex can't be anything else than an incoming c or an outgoing $\bar{c}$. Thus always one fermion-arrow is directed towards the vertex, and one fermion-arrow is directed away from the vertex. If the fermion-arrow of an external line is pointing towards the vertex, then it could symbolize a fermion moving towards the vertex, or an antifermion moving away from the vertex. A unique interpretation of the diagram can be enforced by additional little arrows drawn at the side of the fermion lines, which are indicating the direction of the momentum. Inner lines must according to theorem (23.39) always be interpreted as particles (not antiparticles), whose momentum direction is identical to the direction of the triangular fermion arrows.

The direction of photon momentum is in general not indicated by arrows on the wavy lines. If needed, the momentum direction can be indicated by a little arrow at the side of the wavy line. For inner lines, the direction of photon momentum does not matter, as the photon propagator is a function of k^2, see rule G.

Rule D: The Lagrangian of quantum electrodynamics is

$$\mathcal{L} \overset{(8.21)}{=} \overline{\psi}_{(W)}\Big(i\hbar c\gamma^\nu(\mathrm{d}_\nu + \frac{i}{\hbar}qA_{(W)\nu}) - mc^2\Big)\psi_{(W)} - \frac{1}{4\mu_0}F_{\sigma\tau}F^{\sigma\tau}\ .$$

Comparing this with the Lagrangian

$$\mathcal{L} \overset{(23.8)}{=} \overline{\psi}_{(W)}\Big(i\hbar c\gamma^\nu \mathrm{d}_\nu - g(\hbar c)^{3/2}\phi_{(W)} - mc^2\Big)\psi_{(W)} + \\ + \frac{1}{2}\Big(\hbar^2c^2\mathrm{d}_\mu\mathrm{d}^\mu - M^2c^4\Big)\phi^2_{(W)}$$

of Yukawa-theory, the following replacement in the interaction term is obvious:

Box 24.1 : **Feynman-rules in energy-momentum-space** for the computation of the component $\overline{S}^{(n)}$ of the scattering amplitude in quantum electrodynamics

A $\overline{S}^{(n)}$ is equal to the sum of all connected 1PI graphs with n vertices.

B The symmetry factor is the number of alternatives for the pairwise contraction of the operators, which are building up the graph, to the propagators of the graph. Only one of the equivalent graphs is inserted into the scattering amplitude, and multiplied by the symmetry factor.

B′ When combining the operators to propagators, each permutation of two fermion operators gives a factor (-1).

C The structure of any vertex is  .

D For the n vertices insert a factor

$$\frac{1}{n!}\Big(\frac{-iq\gamma^{\nu}}{\hbar}\Big)^{n} \quad .$$

Contract the index ν with the corresponding index of the docking photon.

E For incoming and outgoing lines with four-wavenumbers k_{in} and k_{out} insert a factor

$$2\pi\Omega\,\delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}}\Big)\,\delta_{\sum_{\text{in}}\boldsymbol{k}_{\text{in}}\,,\,\sum_{\text{out}}\boldsymbol{k}_{\text{out}}} \quad .$$

F For each external line with wavenumber k insert a factor

$$\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}}\cdot\begin{cases} {}^{r}u^{\boldsymbol{k}} \text{ for incoming lepton} \\ {}^{r}\overline{u}^{\boldsymbol{k}} \text{ for outgoing lepton} \\ (-{}^{r}\overline{v}^{\boldsymbol{k}}) \text{ for incoming antilepton} \\ (-{}^{r}v^{\boldsymbol{k}}) \text{ for outgoing antilepton} \\ \epsilon^{(\alpha)}_{\boldsymbol{k}\nu}\hbar c\sqrt{\mu_0} \text{ for incoming photon} \\ \epsilon^{(\alpha)*}_{\boldsymbol{k}\nu}\hbar c\sqrt{\mu_0} \text{ for outgoing photon} \; . \end{cases}$$

continued on next page!

(continued from previous page)

G For each inner photon-line with wavenumber k insert a factor

$$\widetilde{D}_{\nu\mu}(k) \overset{(17.86)}{=} \frac{-ig_{\nu\mu}\,\mu_0\hbar c}{k^2+i\epsilon'}.$$

For each inner fermion-line with wavenumber k insert a factor

$$\widetilde{S}(k) \overset{(12.24)}{=} \frac{i(\gamma^\nu k_\nu + mc/\hbar)}{k^2-(mc/\hbar)^2+i\epsilon'}\ .$$

H Sum and integrate over the wavenumber k of an inner line by

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\ ,$$

if k is not fixed due to conservation of energy and momentum.

J Apply the rearrangement-operator $\mathcal{U}_{\mathrm{S}} = (23.21)$.

K Inner fermion-lines must *always* be interpreted as particles (not as antiparticles).

$$-g(\hbar c)^{3/2}\phi_{(W)} \longrightarrow +i\hbar c\gamma^\nu\frac{i}{\hbar}q\,A_{(W)\nu}$$

$$\Longrightarrow \text{ vertex factor: } -ig\sqrt{\hbar c} \longrightarrow \frac{-iq\gamma^\nu}{\hbar}$$

The index $^\nu$ is contracted by factors, which we will encounter in rules F and G. For q always insert into the vertex factor the charge of the field, not the charge of the antifield, no matter whether the fermion lines which are connected to the vertex are interpreted as fields or as antifields. The vertex factors of an electron current and of a positron current both are $+ie\gamma^\nu/\hbar = +i|e|\gamma^\nu/\hbar$. The vertex factors of an up current and an anti-up current both are $-i2e\gamma^\nu/(3\hbar)$. From the point of view of classical physics it is surprising, that QED is using identical charge parameters, no matter whether one is considering particles or antiparticles. But we will see, that the differences — which of course must exist — are accomplished by a different number of permutations of fermion operators (when applying rule B′) as well as by different spinor functions (when applying rule F).

Rule F: The rules for fermions can be transferred from box 23.1 of Yukawa-theory. To fix the rule for photons, we must go back to the derivation of the LSZ-reduction-formula, which we investigated in section 19.3 for uncharged Klein-Gordon-fields. For that purpose we compare the field-operator of the uncharged Klein-Gordon field

$$\phi(x) \overset{(19.11)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} + a^{\dagger}_{\boldsymbol{k}} \exp\{+ikx\} \Big)$$

with the four space-time components

$$A^{\nu}(x) \overset{(17.79)}{=} \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sqrt{\frac{\mu_0 c^2 \hbar^2}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \cdot \cdot \Big(\epsilon^{(\alpha)\nu}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} \exp\{-ikx\} + \epsilon^{(\alpha)\nu *}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} \exp\{+ikx\} \Big)$$

of the photon field-operator. As usual and defined in (19.21), field-operators with an index $_{(W)}$ are the complete operators of interacting fields. Operators without this index are operators in the interaction picture. In the derivation of the LSZ-formula, the relations (A.187) of the Klein-Gordon field are replaced by the following relations for the ν-component of the photon field:

$$\sum_{\alpha=0}^{3} \epsilon^{(\alpha)\nu}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} = ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega\mu_0 c^2 \hbar^2}} \int_{\Omega} \mathrm{d}^3x \exp\{ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} A^{\nu}(x) \tag{24.1a}$$

$$\sum_{\alpha=0}^{3} \epsilon^{(\alpha)\nu *}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} = -ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega\mu_0 c^2 \hbar^2}} \int_{\Omega} \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} A^{\nu}(x) \tag{24.1b}$$

The second relation is the adjoint of the first. The first can be checked by insertion of the field-operator, as done in (A.187).

Now (24.1a) is multiplied by $\sum_{\nu=0}^{3} \epsilon^{(\beta)*}_{\boldsymbol{k}\nu}$, and (24.1b) is multiplied by $\sum_{\nu=0}^{3} \epsilon^{(\beta)}_{\boldsymbol{k}\nu}$. As the index ν is showing up twice in these product, the sign $\sum_{\nu}$ is dispensable due to the sum-convention. The imaginary parts of the polarization vectors — see (17.9) — disappear in the product. As these

vectors are unit vectors,

$$\sum_{\alpha=0}^{3} \epsilon_{\boldsymbol{k}\nu}^{(\beta)*} \epsilon_{\boldsymbol{k}}^{(\alpha)\nu} = \sum_{\alpha=0}^{3} \epsilon_{\boldsymbol{k}\nu}^{(\beta)} \epsilon_{\boldsymbol{k}}^{(\alpha)\nu*} = 1 \ .$$

Thus one gets the Fourier-operators

$$c_{\boldsymbol{k}}^{(\beta)} = ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int\limits_{\Omega} \mathrm{d}^3x \exp\{ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \frac{A^\nu(x)\, \epsilon_{\boldsymbol{k}\nu}^{(\beta)*}}{c\hbar\sqrt{\mu_0}}$$

$$c_{\boldsymbol{k}}^{(\beta)\dagger} = -ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int\limits_{\Omega} \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \frac{A^\nu(x)\, \epsilon_{\boldsymbol{k}\nu}^{(\beta)}}{c\hbar\sqrt{\mu_0}} \ .$$

These operators are dimension-less scalars, as needed. Comparing these relations with (A.187), one can see that in the following steps of the derivation of the LSZ-formula, which correlate to (A.187)ff, the field-operators of the uncharged Klein-Gordon bosons must be replaced as follows:

$$\begin{aligned} \text{incoming or outgoing KG-boson}: &\quad \phi(x) \\ \text{incoming photon, polarization } \epsilon_{\boldsymbol{k}}^{(\alpha)}: &\quad \frac{A^\nu(x)\, \epsilon_{\boldsymbol{k}\nu}^{(\alpha)}}{c\hbar\sqrt{\mu_0}} \\ \text{outgoing photon, polarization } \epsilon_{\boldsymbol{k}}^{(\alpha)}: &\quad \frac{A^\nu(x)\, \epsilon_{\boldsymbol{k}\nu}^{(\alpha)*}}{c\hbar\sqrt{\mu_0}} \end{aligned} \tag{24.3}$$

After that we inserted

$$\left(k^2 - m^2\frac{c^2}{\hbar^2}\right) = \frac{i}{\hbar c}\, \widetilde{G}^{-1}(k) \tag{24.4}$$

into (A.201). Instead of that we now replace for photons (because their rest-mass is zero)

$$k^2 \overset{(17.86)}{=} -ig_{\sigma\tau}\, \mu_0 \hbar c\, \widetilde{D}_{\sigma\tau}^{-1}(k) \ . \tag{24.5}$$

As an exception, there is *no* automatic summation on the right side over the indices σ and τ. We will return immediately to the question, how the

space-time indices shall be handled. In the LSZ-formula (19.49) there is for each incoming or outgoing Klein-Gordon particle a factor $\widetilde{G}^{-1}(k)\,\phi(x)$. Because of (24.5) and (24.3), this factor must be replaced for each incoming or outgoing photon by

$$\begin{aligned}
&\text{incoming photon, polarization } \epsilon^{(\alpha)}_{\boldsymbol{k}}:\\
&\qquad \widetilde{G}^{-1}(k)\,\phi(x) \;\longrightarrow\; -g_{\sigma\tau}\,\hbar c\,\sqrt{\mu_0}\,\widetilde{D}^{-1}_{\sigma\tau}(k)\,A^{\nu}(x)\,\epsilon^{(\alpha)}_{\boldsymbol{k}\nu}\\
&\text{outgoing photon, polarization } \epsilon^{(\alpha)}_{\boldsymbol{k}}:\\
&\qquad \widetilde{G}^{-1}(k)\,\phi(x) \;\longrightarrow\; -g_{\sigma\tau}\,\hbar c\,\sqrt{\mu_0}\,\widetilde{D}^{-1}_{\sigma\tau}(k)\,A^{\nu}(x)\,\epsilon^{(\alpha)*}_{\boldsymbol{k}\nu}\;. \qquad (24.6)
\end{aligned}$$

Those factors, which are new in comparison to Yukawa-theory, are indicated in rule F.

We spare ourselves the tiresome evaluation, how in (24.5) and in the following formulas the space-time indices of the photon functions and of the factors $g_{\nu\mu}$ shall be handled, and take from the literature the simple and plausible result: The index ν of the vertex-factor (rule D) must be contracted with the index ν of the polarization vector (rule F) resp. with the index ν of the propagator (rule G) of the photon which is docking to this vertex. This is documented in rule D in box 24.1. Due to this rule, the probability amplitude in total is a Lorentz-scalar.

Not indicated in rule F: If the spin resp. the polarization is not observed, then the mean value of the spinor indices $r = 1, 2$ of the fermion-operators resp. of the polarization indices $\alpha = 0, 1, 2, 3$ of the photon-operators must be taken (for incoming particles) resp. their sum must be taken (for outgoing particles).

Rule G: The photon propagator has been computed in section 17.5. The indices $_{\mu}$ and $_{\nu}$ of the photon propagator and the indices $^{\mu}$ and $^{\nu}$ of the vertex factors (rule D) are summed-up according to the summation convention. For that purpose, clearly the vertex factor γ^{μ} at one end of the inner photon line must be indexed by μ, and the vertex factor γ^{ν} at the other end of the inner photon line by ν. Due to this multiplication the product of the photon-propagator and the vertex factors becomes a Lorentz-scalar.

Rule J: This rule is transferred without any change from Yukawa-theory.

Due to this rule, all fermion-products become spinor-scalars. This rule could be "spared", if we would write down all spinor indices explicitly, like we are writing down in the rules all space-time indices explicitly. According to the aesthetic taste of the author, however, rule J is less nasty than lots of spinor indices. The scattering amplitude in total is a scalar, because all vector-factors of the photons and all spinor-factors of the fermions are combined to scalar products.

Rule K: This rule, which is based on theorem (23.39), again was transferred without change from Yukawa-theory.

24.2 Cross-Section and Decay-Rate

Let N_A particles or antiparticles of type A be in a cylindric volume of length L_A and cross-section area F. Let in a second cylindric volume of length L_B and cross-section area F be N_B particles or antiparticles of type B (which may or may not be identical to the particle type A). The two bundles of particles are beamed upon another such, that the both head areas F exactly hit one another, and the both bundles penetrate one another along their lengths. Let the particle density be in both bundles so low, and the lengths L_A and L_B of the bundles so small, that essentially never one particle is covered by another particle of the same bundle (is "in the wind shadow" of another particle of same type). Then the count of scattering events will be proportional to the number of incoming particles Z_A, proportional to the number of incoming particles Z_B, and inversely proportional to the cross section F of the bundles:

$$\text{count of scattering events} = \sigma \frac{Z_A Z_B}{F} \tag{24.7}$$

The proportionality factor σ, whose dimension obviously is area, is simply called "cross section". It is not explicitly mentioned in this equation, that the cross section not only is dependent of the number and types of the particles, but also of their respective energies and their respective polarizations. Note that the count of scattering events is independent of the lengths of the bundles, due to the assumption of very sow particle density. Consequently this formula is invariant under coordinate transformations between different

reference systems, provided that the origin of the systems is on the line of movement of the particle bundles, because only L_A and L_B change under such coordinate transformations due to relativistic length contraction, while the area F is invariant. Often systems are chosen, whose origin is defined by the common center of mass of one particle A and one particle B, or by the center of mass of one of the two bundles.

Equation (24.7) is the definition of the total or integral cross section. Frequently subdivisions ("channels") of the cross section are defined with regard to different types of scattering events. If e. g. a bundle of electrons collides with a bundle of positrons, then — provided the energy is sufficiently high — many different reactions like $e^-e^+ \to e^-e^+$, $e^-e^+ \to \mu^-\mu^+$, $e^-e^+ \to$ hadrons, $e^-e^+ \to \tau^-\tau^+\gamma$, ... may be observed. Separate cross sections can be assigned to each of these scattering events.

Further details of a scattering events may be documented by differential cross sections. If for example in a scattering event (besides other particles) a myon is created, then the differential cross section

$$\frac{\mathrm{d}\sigma_\mu}{\mathrm{d}\widetilde{\Omega}} = \frac{\text{myon count in spherical angle } \mathrm{d}\widetilde{\Omega}}{Z_A Z_B / F} \tag{24.8}$$

may be measured by counting the number of myons spreading from the scattering center into the spherical angle $\mathrm{d}\widetilde{\Omega}$. (We mark the spherical angle $\widetilde{\Omega}$ by a tilde, to avoid confusion with the normalization volume Ω.) If furthermore the energies of these myons are measured, then the double differential cross section

$$\frac{\mathrm{d}^2\sigma_\mu}{\mathrm{d}\widetilde{\Omega}\mathrm{d}E} = \frac{\text{myon count within } (E, E+\mathrm{d}E) \text{ and } \mathrm{d}\widetilde{\Omega}}{N_A N_B / F} \tag{24.9}$$

can be indicated. In spite of the notation as differentials, the quantities $\mathrm{d}\widetilde{\Omega}$ and $\mathrm{d}E$ are not infinitesimally small in these formulas, but denote finite intervals $\Delta\widetilde{\Omega}$ and ΔE, the size of which is dependent of the resolution capabilities of the detectors.

The cross section clearly is closely related to the scattering amplitude, which was computed in the previous chapters. Before explaining the exact

relation, we define the scattering matrix $\mathcal{M}$. In all scattering amplitudes S, a normalization factor is showing up for each incoming or outgoing particle, and furthermore for all particles together a delta function and a Kronecker symbol, which guarantee conservation of energy and momentum. A scattering matrix $\mathcal{M}$ is defined by

$$\overline{S}^{(n)} \overset{(23.27)}{\equiv} \mathcal{M}^{(n)} \cdot \Big(\prod_{\text{in}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{in}}}\Omega}} \Big) \Big(\prod_{\text{out}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{out}}}\Omega}} \Big) \cdot \\ \cdot \; 2\pi\Omega \, \delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}} \Big) \, \delta_{\sum_{\text{in}} \boldsymbol{k}_{\text{in}} \, , \sum_{\text{out}} \boldsymbol{k}_{\text{out}}} \; . \tag{24.10}$$

We might call $\mathcal{M}$ the "interesting" factor of the scattering amplitude, and the remainder the "self-evident" factor. In the slang of high-energy physicists however, $\mathcal{M}$ is called the "dynamic" factor, while the reminder is called the "kinematic" factor.

The S-matrix $$S_{\boldsymbol{k}_3 \ldots \boldsymbol{k}_n, \boldsymbol{k}_1 \boldsymbol{k}_2} \overset{(19.43b)}{=} \langle \boldsymbol{k}_3 \ldots \boldsymbol{k}_n | \, S \, | \boldsymbol{k}_1 \boldsymbol{k}_2 \rangle \tag{24.11}$$

is the probability amplitude of a scattering event, in which two particles with momenta $\hbar\boldsymbol{k}_1$ and $\hbar\boldsymbol{k}_2$ are coming in, and $(n-2)$ particles with momenta $\hbar\boldsymbol{k}_3 \ldots \hbar\boldsymbol{k}_n$ are going out. The probability of this event is

$$|S|^2 = |\mathcal{M}|^2 \cdot \Big(\prod_{\text{ein}} \frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{ein}}}\Omega} \Big) \Big(\prod_{\text{aus}} \frac{1}{2\hbar\omega_{\boldsymbol{k}_{\text{aus}}}\Omega} \Big) \cdot \\ \cdot \; \Big| 2\pi \, \delta\Big(\sum_{\text{ein}} k^0_{\text{ein}} - \sum_{\text{aus}} k^0_{\text{aus}} \Big) \Big|^2 \, \Omega^2 \, \delta_{\sum_{\text{ein}} \boldsymbol{k}_{\text{ein}} \, , \sum_{\text{aus}} \boldsymbol{k}_{\text{aus}}} \; . \tag{24.12}$$

Here already $|\Omega\delta_{\sum_{\text{in}} \boldsymbol{k}_{\text{in}} \, , \sum_{\text{out}} \boldsymbol{k}_{\text{out}}}|^2 = \Omega^2 \delta_{\sum_{\text{in}} \boldsymbol{k}_{\text{in}} \, , \sum_{\text{out}} \boldsymbol{k}_{\text{out}}}$ has been inserted. The modulus-square of the delta function under an integral gives

$$\int_{-\infty}^{+\infty} \mathrm{d}k_{\mathrm{in}}\, \delta(k_{\mathrm{in}} - k_{\mathrm{out}})\, f(k_{\mathrm{out}}) = f(k_{\mathrm{out}}) \quad \Longrightarrow$$

$$\int_{-\infty}^{+\infty} \mathrm{d}k_{\mathrm{in}}\, \delta(k_{\mathrm{in}} - k_{\mathrm{out}})\, \delta(k_{\mathrm{in}} - k_{\mathrm{out}}) = \delta(k_{\mathrm{out}} - k_{\mathrm{out}}) = \delta(0)\ .$$

For a better understanding of the irritating factor $\delta(0)$, remember that we used only for the sake of convenience the notation $t = -\infty$ to $t = +\infty$ for the time interval, during which the scattering is taking place. In the sequel of this definition in (19.48), however, we emphasized that actually a finite time interval $T = t_a - t_e$ is meant, with T being very large as compared to all other time intervals which are characteristic for this scattering event. We define $\Delta k^0 \equiv \sum_{\mathrm{in}} k^0_{\mathrm{in}} - \sum_{\mathrm{out}} k^0_{\mathrm{out}}$ as a shorthand notation, and compute the delta function

$$\begin{aligned}\delta(\Delta k^0) &\overset{(7.16b)}{=} \lim_{T\to\infty} \frac{1}{2\pi} \int_{-cT/2}^{+cT/2} \mathrm{d}x^0\, \exp\{+i\Delta k^0 x^0\} \\ &= \lim_{T\to\infty} \frac{1}{2\pi}\, \frac{2\sin(\Delta k^0 cT/2)}{\Delta k^0}\ . \end{aligned} \tag{24.13}$$

Using the rule of l'Hospital, this results into

$$\begin{aligned}\delta(0) &= \lim_{\Delta k^0\to 0} \delta(\Delta k^0) = \lim_{T\to\infty} \frac{1}{\pi} \lim_{\Delta k^0\to 0} \frac{cT\cos(\Delta k^0 cT/2)}{2} \\ &\approx \frac{cT}{2\pi} \text{ for very large } T\ . \end{aligned} \tag{24.14}$$

Consequently the modulus square of the scattering amplitude is

$$\begin{aligned}|S|^2 = |\mathcal{M}|^2 \cdot \Big(\prod_{\mathrm{in}} \frac{1}{2\hbar\omega_{\boldsymbol{k}_{\mathrm{in}}}\Omega}\Big)\Big(\prod_{\mathrm{out}} \frac{1}{2\hbar\omega_{\boldsymbol{k}_{\mathrm{out}}}\Omega}\Big) \cdot \\ \cdot\, \Omega cT\, 2\pi\Omega\, \delta\Big(\sum_{\mathrm{in}} k^0_{\mathrm{in}} - \sum_{\mathrm{out}} k^0_{\mathrm{out}}\Big)\, \delta_{\sum_{\mathrm{in}}\boldsymbol{k}_{\mathrm{in}},\, \sum_{\mathrm{out}}\boldsymbol{k}_{\mathrm{out}}}\ . \end{aligned} \tag{24.15}$$

According to (24.11), this is the probability that $(n-2)$ scattered particles with wave numbers $\boldsymbol{k}_3 \dots \boldsymbol{k}_n$ will go out, if two particles with wavenumbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ come in. As we don't have detectors of infinite precision at our disposal, this modulus square must be multiplied by the count of possible wavenumbers in unmeasurable close neighborhood of $\boldsymbol{k}_3 \dots \boldsymbol{k}_n$. With a resolution capability $\Delta\boldsymbol{k}$ of the detectors, we call that neighborhood

$$U(\boldsymbol{k}_j) \equiv \text{set of all } \boldsymbol{k} \text{ with } |\boldsymbol{k} - \boldsymbol{k}_j| \leq |\Delta\boldsymbol{k}| \; . \tag{24.16}$$

Due to the finite normalization volume Ω, any wavenumber $\boldsymbol{k}_j$ meets the condition

$$k_j^i \, \Omega^{1/3} = n_j^i \, 2\pi \quad \text{with } i = 1, 2, 3 \text{ and } n_j^i \in \mathbb{N} \; . \tag{24.17}$$

Therefore an outgoing particle, whose momentum is measured to be $\hbar\boldsymbol{k}_j$, may actually have one particular of

$$\prod_{i=1}^{3} n_j^i = \sum_{U(\boldsymbol{k}_j)} \frac{\Omega}{(2\pi)^3} \Big(\prod_{i=1}^{3} k^i \Big) \tag{24.18}$$

different momenta. Consequently

$$\begin{aligned} |S_{\boldsymbol{k}_3 \dots \boldsymbol{k}_n, \boldsymbol{k}_1 \boldsymbol{k}_2}|^2 \cdot \Bigg(\prod_{j=3}^{n} \sum_{U(\boldsymbol{k}_j)} \frac{\Omega \, (\prod_{i=1}^{3} k_j^i)}{(2\pi)^3} \Bigg) = \\ = |\mathcal{M}|^2 \, \Omega c T \Big(\prod_{j=1}^{2} \frac{1}{2\hbar\omega_{\boldsymbol{k}_j} \Omega} \Big) \Bigg(\prod_{j=3}^{n} \sum_{U(\boldsymbol{k}_j)} \frac{(\prod_{i=1}^{3} k_j^i)}{2\hbar\omega_{\boldsymbol{k}_j} (2\pi)^3} \Bigg) \cdot \\ \cdot \; 2\pi\Omega \, \delta\Big(\sum_{\text{in}} k^0_{\text{in}} - \sum_{\text{out}} k^0_{\text{out}} \Big) \, \delta_{\sum_{\text{in}} \boldsymbol{k}_{\text{in}} \, , \, \sum_{\text{out}} \boldsymbol{k}_{\text{out}}} \end{aligned} \tag{24.19}$$

is the probability of a scattering event, which can not be discerned from $S_{\boldsymbol{k}_3 \dots \boldsymbol{k}_n, \boldsymbol{k}_1 \boldsymbol{k}_2}$ due to the finite resolution of the instruments. Our formulas now are becoming quite laborious and confusing, because on the one hand we are assuming a discrete spectrum of possible wavenumbers (caused by the finite normalization volume Ω), while on the other hand we are explicitly

considering by summation over $U(\boldsymbol{k}_j)$ that the wavenumber spectrum is seemingly continuous due to the finite resolution of the detectors. Therefore it is better at this point, to change to a throughout continuum-notation due to

$$\begin{aligned}\sum_{U(\boldsymbol{k}_j)} \frac{\left(\prod_{i=1}^{3} k_j^i\right)}{2\hbar\omega_{\boldsymbol{k}_j}(2\pi)^3} \quad &\to \quad \int\limits_{U(\boldsymbol{k}_j)} \mathrm{d}^3k_j \, \frac{1}{2\hbar\omega_{\boldsymbol{k}_j}(2\pi)^3} \\ \Omega\, \delta_{\sum_{\text{in}}\boldsymbol{k}_{\text{in}}\,,\,\sum_{\text{out}}\boldsymbol{k}_{\text{out}}} \quad &\to \quad (2\pi)^3\delta^{(3)}\Big(\sum_{\text{in}}\boldsymbol{k}_{\text{in}} - \sum_{\text{out}}\boldsymbol{k}_{\text{out}}\Big)\,.\end{aligned}$$

The $(n-2)$-dimensional lorentzinvariant "phase-space-volume" is defined by

$$\begin{aligned}\Phi_{\text{LIPS}}^{(n-2)} &\equiv \Big(\prod_{j=3}^{n}\int\limits_{U(\boldsymbol{k}_j)} \frac{\mathrm{d}^3k_j}{2\hbar\omega_{\boldsymbol{k}_j}(2\pi)^3}\Big)\,(2\pi)^4\,\delta^{(4)}\Big(\sum_{\text{in}}k_{\text{in}} - \sum_{\text{out}}k_{\text{out}}\Big) \;= \\ &= \Big(\prod_{j=3}^{n}\int\limits_{U(\boldsymbol{k}_j)} \frac{\mathrm{d}\widetilde{\Omega}\,\mathrm{d}|\boldsymbol{k}_j|\,|\boldsymbol{k}_j|^2}{2\hbar\omega_{\boldsymbol{k}_j}(2\pi)^3}\Big)\,(2\pi)^4\,\delta^{(4)}\Big(\sum_{\text{in}}k_{\text{in}} - \sum_{\text{out}}k_{\text{out}}\Big)\,. \qquad (24.20)\end{aligned}$$

The spherical angle $\widetilde{\Omega}$ must not be confused with the normalization volume Ω. The index $_{\text{LIPS}}$ stands for "lorentz-invariant phase space". $\Phi_{\text{LIPS}}^{(n-2)}$ is lorentzinvariant, because the four-dimensional delta function enforces the Lorentz invariance of the three-dimensional integrals. $\Phi_{\text{LIPS}}^{(m)}$ contains the factor (24.18). Thus it is proportional to the phase-space volume accessible to the outgoing particles. But $\Phi_{\text{LIPS}}^{(m)}$ itself is no phase-space volume. If the normalization factors N_j are dimension-less, then instead it has the strange dimension

$$\Big[\Phi_{\text{LIPS}}^{(m)}\Big] = \frac{\text{wavenumber}^{3m-4}}{\text{energy}^m}\,. \qquad (24.21)$$

$\Phi_{\text{LIPS}}^{(n-2)}$ contains the factor

$$\prod_{j=3}^{n} \frac{\mathrm{d}^3 k_j}{\hbar\omega_{\boldsymbol{k}_j}} = 4\pi \prod_{j=3}^{n} \frac{\mathrm{d}|\boldsymbol{k}_j|\,|\boldsymbol{k}_j|^2}{\hbar\omega_{\boldsymbol{k}_j}} \ .$$

Due to this factor, the scattering products with energy $\hbar\omega_{\boldsymbol{k}_j}$ will be more likely light particles with large momenta than heavy particles with small momenta. This is caused by the fact, that n_j in (24.17) is small for small $|\boldsymbol{k}_j|$, i. e. that only relatively few finite states are available for this scattering event. We say that the creation of heavy scattering products is "phase-space-suppressed".

The normalization volume Ω does not show up any more in $\Phi_{\mathsf{LIPS}}^{(n-2)}$. We also must remove it somehow from the normalization factors of the two incoming particles for the following reason. Though we are describing the incoming particles as plain waves, which may be detected anywhere in Ω with same probability, we know (because the experimental setup is arranged accordingly) that the both particles will be during the time interval $T = t_a - t_e$ of the scattering in the much smaller scattering volume

$$|v_{\mathrm{rel}}|\,TF \ll \Omega \tag{24.22}$$

with v_{rel} = relative velocity of the two incoming particles. In a laboratory system, in which the incoming particle$_1$ is at rest, $|v_{\mathrm{rel}}| = |v_2|$ is equal to the velocity of the other incoming particle. From the literature [48, §12] we take the lorentz-invariant generalization

$$\hbar\omega_{\boldsymbol{k}_1}\hbar\omega_{\boldsymbol{k}_2}|v_{\mathrm{rel}}| = \hbar\omega_{\boldsymbol{k}_1}\hbar\omega_{\boldsymbol{k}_2}\sqrt{(\boldsymbol{v}_1-\boldsymbol{v}_2)^2 - (\boldsymbol{v}_1\times\boldsymbol{v}_2)^2}\ , \tag{24.23}$$

which sometimes is called flow-factor. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are antiparallel, then

$$|v_{\mathrm{rel}}| = |v_1 - v_2| \qquad \text{if } \boldsymbol{v}_1 = r\boldsymbol{v}_2 \text{ with } r \in \mathbb{R}\ . \tag{24.24}$$

We spare ourselves the tedious task to describe the incoming particles as wave packets, with their wavenumbers centered around $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Instead for the sake of simplicity we continue to write them as plain waves $\boldsymbol{k}_1$ aund $\boldsymbol{k}_2$, and only replace the factor Ω by $|v_{\mathrm{rel}}|\,TF$:

$$|S_{\boldsymbol{k}_3\dots\boldsymbol{k}_n,\boldsymbol{k}_1\boldsymbol{k}_2}|^2 \cdot \left(\prod_{j=3}^{n} \int_{U(\boldsymbol{k}_j)} \mathrm{d}^3k_j\right) Z|\mathcal{M}|^2 \, \frac{c}{|v_{\mathrm{rel}}|\,F} \left(\prod_{j=1}^{2} \frac{1}{2\hbar\omega_{\boldsymbol{k}_j}}\right) \Phi_{\mathsf{LIPS}}^{(n-2)}$$
$$\text{with } \Phi_{\mathsf{LIPS}}^{(n-2)} = (24.20) \text{ and } v_{\mathrm{rel}} = (24.23) \tag{24.25}$$

The cross section

$$\sigma \overset{(24.7)}{=} \frac{\text{count of scattering events} \cdot F}{Z_A Z_B}$$

has been indicated above for Z_A and Z_B incoming particles. In equation (24.25), the

$$\frac{\text{scattering probability}_{\boldsymbol{k}_3\dots\boldsymbol{k}_n,\boldsymbol{k}_1\boldsymbol{k}_2}}{Z_A Z_B} \quad \text{with } Z_A = Z_B = 1$$

is computed. Thus the cross section of this scattering event is

$$\boxed{\begin{aligned} \sigma_{\boldsymbol{k}_3\dots\boldsymbol{k}_n,\boldsymbol{k}_1\boldsymbol{k}_2} &= |\mathcal{M}|^2 \, \frac{c}{|v_{\mathrm{rel}}|} \left(\prod_{j=1}^{2} \frac{1}{2\hbar\omega_{\boldsymbol{k}_j}}\right) \Phi_{\mathsf{LIPS}}^{(n-2)} \\ &\text{with } v_{\mathrm{rel}} = (24.23) \text{ and } \Phi_{\mathsf{LIPS}}^{(n-2)} = (24.20)\ . \end{aligned}} \tag{24.26}$$

The cross section can be measured, and the theory can be tested by means of this formula. As the scattering matrix $\mathcal{M}$, the phase-space volume $\Phi_{\mathsf{LIPS}}^{(n-2)} = (24.20)$, and the flow-factor (24.23) all are lorentz-invariant, the cross section (24.26) is lorentz-invariant as well.

In case $n = 4$, i. e. if two particles only are going out, then in the center-of-mass system (cms) $\boldsymbol{k}_2 = -\boldsymbol{k}_1$ and $\boldsymbol{k}_4 = -\boldsymbol{k}_3$. Thereby the integral over $\boldsymbol{k}_4$ in the phase-space of the outgoing particles can be computed:

$$\begin{aligned} \Phi_{\mathsf{LIPS}}^{(2)} &\overset{(24.20)}{=} \int_{U(\boldsymbol{k}_4)} \mathrm{d}^3k_4 \int_{U(\boldsymbol{k}_3)} \mathrm{d}^3k_3 \, \frac{(2\pi)^4\, \delta(k_1^0 + k_2^0 - k_3^0 - k_4^0)\, \delta^{(3)}(\boldsymbol{k}_3 + \boldsymbol{k}_4)}{4\hbar^2\omega_{\boldsymbol{k}_4}\omega_{\boldsymbol{k}_3}(2\pi)^6} \\ &= \int_{U(\boldsymbol{k}_3)} \mathrm{d}^3k_3 \, \frac{\delta(k_1^0 + k_2^0 - k_3^0 - k_4^0)}{16\pi^2\hbar^2\omega_{\boldsymbol{k}_4}\omega_{\boldsymbol{k}_3}} \end{aligned} \tag{24.27}$$

In this integral, $k_3^0, k_4^0, \omega_{\boldsymbol{k}_3}, \omega_{\boldsymbol{k}_4}$ are not independent because of

$$k_j^0 = \frac{\omega_{\boldsymbol{k}_j}}{c} \overset{(7.18)}{=} \sqrt{\boldsymbol{k}_3^2 + m_j^2 \frac{c^2}{\hbar^2}} \quad \text{for } j = 3, 4 \ . \tag{24.28}$$

Instead they are all uniquely fixed by $|\boldsymbol{k}_3|$. As the total energy in the cms (center-of-mass system)

$$E_{\text{CM}} = \hbar c k_{\text{CM}}^0 \equiv \hbar c (k_1^0 + k_2^0) \tag{24.29}$$

is to be considered constant as well, $\boldsymbol{k}_3$ is the only variable in the integral (24.27). Defining

$$\begin{aligned} k_{34}^0 &\equiv k_3^0 + k_4^0 \\ \frac{\mathrm{d}\, k_{34}^0}{\mathrm{d}\,|\boldsymbol{k}_3|} &\overset{(24.28)}{=} \frac{|\boldsymbol{k}_3|}{k_3^0} + \frac{|\boldsymbol{k}_3|}{k_4^0} \\ \mathrm{d}|\boldsymbol{k}_3| &= \frac{\mathrm{d}k_{34}^0}{|\boldsymbol{k}_3| \left((k_3^0)^{-1} + (k_4^0)^{-1} \right)} \ , \end{aligned} \tag{24.30}$$

and using

$$\int\limits_{U(\boldsymbol{k}_3)} \mathrm{d}^3 k = \int\limits_{U(\boldsymbol{k}_3)} \mathrm{d}\widetilde{\Omega}_3 \, \mathrm{d}|\boldsymbol{k}_3| \, \boldsymbol{k}_3^2 \ ,$$

$\Phi_{\text{LIPS}}^{(2)}$ may be written in the form

$$\begin{aligned} \Phi_{\text{LIPS}}^{(2)} &= \int\limits_{U(\boldsymbol{k}_3)} \mathrm{d}\widetilde{\Omega}_3 \, \mathrm{d}k_{34}^0 \, \frac{|\boldsymbol{k}_3| \, \delta(k_{\text{CM}}^0 - k_{34}^0)}{16\pi^2 \hbar^2 c^2 k_4^0 k_3^0 \left((k_3^0)^{-1} + (k_4^0)^{-1} \right)} \\ &= \int\limits_{U(\boldsymbol{k}_3)} \mathrm{d}\widetilde{\Omega}_3 \, \mathrm{d}k_{34}^0 \, \frac{|\boldsymbol{k}_3| \, \delta(k_{\text{CM}}^0 - k_{34}^0)}{16\pi^2 \hbar^2 c^2 k_{34}^0} \\ &= \int\limits_{U(\boldsymbol{k}_3)} \mathrm{d}\widetilde{\Omega}_3 \, \frac{|\boldsymbol{k}_3|}{16\pi^2 \hbar c E_{\text{CM}}} \ . \end{aligned} \tag{24.31}$$

Thus the differential cross section is

$$\boxed{\frac{\mathrm{d}\sigma_{\boldsymbol{k}_3\boldsymbol{k}_4,\boldsymbol{k}_1\boldsymbol{k}_2}}{\mathrm{d}\widetilde{\Omega}_3} = \frac{|\mathcal{M}|^2\,|\boldsymbol{k}_3|}{|v_{\mathrm{rel}}|\,\hbar\,16\pi^2 E_{\mathrm{CM}}^3} \quad \text{in the cms with } v_{\mathrm{rel}} = (24.23)\ .} \tag{24.32}$$

In appendix A.22 the useful relation

$$|\boldsymbol{k}_3| \overset{(A.154)}{=} \frac{S_{34}}{2E_{\mathrm{CM}}\hbar c} \tag{24.33a}$$

$$S_{34} \equiv \sqrt{\left(E_{\mathrm{CM}}^2 - (m_3+m_4)^2c^4\right)\left(E_{\mathrm{CM}}^2 - (m_3-m_4)^2c^4\right)} \tag{24.33b}$$

is proved, by which the cross section can be written in the form

$$\frac{\mathrm{d}\sigma_{\boldsymbol{k}_3\boldsymbol{k}_4,\boldsymbol{k}_1\boldsymbol{k}_2}}{\mathrm{d}\widetilde{\Omega}_3} = \frac{|\mathcal{M}|^2\,S_{34}}{|v_{\mathrm{rel}}|\,\hbar^2 c\,32\pi^2 E_{\mathrm{CM}}^4} \quad \text{in the cms with } v_{\mathrm{rel}} = (24.23) \text{ and } S_{34} = (24.33b) \tag{24.34}$$

In this notation, the momenta of the outgoing particles do not show up explicitly any more.

If the rest masses of the two incoming particles are identical (which is quite often the case in everyday laboratory work), then in the cms

$$\hbar\omega_{\boldsymbol{k}_1} = \hbar\omega_{\boldsymbol{k}_2} = \tfrac{1}{2}E_{\mathrm{CM}}\ , \tag{24.35}$$

and furthermore — as the particle flavor is conserved at each vertex —

$$m_3 = m_4 \quad \text{and} \quad \hbar\omega_{\boldsymbol{k}_3} = \hbar\omega_{\boldsymbol{k}_4} = \tfrac{1}{2}E_{\mathrm{CM}} \tag{24.36}$$

must hold as well. Thereby the cross section in the cms simplifies to

$$\frac{\mathrm{d}\sigma_{\boldsymbol{k}_3\boldsymbol{k}_4,\boldsymbol{k}_1\boldsymbol{k}_2}}{\mathrm{d}\widetilde{\Omega}_3} = \frac{|\mathcal{M}|^2\,\sqrt{E_{\mathrm{CM}}^2 - (2m_3c^2)^2}}{|v_{\mathrm{rel}}|\,\hbar^2 c\,32\pi^2 E_{\mathrm{CM}}^3}$$
$$\text{in the cms with } v_{\mathrm{rel}} = (24.23),\ \text{if } m_1 = m_2 \text{ and } m_3 = m_4\ .$$

In this result it is clearly visible, that heavy scattering products $m_3 + m_4 =$

$2m_3 \to E_{\rm CM}/c^2$ are phase-space-suppressed. Usually the incoming particles are accelerated to high-relativistic velocities. Then the formula simplifies further because of $E_{\rm CM} \gg m_1 c^2 \implies |v_{\rm rel}| \approx 2c$ to

$$\boxed{\begin{aligned} &\frac{\mathrm{d}\sigma_{\boldsymbol{k}_3\boldsymbol{k}_4,\boldsymbol{k}_1\boldsymbol{k}_2}}{\mathrm{d}\widetilde{\Omega}_3} = \frac{|\mathcal{M}|^2 \ \sqrt{E_{\rm CM}^2 - (2m_3c^2)^2}}{\hbar^2 c^2\, 64\pi^2 E_{\rm CM}^3} \\ &\text{in the cms (= center of mass system) ,} \\ &\text{if } m_1 = m_2 \text{ and } m_3 = m_4 \text{ and } E_{\rm CM} \gg m_1 c^2 \ . \end{aligned}} \tag{24.37}$$

A final possible simplification, which typically becomes possible in case $m_3 = m_1$, can be made in case $m_3 c^2 \ll E_{\rm CM}$. Then one finds

$$\begin{aligned} &\frac{\mathrm{d}\sigma_{\boldsymbol{k}_3\boldsymbol{k}_4,\boldsymbol{k}_1\boldsymbol{k}_2}}{\mathrm{d}\widetilde{\Omega}_3} = \frac{|\mathcal{M}|^2}{\hbar^2 c^2 64\pi^2 E_{\rm CM}^2} \\ &\text{in the cms, if } m_1 = m_2 \text{ and } m_3 = m_4 \\ &\text{and } E_{\rm CM} \gg m_1 c^2 \text{ and } E_{\rm CM} \gg m_3 c^2 \ . \end{aligned} \tag{24.38}$$

So far we always considered the case, that 2 particles are coming in, and $(n-2)$ scattered particles are going out. The S-matrix (24.10) holds with no changes as well in case that only 1 particle is coming in, and $(n-1)$ particle are going out. This case is not called scattering, but decay of the particle number 1. The probability for an incoming particle with wavenumber $\boldsymbol{k}_1$ to decay within the time interval T into $(n-1)$ particles with wavenumbers $\boldsymbol{k}_2 \ldots \boldsymbol{k}_n$, is

$$|S_{\boldsymbol{k}_2 \ldots \boldsymbol{k}_n, \boldsymbol{k}_1}|^2 \cdot \left(\prod_{j=2}^{n} \int\limits_{U(\boldsymbol{k}_j)} \mathrm{d}^3 k_j \right) \overset{(24.19)}{=} |\mathcal{M}|^2 \, \Omega c T \, \frac{1}{2\hbar\omega_{\boldsymbol{k}_1} \Omega} \, \Phi_{\rm LIPS}^{(n-1)}$$

with

$$\Phi_{\rm LIPS}^{(n-1)} \overset{(24.20)}{=} \left(\prod_{j=2}^{n} \int\limits_{U(\boldsymbol{k}_j)} \frac{\mathrm{d}^3 k_j}{2\hbar\omega_{\boldsymbol{k}_j} (2\pi)^3} \right) (2\pi)^4 \, \delta^{(4)}\Big(k_1 - \sum_{l=2}^{n} k_l \Big) \ . \tag{24.39}$$

The probability per time for this decay is called decay rate:

$$\boxed{\Gamma_{\boldsymbol{k}_2\ldots\boldsymbol{k}_n,\boldsymbol{k}_1} = |\mathcal{M}|^2 \frac{c}{2\hbar\omega_{\boldsymbol{k}_1}} \Phi^{(n-1)}_{\text{LIPS}} \text{ with } \Phi^{(n-1)}_{\text{LIPS}} = (24.39)} \tag{24.40}$$

Often there exist different "channels" for the decay of a particle, i. e. different types of decay. The sum of all types of decay rates is the total decay rate, the inverse of which is called the lifetime of the particle:

$$\text{lifetime} \equiv \tau \equiv \frac{1}{\Gamma_{\text{total}}} \equiv \frac{1}{\sum_{\text{all channels}} \Gamma} \tag{24.41}$$

While the factors $\mathcal{M}$ and $\Phi^{(n-1)}_{\text{LIPS}}$ are lorentz-invariant, the factor

$$\hbar\omega_{\boldsymbol{k}_1} = \frac{m_1 c^2}{\sqrt{1-(v/c)^2}} \,, \tag{24.42}$$

in which v is the decaying particle's velocity, is not. The lifetime of an unstable particle becomes arbitrarily long in a reference frame, in which the velocity of the particle approaches the speed of light. The first experimentally confirmed examples for this relativistic effect are the myons, which are generated by the cosmic radiation in the upper earth atmosphere ($\approx 10\,\text{km}$ above ground). In the rest system their lifetime[2] is $\tau_0 = 2.2\,\mu\text{s}$. To cover the distance to ground, they need approximately the time $10\,\text{km}/c \approx 150 \cdot \tau_0$, if they are moving with almost the speed of light. From observation [49] it became obvious, that a much higher percentage of them is reaching ground, than could be explained without the realistic effect (24.42).

If a particle decays into only 2 particles, then in the cms, with $\boldsymbol{k}_1 = 0$ and $\boldsymbol{k}_3 = -\boldsymbol{k}_2$, the invariant phase-space $\Phi^{(2)}_{\text{LIPS}}$ can be read from (24.31). Only E_{CM} needs to be replaced by $m_1 c^2$, and the indices 3 and 4 must be replaced by 2 and 3:

[2] Myons decay due to weak interaction. Insofar the example does not really fit into this chapter dealing with QED.

$$\Phi_{\text{LIPS}}^{(2)} \overset{(24.31)}{=} \int\limits_{U(\boldsymbol{k}_2)} \mathrm{d}\widetilde{\Omega}_2 \, \frac{|\boldsymbol{k}_2|}{16\pi^2\hbar c\, m_1 c^2} \overset{(24.33)}{=}$$

$$= \int\limits_{U(\boldsymbol{k}_2)} \mathrm{d}\widetilde{\Omega}_2 \, \frac{c^4 \sqrt{\left(m_1^2 - (m_2+m_3)^2\right)\left(m_1^2 - (m_2-m_3)^2\right)}}{32\pi^2\hbar^2 c^2 \, (m_1 c^2)^2} \tag{24.43}$$

Thereby on finds the differential decay rate

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\widetilde{\Omega}_2} \overset{(24.40)}{=} |\mathcal{M}|^2 \, \frac{c^5 \sqrt{\left(m_1^2 - (m_2+m_3)^2\right)\left(m_1^2 - (m_2-m_3)^2\right)}}{64\pi^2\hbar^2 c^2 \, (m_1 c^2)^3}$$

decay $m_1 \to m_2, m_3$ in the cms . (24.44)

Attention: Some authors emphasize, that a factor $1/n!$ shall be inserted when computing the phase-space volume Φ_{LIPS}, if the decay should result into n scattering- or decay-products of same type, which are not discernible experimentally. But this is correct only, if the scattering amplitude $\mathcal{M}$ is defined in a less accurate manner than according to our rules. According to our definitions, all possible cases with regard to identical or discernible particles are already considered in the computation of $\mathcal{M}$, and thus must not be inserted a second time as a modification of Φ_{LIPS}.

24.3 Tree-Graphs

24.3.1 Lepton-Lepton Scattering

As a first example we will compute the scattering of a lepton by another lepton (not antilepton). Let two leptons with wavenumbers k_1 and k_2, spin variables r_1 and r_2, and mass parameters m_1 and m_2 come in, and two leptons with wave numbers k_3 and k_4 and spin variables r_3 and r_4 go out after the scattering event. As the fermion flavor is conserved at each vertex, the masses of the outgoing particles must be identical to the masses m_1 and m_2 of the incoming particles. The computation of the scattering amplitude is almost identical to that which we computed on pages 490ff for

the scattering of nucleons in Yukawa-theory. Just the Klein-Gordon bosons of Yukawa-theory must be replaced by photons.

The probability amplitude $\overline{S}^{(0)}$ in first order of perturbation computation is zero, because — just as in (20.14) — the two propagators $S(x_3 - x_1)$ and $S(x_4 - x_2)$, which are constructed in zeroth order from the matrix element

$$\langle 0|\, T\psi(x_3)\, \psi(x_4)\, \overline{\psi}(x_1)\, \overline{\psi}(x_2)\, |0\rangle_c \ ,$$

are not sufficient to compensate the four zero-factors $\widetilde{S}^{-1}$ in the LSZ-formula. In first order, the matrix element

$$\langle 0|\, T\psi(x_3)\, \psi(x_4)\, \overline{\psi}(x_1)\, \overline{\psi}(x_2)\, \overline{\psi}(y)\, \gamma^\mu A_\mu(y)\, \psi(y)\, |0\rangle_c$$

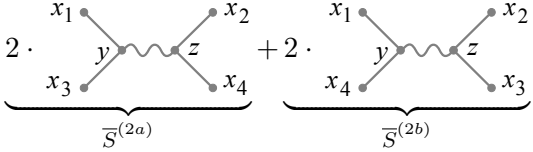
contains one operator $\gamma^\nu A_\nu(y)$, and again is zero because of (23.13). The same holds for all odd orders $n = 1, 3, 5, \ldots$ of perturbation theory. Only even orders $n = 2, 4, \ldots$ can contribute to the scattering amplitude.

The leading non-vanishing term is found in second order perturbation theory, namely the matrix element

$$\begin{aligned} &\langle 0|\, T\psi(x_3)\, \psi(x_4)\, \overline{\psi}(x_1)\, \overline{\psi}(x_2)\, \overline{\psi}(y) \cdot \\ &\qquad \cdot \gamma^\mu A_\mu(y)\, \psi(y)\, \overline{\psi}(z)\, \gamma^\nu A_\nu(z)\, \psi(z)\, |0\rangle_c \ . \end{aligned} \tag{24.45}$$

It can be contracted to these graphs:

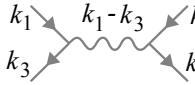
$$\overline{S}^{(2)} \mathrel{\widehat{=}} \underbrace{2 \cdot}_{\overline{S}^{(2a)}} \quad \underbrace{+2 \cdot}_{\overline{S}^{(2b)}} \tag{24.46}$$

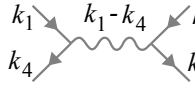


The second graph only exists, if both leptons are of same flavor. In other cases this term vanishes, because the scattered particles are discernible, see the arguments in the sequel of (23.28). Without any changes, the justification of the two symmetry factors can be extracted from the text following (23.24). From the same text we can take over the consideration that there is a factor (-1) in $\overline{S}^{(2a)}$ but not in $\overline{S}^{(2b)}$ due to the number of

permutations of fermion operators.

Thus far we worked through rules A, B, and B′ of box 24.1. By means of the further rules in this box we arrive at the following scattering amplitude:

$2 \cdot$ 

$+\, 2 \cdot$ 

$\widehat{=}$

$$\widehat{=}\; \overline{S}^{(2)} = \Big(\prod_{j=1}^{4} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}_j}\Omega}} \Big)\, \mathcal{M}^{(2)} \cdot$$
$$\cdot\, 2\pi\Omega\, \delta\Big(k_1^0 + k_2^0 - k_3^0 - k_4^0\Big)\, \delta_{(\boldsymbol{k}_1+\boldsymbol{k}_2),(\boldsymbol{k}_3+\boldsymbol{k}_4)} \tag{24.47a}$$

$$\mathcal{M}^{(2)} = 2 \cdot \frac{1}{2!}\Big(\frac{+ie}{\hbar}\Big)^2 \cdot$$
$$\cdot \Big(-\, (^{r_3}\overline{u}^{\boldsymbol{k}_3}\, \gamma^{\mu}\, ^{r_1}u^{\boldsymbol{k}_1})\, \frac{(-ig_{\mu\nu}\, \mu_0\hbar c)}{(k_1-k_3)^2 + i\epsilon'}\, (^{r_4}\overline{u}^{\boldsymbol{k}_4}\, \gamma^{\nu}\, ^{r_2}u^{\boldsymbol{k}_2}) +$$
$$+\, (^{r_4}\overline{u}^{\boldsymbol{k}_4}\, \gamma^{\mu}\, ^{r_1}u^{\boldsymbol{k}_1})\, \frac{(-ig_{\mu\nu}\, \mu_0\hbar c)}{(k_1-k_4)^2 + i\epsilon'}\, (^{r_3}\overline{u}^{\boldsymbol{k}_3}\, \gamma^{\nu}\, ^{r_2}u^{\boldsymbol{k}_2}) \Big) \tag{24.47b}$$

Here the replacement-operator (23.21) has already been applied and therefore does not need to be stated explicitly again. The spinor factors belonging to the second graph have been colored green. If the two outgoing particles are of different flavors and thus are discernible, then this graph is not applicable, and all green factors must be set to zero.

In the propagators of the intermediary photons, the variables

$$s \equiv (k_1 + k_2)^2\ ,\ t \equiv (k_1 - k_3)^2\ ,\ u \equiv (k_1 - k_4)^2 \tag{24.48}$$

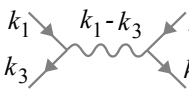
defined by Mandelstam[3] are showing up. Therefore the first diagram in (24.47b) sometimes is called t-channel, and the second is called u-channel of the scattering event. (Examples for s-channel scattering will be encountered below.) In general no conservation law holds for the spin variable r. But in the non-relativistic limit, spin is conserved:

[3] Stanley Mandelstam (1928 – 2016)

$$({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^{\mu\,r_1}u^{\boldsymbol{k}_1})\,g_{\mu\nu}\,({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^{\nu\,r_2}u^{\boldsymbol{k}_2}) \overset{(A.153)}{\approx} 4m_1m_2c^4\,\delta_{r_3r_1}\,\delta_{r_4r_2}$$

$$({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^{\mu\,r_1}u^{\boldsymbol{k}_1})\,g_{\mu\nu}\,({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^{\nu\,r_2}u^{\boldsymbol{k}_2}) \overset{(A.153)}{\approx} 4m_1m_2c^4\,\delta_{r_4r_1}\,\delta_{r_3r_2}$$
$$\text{if } |c\hbar\boldsymbol{k}_1| \ll m_1c^2 \text{ and } |c\hbar\boldsymbol{k}_2| \ll m_2c^2 \tag{24.49}$$

Thus lepton spin is conserved at each vertex. In case $r_1 \neq r_2$ the particles are discernible even for $m_1 = m_2$, and the second term in (24.47b) will vanish even in case of same flavor. (In case $m_1 \neq m_2$ it is always zero, because then the particles are discernible, independent of their polarization.) Consequently in the non-relativistic limit the matrix for the scattering of two leptons is

$2 \cdot$ 

$$\approx\!\approx \mathcal{M}^{(2)} \approx \frac{-ie^2\,4m_1m_2c^4\,\mu_0c\hbar^{-1}\,\delta_{r_3r_1}\,\delta_{r_4r_2}}{(k_1-k_3)^2+i\epsilon'}$$
$$\text{if } \Big(|c\hbar\boldsymbol{k}_j| \ll m_jc^2 \text{ for } j=1,2\Big) \text{ and } \Big(r_1 \neq r_2 \text{ if } m_1 = m_2\Big)\,. \tag{24.50}$$

We emphasize again, that we always put the "corresponds to"-sign, but not the equal-sign between the graphs and the algebraic formulas, because the interpretation of a graph does always depend on the context. Here the graph corresponds to $\mathcal{M}$, while in (24.47a) the graphs correspond to $\overline{S}$. It is instructive, to compare the result (24.50) with equation (23.29), which is describing the scattering of a nucleon or antinucleon by a nucleon or antinucleon due to Yukawa-interaction. The signs of (24.50) and (23.29) are different. This difference can be traced back to the different signs of the boson-propagators

$$\widetilde{G}(k) = \frac{i}{\hbar c(k^2-(Mc/\hbar)^2+i\epsilon')} \quad\longleftrightarrow\quad \widetilde{D}_{\nu\mu}(k) = \frac{-ig_{\nu\mu}\,\mu_0\hbar c}{k^2+i\epsilon'}$$

in rule G of the boxes 23.1 and 24.1. Note that in the non-relativistic approximation because of (A.153) only the term with $-g_{00} = -1$ is different from zero.

Not the scattering amplitude, but only the cross section

$$\frac{\mathrm{d}\sigma_{\boldsymbol{k}_3\boldsymbol{k}_4,\boldsymbol{k}_1\boldsymbol{k}_2}}{\mathrm{d}\widetilde{\Omega}_3} \overset{(24.32)}{=} \frac{|\mathcal{M}|^2\,|\boldsymbol{k}_3|}{|v_{\mathrm{rel}}|\,\hbar 64\pi^2\hbar\omega_{\boldsymbol{k}_1}\hbar\omega_{\boldsymbol{k}_2}E_{\mathrm{CM}}}$$
$$\text{in the cms with } v_{\mathrm{rel}} = (24.23)$$

for the scattering event with two incoming and two outgoing particles can be checked experimentally. To compute the cross section, the modulus square $|\mathcal{M}|^2 = \mathcal{M}^*\mathcal{M}$ of the scattering matrix (24.47b) must be known. To find it, we first compute the complex-conjugated of the spinor product

$$\begin{aligned}(\bar{u}\gamma^\mu v)^* &= v^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u \overset{(8.12)}{=} v^\dagger g^{\mu\nu}\gamma^\nu\gamma^0 u \overset{(8.9)}{=} \\ &= g^{\mu\nu}v^\dagger(2g^{\nu 0}\mathbb{1} - \gamma^0\gamma^\nu)u = v^\dagger\gamma^0\gamma^\mu u = \bar{v}\gamma^\mu u\ . \end{aligned} \tag{24.51}$$

Consequently we have:

$$\begin{aligned}&\left|\mathcal{M}^{(2)}\right|^2 \overset{(24.47b)}{=} \frac{e^4\mu_0^2c^2}{\hbar^2}\cdot \\ &\cdot\Bigg(\frac{({}^{r_1}\bar{u}^{\boldsymbol{k}_1}\,\gamma^\mu\,{}^{r_3}u^{\boldsymbol{k}_3})({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\nu\,{}^{r_1}u^{\boldsymbol{k}_1})({}^{r_2}u^{\boldsymbol{k}_2}\,\gamma_\mu\,{}^{r_4}\bar{u}^{\boldsymbol{k}_4})({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma_\nu\,{}^{r_2}u^{\boldsymbol{k}_2})}{(k_1-k_3)^4+\epsilon'^2} \\ &\quad - \frac{({}^{r_1}\bar{u}^{\boldsymbol{k}_1}\,\gamma^\mu\,{}^{r_3}u^{\boldsymbol{k}_3})({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\tau\,{}^{r_1}u^{\boldsymbol{k}_1})({}^{r_2}u^{\boldsymbol{k}_2}\,\gamma_\mu\,{}^{r_4}\bar{u}^{\boldsymbol{k}_4})({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma_\tau\,{}^{r_2}u^{\boldsymbol{k}_2})}{(k_1-k_3)^2(k_1-k_4)^2+\epsilon'^2} \\ &\quad - \frac{({}^{r_1}\bar{u}^{\boldsymbol{k}_1}\,\gamma^\sigma\,{}^{r_4}u^{\boldsymbol{k}_4})({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\nu\,{}^{r_1}u^{\boldsymbol{k}_1})({}^{r_2}\bar{u}^{\boldsymbol{k}_2}\,\gamma_\sigma\,{}^{r_3}u^{\boldsymbol{k}_3})({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma_\nu\,{}^{r_2}u^{\boldsymbol{k}_2})}{(k_1-k_4)^2(k_1-k_3)^2+\epsilon'^2} \\ &\quad + \frac{({}^{r_1}\bar{u}^{\boldsymbol{k}_1}\,\gamma^\sigma\,{}^{r_4}u^{\boldsymbol{k}_4})({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\tau\,{}^{r_1}u^{\boldsymbol{k}_1})({}^{r_2}\bar{u}^{\boldsymbol{k}_2}\,\gamma_\sigma\,{}^{r_3}u^{\boldsymbol{k}_3})({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma_\tau\,{}^{r_2}u^{\boldsymbol{k}_2})}{(k_1-k_4)^4+\epsilon'^2}\Bigg)\end{aligned} \tag{24.52}$$

In this formula, some certain spin orientation is specified for each incoming or outgoing particle due to ^{r_j}. If in an experiment the spins of the incoming particles are not controlled, then the measured cross section is equal to the mean value of the cross sections with all possible orientations of the spins of the incoming particles. If furthermore the spins of the outgoing particles is not controlled, then the measured cross section is equal to the sum of all cross sections with all possible orientations of the spins of the outgoing particles. Thus the cross section must be computed by

$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} = \frac{1}{2}\sum_{r_1=1}^{2}\frac{1}{2}\sum_{r_2=1}^{2}\sum_{r_3=1}^{2}\sum_{r_4=1}^{2}\left|\mathcal{M}^{(2)}\right|^2 \tag{24.53}$$

if the spins of the incoming and outgoing leptons are unknown. The index SNO means "spins not observed". In section 8.5 we already computed the spin-sums

$$\sum_{r=1}^{2} {}^{r}u^{\boldsymbol{k}}\,{}^{r}\bar{u}^{\boldsymbol{k}} \overset{(8.77a)}{=} c\gamma^{\mu}p_{\mu} + mc^2 \tag{24.54a}$$

$$\sum_{r=1}^{2} {}^{r}v^{\boldsymbol{k}}\,{}^{r}\bar{v}^{\boldsymbol{k}} \overset{(8.77b)}{=} c\gamma^{\mu}p_{\mu} - mc^2\ . \tag{24.54b}$$

Using the

$$\text{Feynman-dagger} \equiv \not{b} \equiv \gamma^{\mu}b_{\mu} \tag{24.55}$$

as a shorthand notation, we get for the first half of the first spinor product in (24.52)

$$\begin{aligned}\sum_{r_1,r_3} {}^{r_1}\bar{u}_a^{\boldsymbol{k}_1}\,\gamma^{\mu}_{ab}\,{}^{r_3}u_b^{\boldsymbol{k}_3}\,{}^{r_3}\bar{u}_c^{\boldsymbol{k}_3}\,\gamma^{\nu}_{cd}\,{}^{r_1}u_d^{\boldsymbol{k}_1} &= \sum_{r_1,r_3} {}^{r_1}u_d^{\boldsymbol{k}_1}\,{}^{r_1}\bar{u}_a^{\boldsymbol{k}_1}\,\gamma^{\mu}_{ab}\,{}^{r_3}u_b^{\boldsymbol{k}_3}\,{}^{r_3}\bar{u}_c^{\boldsymbol{k}_3}\,\gamma^{\nu}_{cd} = \\ &= (c\hbar\not{k}_1 + m_1c^2)_{da}\gamma^{\mu}_{ab}(c\hbar\not{k}_3 + m_3c^2)_{bc}\gamma^{\nu}_{cd} = \\ &= \hbar^2c^2\,\mathrm{tr}\Big\{(\not{k}_1 + m_1\frac{c}{\hbar})\,\gamma^{\mu}(\not{k}_3 + m_3\frac{c}{\hbar})\,\gamma^{\nu}\Big\}\ .\end{aligned} \tag{24.56}$$

Here tr{ } is the trace of the spinor matrix, i. e. the sum of the elements in the matrix diagonal. The second half of the first spinor product in (24.52) is formed into a trace by the same method, and the same is done with the both halfs of the fourth spinor product in (24.52). The second and the third spinor product can be formed by this method to only one trace each. Thereby we get

$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} = \frac{1}{4}\sum_{r_1,r_2,r_3,r_4}\left|\mathcal{M}^{(2)}\right|^2 = \frac{e^4\hbar^2c^6\mu_0^2}{4}\cdot$$
$$\cdot\Bigg(\operatorname{tr}\Big\{(\not{k}_1 + m_1\tfrac{c}{\hbar})\,\gamma^\mu(\not{k}_3 + m_3\tfrac{c}{\hbar})\,\gamma^\nu\Big\}\cdot$$
$$\cdot\frac{\operatorname{tr}\Big\{(\not{k}_2 + m_2\tfrac{c}{\hbar})\,\gamma_\mu(\not{k}_4 + m_4\tfrac{c}{\hbar})\,\gamma_\nu\Big\}}{(k_1-k_3)^4+\epsilon'^2} -$$
$$-\frac{\operatorname{tr}\Big\{(\not{k}_1 + m_1\tfrac{c}{\hbar})\,\gamma^\mu(\not{k}_3 + m_3\tfrac{c}{\hbar})\,\gamma_\tau(\not{k}_2 + m_2\tfrac{c}{\hbar})\,\gamma_\mu(\not{k}_4 + m_4\tfrac{c}{\hbar})\,\gamma^\tau\Big\}}{(k_1-k_3)^2(k_1-k_4)^2+\epsilon'^2} -$$
$$-\frac{\operatorname{tr}\Big\{(\not{k}_1 + m_1\tfrac{c}{\hbar})\,\gamma^\sigma(\not{k}_4 + m_4\tfrac{c}{\hbar})\,\gamma_\nu(\not{k}_2 + m_2\tfrac{c}{\hbar})\,\gamma_\sigma(\not{k}_3 + m_3\tfrac{c}{\hbar})\,\gamma^\nu\Big\}}{(k_1-k_4)^2(k_1-k_3)^2+\epsilon'^2} +$$
$$+\frac{\operatorname{tr}\Big\{(\not{k}_1 + m_1\tfrac{c}{\hbar})\,\gamma^\sigma(\not{k}_4 + m_4\tfrac{c}{\hbar})\,\gamma^\tau\Big\}}{(k_1-k_4)^4+\epsilon'^2}\cdot$$
$$\cdot\operatorname{tr}\Big\{(\not{k}_2 + m_2\tfrac{c}{\hbar})\,\gamma_\sigma(\not{k}_3 + m_3\tfrac{c}{\hbar})\,\gamma_\tau\Big\}\Bigg) \tag{24.57}$$

The trace of a sum is equal to the sum of the traces of each single summand. As furthermore numeric factors like k_1^3, $g^{\mu\nu}$, $m_1\frac{c}{\hbar}$, ... commute with the γ-matrices, and because numeric factors can be factored out of the trace, the first trace becomes

$$\begin{aligned} A &\equiv \operatorname{tr}\Big\{(k_1^\alpha\gamma_\alpha + m_1\tfrac{c}{\hbar})\,\gamma^\mu(k_3^\beta\gamma_\beta + m_3\tfrac{c}{\hbar})\,\gamma^\nu\Big\} = \\ &= k_1^\alpha k_3^\beta\operatorname{tr}\Big\{\gamma_\alpha\gamma^\mu\gamma_\beta\gamma^\nu\Big\} + k_1^\alpha m_3\tfrac{c}{\hbar}\operatorname{tr}\Big\{\gamma_\alpha\gamma^\mu\gamma^\nu\Big\} + \\ &\quad + m_1\tfrac{c}{\hbar}k_3^\beta\operatorname{tr}\Big\{\gamma^\mu\gamma_\beta\gamma^\nu\Big\} + m_1\tfrac{c}{\hbar}m_3\tfrac{c}{\hbar}\operatorname{tr}\Big\{\gamma^\mu\gamma^\nu\Big\} \end{aligned} \tag{24.58}$$

We need to compute the traces of the products of 2, 3, and 4 γ-matrices. In the equivalent terms in the second and third line of (24.57) there are traces of products of up to 8 γ-matrices. For their computation, the use of algebraic computer programs like Reduce, or Maple, or Mathematica is advisable. Furthermore there exist some useful theorems for the simplification of these traces. We list the most important of them without proof. A recommendable

reference, in which these theorems are discussed and motivated in more detail, is [3, section 5.1].

Theorems on the traces of γ-matrices:

$$\mathrm{tr}\{\underbrace{\gamma^\alpha \dots \gamma^\tau}_{n \text{ matrices}}\} = 0 \text{ if } n \text{ is odd} \tag{24.59a}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\} = 4g^{\alpha\beta} \tag{24.59b}$$

$$\mathrm{tr}\{\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\} = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}) \tag{24.59c}$$

$$\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\} = \mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\delta\gamma^\gamma\gamma^\beta\gamma^\alpha\} \tag{24.59d}$$

$$\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\alpha\gamma_\alpha\} = 4\,\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\} \tag{24.59e}$$

$$\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\alpha\gamma^\beta\gamma_\alpha\} = -2\,\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\beta\} \tag{24.59f}$$

$$\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\alpha\} = 4g^{\beta\gamma}\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\} \tag{24.59g}$$

$$\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma_\alpha\} = -2\,\mathrm{tr}\{\gamma^\mu \dots \gamma^\nu\gamma^\delta\gamma^\gamma\gamma^\beta\} \tag{24.59h}$$

As factors may be permuted cyclically under the trace, the last five theorems still are valid if the factors $\gamma^\mu \dots \gamma^\nu$ are not at the very left of the products. Furthermore $\gamma^\mu \dots \gamma^\nu = \mathbb{1}$ is allowed. Clearly $\mathrm{tr}\{\mathbb{1}\} = 4$.

By means of these theorems we find

$$\begin{aligned} A &\overset{(24.58)}{=} k_1^\alpha k_3^\beta\, 4(g_\alpha{}^\mu g_\beta{}^\nu - g_{\alpha\beta}g^{\mu\nu} + g_\alpha{}^\nu g^\mu{}_\beta) + m_1\tfrac{c}{\hbar}m_3\tfrac{c}{\hbar}\,4g^{\mu\nu} \\ &= 4(k_1^\mu k_3^\nu - k_1k_3g^{\mu\nu} + k_1^\nu k_3^\mu + m_1\tfrac{c}{\hbar}m_3\tfrac{c}{\hbar}\,g^{\mu\nu}) \end{aligned} \tag{24.60}$$

$$\begin{aligned} &\mathrm{tr}\Big\{(\not{k}_1 + m_1\tfrac{c}{\hbar})\,\gamma^\mu(\not{k}_3 + m_3\tfrac{c}{\hbar})\,\gamma^\nu\Big\} \cdot \mathrm{tr}\Big\{(\not{k}_2 + m_2\tfrac{c}{\hbar})\,\gamma_\mu(\not{k}_4 + m_4\tfrac{c}{\hbar})\,\gamma_\nu\Big\} = \\ &= 4\,(k_1^\mu k_3^\nu - k_1k_3g^{\mu\nu} + k_1^\nu k_3^\mu + m_1\tfrac{c}{\hbar}m_3\tfrac{c}{\hbar}\,g^{\mu\nu}) \cdot \\ &\quad \cdot 4\,(k_{2\mu}k_{4\nu} - k_2k_4g^{\mu\nu} + k_{2\nu}k_{4\mu} + m_2\tfrac{c}{\hbar}m_4\tfrac{c}{\hbar}\,g^{\mu\nu}) = \\ &= 32\,\Big((k_1k_2)(k_3k_4) + (k_1k_4)(k_2k_3) - k_1k_3m_2m_4c^2/\hbar^2 - \\ &\qquad - k_2k_4m_1m_3c^2/\hbar^2 + 2m_1m_3m_2m_4c^4/\hbar^4\Big) \end{aligned} \tag{24.61}$$

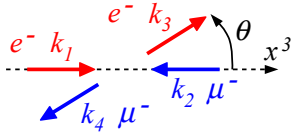
To spare ourselves the tedious computations of the last three terms in (24.57), we restrict our further considerations to the concrete case of the

scattering of an electron (mass m_e, momentum $\hbar\boldsymbol{k}_1$) by a myon (mass m_μ, momentum $\hbar\boldsymbol{k}_2$). Due to flavor conservation at each vertex, one electron with momentum $\hbar\boldsymbol{k}_3$ and one myon with momentum $\hbar\boldsymbol{k}_4$ are going out in this reaction. As the outgoing particles are discernible, all green factors in (24.57) must be set to zero. As the myon is about 207 times as heavy as the electron, furthermore the terms in the last line of (24.61), in which the electron mass even is showing up quadratically, may be neglected versus the mass term in the second-last line. Thereby one gets, if the spins of the electrons and myons are not controlled, as modulus square of the scattering matrix:

$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} = \frac{1}{4}\sum_{r_1,r_2,r_3,r_4}\left|\mathcal{M}^{(2)}\right|^2 \overset{(24.57)}{=} \tag{24.62}$$

$$= \frac{8e^4\hbar^2c^6\mu_0^2}{(k_1-k_3)^4+\epsilon'^2}\left((k_1k_2)(k_3k_4)+(k_1k_4)(k_2k_3)-k_1k_3m_\mu^2c^2/\hbar^2\right)$$

Before we apply this result to find the scattering cross section, we modify it such, that instead of the four-wavenumbers the energies, masses, and scattering angles of the particles become explicitly visible. In the center-of-mass system



$$\text{cms:} \qquad \boldsymbol{k}_2 = -\boldsymbol{k}_1 \quad , \quad \boldsymbol{k}_4 = -\boldsymbol{k}_3$$

we have

$$|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = |\boldsymbol{k}_3| = |\boldsymbol{k}_4| = \sqrt{\left(\frac{E_1}{\hbar c}\right)^2 - \left(\frac{m_e c}{\hbar}\right)^2} = \sqrt{\left(\frac{E_2}{\hbar c}\right)^2 - \left(\frac{m_\mu c}{\hbar}\right)^2}$$

$$E_3 = E_1 \quad , \quad E_4 = E_2 \ . \tag{24.63}$$

In the sequel, we will assume that the electrons are high-relativistic:

$$E_1 \gg m_e c^2 \tag{24.64}$$
$$\Longrightarrow \quad |\boldsymbol{k}_1| = \frac{E_1}{\hbar c} \quad \Longrightarrow \quad k_1 k_1 = k_1^0 k_1^0 - \boldsymbol{k}_1^2 = 0 \ .$$

Then the factors in equation (24.62) can be written in the following form:

$$\begin{aligned}
k_1 k_2 &= k_1^0 k_2^0 - \boldsymbol{k}_1 \boldsymbol{k}_2 = \frac{E_1}{(\hbar c)^2}\,(E_2 + E_1) \\
k_1 k_3 &= \frac{E_1^2}{(\hbar c)^2}\,(1 - \cos\theta) \\
k_1 k_4 &= \frac{E_1 E_2}{(\hbar c)^2} - \frac{E_1^2}{(\hbar c)^2}\cos(\pi - \theta) = \frac{E_1}{(\hbar c)^2}\,(E_2 + E_1 \cos\theta) \\
k_2 k_3 &= k_1 k_4 \quad , \quad k_3 k_4 = k_1 k_2 \\
(k_1 - k_3)^4 &\overset{(24.48)}{=} t^2 = 4(k_1 k_3)^2
\end{aligned} \tag{24.65}$$

In the next step, the first of the relations

$$1 - \cos\theta = 2\sin^2(\theta/2) \quad , \quad 1 + \cos\theta = 2\cos^2(\theta/2) \tag{24.66}$$

will be used. We will make use of the second in (24.69). With (24.65) the modulus square of the scattering matrix becomes, if the spins are not observed,

$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} \overset{(24.62)}{=} e^4 \hbar^2 c^6 \mu_0^2 \, \frac{(E_2 + E_1)^2 + (E_2 + E_1\cos\theta)^2 - (1 - \cos\theta) m_\mu^2 c^4}{2E_1^2 \sin^4(\theta/2)}$$
$$\text{if } m_e c^2 \ll E_1 \ . \tag{24.67}$$

Here $\theta \neq 0$ was assumed, such that the ϵ'^2 in the denominator could be skipped. Insertion of this scattering matrix into (24.32) and use of $\epsilon_0 \mu_0 = c^{-2}$ results, if the spins are not observed, into the following cross section for the scattering of an electron by a myon:

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\widetilde{\Omega}_3}\Big)_{\text{SNO}} \overset{(24.32)}{=}$$
$$= \Big(\frac{e^2}{4\pi\epsilon_0\hbar c}\Big)^2 \hbar^2 c^3 \, \frac{(E_2+E_1)^2 + (E_2 + E_1\cos\theta)^2 - (1-\cos\theta)m_\mu^2 c^4}{|v_{\text{rel}}|\, 8E_1^2 E_2 (E_1+E_2)\sin^4(\theta/2)}$$
in the cms with v_{rel} = (24.23) and $m_e c^2 \ll E_1$ (24.68)

In the high-relativistic case $m_\mu c^2 \ll E_2$ furthermore the third term in the numerator may be neglected, and $|v_{\text{rel}}| = 2c$ and $E_1 = E_2 \equiv E$ may be assumed:

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\widetilde{\Omega}_3}\Big)_{\text{SNO}} \overset{(24.32)}{=} \Big(\frac{e^2}{4\pi\epsilon_0\hbar c}\Big)^2 \hbar^2 c^2 \, \frac{1+\cos^4(\theta/2)}{8E^2\sin^4(\theta/2)}$$
in the cms, if $m_\mu c^2 \ll E$ (24.69)

We check the dimensions: $e^2/4\pi\epsilon_0\hbar c \approx 1/137$ is the dimension-less fine structure constant of quantum electrodynamics. (It is a product of the characteristic constant $e^2/4\pi\epsilon_0$ of classical electrodynamics, the characteristic constant c of special relativity theory, and the characteristic constant $\hbar$ of quantum theory.) Therefore

$$[\sigma] = \frac{(\text{energy}\cdot\text{time})^2 \cdot \text{length}^2}{\text{time}^2\cdot\text{energy}^2} = \text{area}\ ,$$

which is correct.

At the beginning of this section, we started to compute the scattering amplitude for leptons of same flavor. But then we evaded the tedious computation of the last three lines in (24.57), and restricted the evaluation to the scattering of an electron by a myon. The complete amplitude for the scattering of two leptons of same flavor in second-order perturbation theory — including the interference-terms of (24.57) — has been published in 1932 by Møller[4] [50]. He found the result

[4] Christian Møller (1904 - 1980)

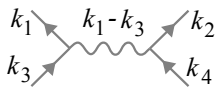
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\widetilde{\Omega}_3}\right)_{\text{SNO}} = \left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 \frac{\hbar^2 c^2}{8E^2} \cdot \cdot \left(\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\,\cos^2(\theta/2)} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\right) \quad \text{in the cms, if } mc^2 \ll E \;, \tag{24.70}$$

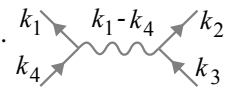
with E being the energy of each incoming or outgoing lepton in the center-of-mass system. In the first term, which is dominating the result for small scattering angle θ, the t-channel scattering (24.69) is discernible. The third term is describing the u-channel scattering. The second term, which — same as the two other terms — is ≥ 0 for arbitrary scattering angles, is brought about by interference of the both channels.

In case of $\theta \to 0$, which is called forward-scattering, the first two terms are diverging. As the frequency of the intermediary photon in the left graph (24.47a) becomes unmeasurable small in the case of forward-scattering, this divergence is called infrared divergence. In section 24.3.6 we will discuss the problem of IR-divergences, which is encountered at many places in QED.

24.3.2 Antilepton-Antilepton Scattering

For the scattering of an antilepton by an antilepton, the symmetry factors are the same as indicated in (23.31) for the scattering of an antinucleon by an antinucleon in Yukawa-theory. Also the number of required permutations of fermion operators is the same as in the matrix elements (23.30); therefore the two terms have same signs as in (23.31). The scattering matrix

$\mathcal{M}^{(2)} \mathrel{\widehat{=}} 2 \cdot$ 

$+\, 2 \cdot$ 

$$\mathcal{M}^{(2)} = 2 \cdot \frac{1}{2!}\Big(\frac{+ie}{\hbar}\Big)^2 \cdot$$
$$\cdot \Bigg(- \, (\,{}^{r_3}\bar{v}^{\boldsymbol{k}_3}\, \gamma^{\mu\, r_1} v^{\boldsymbol{k}_1}) \, \frac{(-ig_{\mu\nu}\, \mu_0 \hbar c)}{(k_1 - k_3)^2 + i\epsilon'} \, (\,{}^{r_4}\bar{v}^{\boldsymbol{k}_4}\, \gamma^{\nu\, r_2} v^{\boldsymbol{k}_2}) +$$
$$+ \, (\,{}^{r_4}\bar{u}^{\boldsymbol{k}_4}\, \gamma^{\mu\, r_1} u^{\boldsymbol{k}_1}) \, \frac{(-ig_{\mu\nu}\, \mu_0 \hbar c)}{(k_1 - k_4)^2 + i\epsilon'} \, (\,{}^{r_3}\bar{v}^{\boldsymbol{k}_3}\, \gamma^{\nu\, r_2} v^{\boldsymbol{k}_2}) \Bigg) \qquad (24.71)$$

of the antileptons differs from (24.47b) by nothing else than the replacement of the spinors $\bar{u}$ and u by the spinors $\bar{v}$ and v. In particular the signs of both equations are identical. Also the modulus square of the scattering matrix differs from (24.52) only by the replacement of all spinors $\bar{u}$ and u by the spinors $\bar{v}$ and v.

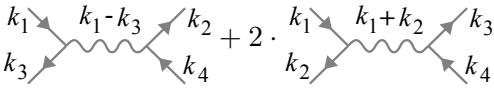
If the scattering matrix and the cross section are computed for the case that the spins of incoming and outgoing particles are not observed, then the masses get the inverse signs as in case of lepton scattering because of (24.54). But when computing the traces in (24.57) one gets in all terms with an odd number of mass factors also the trace of an odd number of γ-matrices, which are zero according to the theorems (24.59). Only terms with an even number of mass factors survive, and in these terms the negative signs mutually compensate. Consequently one finds, if the spins of the incoming and outgoing particles are not observed, for the scattering of antileptons by antileptons *exactly* the same scattering matrix and the same cross section as for the scattering of leptons by leptons, namely (24.68) resp. (24.69) for the scattering of antileptons of different flavors, and (24.70) for the scattering of antileptons of same flavor.

The second term in (24.71) vanishes if the scattering particles are discernible. This is always the case for particles of different flavor, but for particles of same flavor only in the non-relativistic limit in case of different polarizations. Due to (A.153h) = (A.153e), the non-relativistic scattering matrix (24.50) holds as well for antilepton-antilepton scattering.

24.3.3 Lepton-Antilepton Scattering

Let a lepton with wavenumber k_1, spin variable r_1, and mass m_1, and an antilepton with wavenumber k_2, spin variable r_2, and mass m_2 come in. And let after the scattering event a lepton with wavenumber k_3, mass m_3 and spin variable r_3, and an antilepton with wavenumber k_4, spin variable r_4, and mass m_4 go out. The symmetry factors and signs are identical to those found in Yukawa-theory for the scattering of a nucleon by an antinucleon, see (23.34). Application of the rules of box 24.1 results into

$$\mathcal{M}^{(2)} \mathrel{\widehat{=}} 2 \cdot \quad + 2 \cdot$$



$$\mathcal{M}^{(2)} = 2 \cdot \frac{1}{2!} \Big(\frac{+ie}{\hbar} \Big)^2 \Big(({}^{r_3}\bar{u}^{\boldsymbol{k}_3} \, \gamma^\mu \, {}^{r_1}u^{\boldsymbol{k}_1}) \, \frac{(-ig_{\mu\nu} \, \mu_0 \hbar c)}{(k_1 - k_3)^2 + i\epsilon'} \, ({}^{r_2}\bar{v}^{\boldsymbol{k}_2} \, \gamma^\nu \, {}^{r_4}v^{\boldsymbol{k}_4}) -$$
$$- ({}^{r_2}\bar{v}^{\boldsymbol{k}_2} \, \gamma^\mu \, {}^{r_1}u^{\boldsymbol{k}_1}) \, \frac{(-ig_{\mu\nu} \, \mu_0 \hbar c)}{(k_1 + k_2)^2 + i\epsilon'} \, ({}^{r_3}\bar{u}^{\boldsymbol{k}_3} \, \gamma^\nu \, {}^{r_4}v^{\boldsymbol{k}_4}) \Big) \, . \tag{24.72}$$

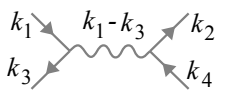
The second term is describing "s-channel" scattering, compare (24.48). In QED this term is different from zero only in case $m_1 = m_2$, because fermion flavor is conserved at each vertex. The by far most important example in experimental practice is the scattering of electrons by positrons, which has been intensively investigated in the last decades of the recent century. If after the scattering again an electron and a positron are going out, then both the t-channel and the s-channel are contributing to the cross section, and there will be interferences between both types of scattering. To evade the quite extensive formulas of that interesting case, we will instead consider two simpler scenarios: First the scattering of an electron by an antimyon (to which only the t-channel is contributing, while the s-channel is zero), and then the annihilation of an electron and a positron into a photon, from which due to pair creation a myon and an antimyon are generated (only the s-channel is contributing to this process, while the t-channel is zero).

Only the first graph in (24.72) is contributing to the scattering of an electron by an antimyon. As

$$({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^{\mu\,r_1}u^{\boldsymbol{k}_1})g_{\mu\nu}({}^{r_2}\bar{v}^{\boldsymbol{k}_2}\,\gamma^{\nu\,r_4}v^{\boldsymbol{k}_4}) \overset{(A.153)}{\approx} 4m_1m_2c^4\,\delta_{r_3r_1}\,\delta_{r_2r_4}$$
$$\text{if } |c\hbar\boldsymbol{k}_j| \ll m_jc^2 \text{ for } j=1,2 \quad ,$$

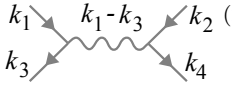
the scattering matrix becomes in non-relativistic approximation

$$\mathcal{M}^{(2)} \approx \frac{ie^2\,4m_1m_2c^4\,\delta_{r_3r_1}\,\delta_{r_2r_4}\,\mu_0\hbar^{-1}c}{(k_1-k_3)^2+i\epsilon'} \lessapprox 2\cdot$$

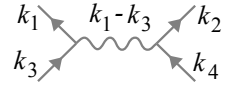


$$\approx$$

$$\overset{(24.50)}{\approx} -2\cdot$$



$$\overset{(24.50)}{\approx} -2\cdot$$



$$\text{if } |c\hbar\boldsymbol{k}_j| \ll m_jc^2 \text{ for } j=1,2\ . \tag{24.73}$$

The sign of the scattering matrix is negative if the signs of the charges of the scattering particles are equal. If the signs of the charges of the scattering particles are different, then the sign of the scattering matrix is positive. We know from classical electrostatic theory, that charges with equal signs are mutually repelling, while charges with different signs are mutually attracting. Thus negative sign of the scattering matrix is indicating repulsion, while positive sign of the scattering matrix is indicating attraction. In equation (23.37) of Yukawa-theory, the sign of the amplitude is always positive, indicating that Yukawa-interactions always are attractive, no matter whether nucleons are interacting with nucleons, or antinucleons with antinucleons, or nucleons with antinucleons. (In the sequel of (23.25b) we had noted that the value of a matrix element in quantum theory is fixed only up to a phase factor which may be chosen arbitrarily. Our comparison between the signs of scattering matrices in Yukawa-theory and QED still is sensible and correct, because we applied for their computation exactly the same matrix elements, which are differing merely by the boson-operator ϕ resp. A, but not by any phase factor.)

Not the charge parameters (e or g) cause attraction or repulsion between fermions in quantum field theory, as the charge parameters of particles and antiparticles are inserted into the formulas with same signs. Whether an interaction is attractive or repulsive is instead fixed first by the number (which is identical in Yukawa-theory and in QED) of permutations of fermion

operators needed for the contraction of matrix elements to propagators, and second by the different signs of the products

$$
\begin{aligned}
{}^{r}\bar{u}^{\boldsymbol{f}}\; {}^{s}u^{\boldsymbol{k}} &\overset{(A.153a)}{\approx} 2mc^2\,\delta_{rs} & {}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^0\,{}^{s}u^{\boldsymbol{k}} &\overset{(A.153e)}{\approx} 2mc^2\,\delta_{rs} \\
{}^{r}\bar{v}^{\boldsymbol{f}}\; {}^{s}v^{\boldsymbol{k}} &\overset{(A.153d)}{\approx} -2mc^2\,\delta_{rs} & {}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^0\,{}^{s}v^{\boldsymbol{k}} &\overset{(A.153h)}{\approx} 2mc^2\,\delta_{rs}\ ,
\end{aligned}
$$

(which in case of differently signed charges of the scattering particles result in different signs of the total products in Yukawa-theory versus QED). In this respect, quantum field theory is by far more intricate than classical theory.

Now we assume $t \overset{(24.48)}{=} (k_1 - k_3)^2 \neq 0$. Then the term $i\epsilon'$ in the denominator of the scattering matrix may be skipped. We compute the modulus square of the scattering matrix for the t-channel at arbitrary energy:

$$
\begin{aligned}
\left|\mathcal{M}^{(2)}\right|^2 &\overset{(24.51)}{=} \frac{e^4\mu_0^2c^2}{\hbar^2(k_1-k_3)^4}\cdot \qquad (24.74)\\
&\cdot\ ({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\mu\,{}^{r_1}u^{\boldsymbol{k}_1})({}^{r_1}\bar{u}^{\boldsymbol{k}_1}\,\gamma^\nu\,{}^{r_3}u^{\boldsymbol{k}_3})({}^{r_2}\bar{v}^{\boldsymbol{k}_2}\,\gamma_\mu\,{}^{r_4}v^{\boldsymbol{k}_4})({}^{r_4}\bar{v}^{\boldsymbol{k}_4}\,\gamma_\nu\,{}^{r_2}v^{\boldsymbol{k}_2})
\end{aligned}
$$

If the spins of the incoming and outgoing particles are not observed, then

$$
\begin{aligned}
\left|\mathcal{M}^{(2)}\right|^2_{\mathrm{SNB}} &= \frac{1}{4}\sum_{r_1,r_2,r_3,r_4}\left|\mathcal{M}^{(2)}\right|^2 = \\
&= \frac{e^4\mu_0^2c^2}{4\hbar^2(k_1-k_3)^4}\sum_{r_1,r_3}({}^{r_3}u_d^{\boldsymbol{k}_3}\,{}^{r_3}\bar{u}_a^{\boldsymbol{k}_3}\,\gamma^\mu_{ab}\,{}^{r_1}u_b^{\boldsymbol{k}_1}\,{}^{r_1}\bar{u}_c^{\boldsymbol{k}_1}\,\gamma^\nu_{cd})\cdot \\
&\cdot \sum_{r_2,r_4}({}^{r_2}v_d^{\boldsymbol{k}_2}\,{}^{r_2}\bar{v}_a^{\boldsymbol{k}_2}\,\gamma_{\mu\,ab}\,{}^{r_4}v_b^{\boldsymbol{k}_4}\,{}^{r_4}\bar{v}_c^{\boldsymbol{k}_4}\,\gamma_{\nu\,cd}) = \\
&\overset{(24.54)}{=} \frac{e^4\mu_0^2\hbar^2c^6}{4(k_1-k_3)^4}\,\mathrm{tr}\Big\{(\not{k}_3 + m_3\tfrac{c}{\hbar})\,\gamma^\mu(\not{k}_1 + m_1\tfrac{c}{\hbar})\,\gamma^\nu\Big\}\cdot \\
&\cdot\ \mathrm{tr}\Big\{(\not{k}_2 - m_2c/\hbar)\,\gamma_\mu(\not{k}_4 - m_4c/\hbar)\,\gamma_\nu\Big\}\ . \qquad (24.75)
\end{aligned}
$$

The only differences of this result versus the first term in (24.57) are, that the indices 1 and 3 are permuted under the trace, and that the signs of the

masses m_2 and m_4 are negative. From (24.61) it is visible, that these both differences will exactly compensate in the computation. Consequently the cross section of the scattering of an electron by an antimyon is exactly equal to the cross section (24.68) resp. (24.69) for the scattering of an electron by a myon.

Now we are going to consider lepton-antilepton s-channel scattering. To avoid interferences with the t-channel, we will investigate the annihilation of an electron and a positron into a virtual photon, from which then due to pair-creation a myon and an antimyon are generated, see the second graph in (24.72). For this reaction the center-of-mass energy must be minimum $2m_\mu c^2 \approx 414 m_e c^2$. Thus there is no low-energy limit. In the sequel we assume $s \overset{(24.48)}{=} (k_1+k_2)^2 \neq 0$. Therefore the term $i\epsilon'$ in the denominator of the scattering matrix may be skipped. We compute the modulus square of the scattering matrix for the s-channel:

$$\left|\mathcal{M}^{(2)}\right|^2 \overset{(24.51)}{=} \frac{e^4\mu_0^2 c^2}{\hbar^2(k_1+k_2)^4} \cdot \tag{24.76}$$
$$\cdot \, ({}^{r_1}\bar{u}^{\boldsymbol{k}_1}\,\gamma^\nu\,{}^{r_2}v^{\boldsymbol{k}_2})({}^{r_4}\bar{v}^{\boldsymbol{k}_4}\,\gamma_\nu\,{}^{r_3}u^{\boldsymbol{k}_3})({}^{r_2}\bar{v}^{\boldsymbol{k}_2}\,\gamma^\mu\,{}^{r_1}u^{\boldsymbol{k}_1})({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma_\mu\,{}^{r_4}v^{\boldsymbol{k}_4})$$

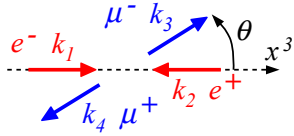
If the spins are not observed, then the same manipulations which we already applied in (24.54)ff lead to

$$\begin{aligned}
\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} &= \frac{1}{4}\sum_{r_1,r_2,r_3,r_4}\left|\mathcal{M}^{(2)}\right|^2 = \\
&= \frac{e^4\mu_0^2\hbar^2c^6}{4(k_1+k_2)^4}\,\mathrm{tr}\Big\{(\not{k}_1 + m_1c/\hbar)\,\gamma^\nu(\not{k}_2 - m_2c/\hbar)\,\gamma^\mu\Big\}\cdot \\
&\quad\cdot\,\mathrm{tr}\Big\{(\not{k}_4 - m_4c/\hbar)\,\gamma_\nu(\not{k}_3 + m_3c/\hbar)\,\gamma_\mu\Big\} \\
&= \frac{e^4\mu_0^2\hbar^2c^6}{4(k_1+k_2)^4}\left(k_1^\alpha k_2^\beta\,\mathrm{tr}\Big\{(\gamma_\alpha\gamma^\nu\gamma_\beta\gamma^\mu\Big\} - m_1\tfrac{c}{\hbar}m_2\tfrac{c}{\hbar}\,\mathrm{tr}\Big\{\gamma^\nu\gamma^\mu\Big\}\right)\cdot \\
&\quad\cdot\left(k_4^\gamma k_3^\delta\,\mathrm{tr}\Big\{\gamma_\gamma\gamma_\nu\gamma_\delta\gamma_\mu\Big\} - m_4\tfrac{c}{\hbar}m_3\tfrac{c}{\hbar}\,\mathrm{tr}\Big\{\gamma_\nu\gamma_\mu\Big\}\right).
\end{aligned} \tag{24.77}$$

Terms with the traces of three γ-matrices have been skipped from the outset because of (24.59). Using (24.59), we then find:

$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} = \frac{e^4\mu_0^2\hbar^2c^6}{4(k_1+k_2)^4}\cdot \tag{24.78}$$
$$\cdot\left(k_1^\alpha k_2^\beta\, 4(g_\alpha{}^\nu g_\beta{}^\mu - g_{\alpha\beta}g^{\nu\mu} + g_\alpha{}^\mu g^\nu{}_\beta) - 4g^{\nu\mu}m_1m_2c^2/\hbar^2\right)\cdot$$
$$\cdot\left(k_4^\gamma k_3^\delta\, 4(g_{\gamma\nu}g_{\delta\mu} - g_{\gamma\delta}g_{\nu\mu} + g_{\gamma\mu}g_{\nu\delta}) - 4g_{\nu\mu}m_4m_3c^2/\hbar^2\right)$$
$$= \frac{8e^4\mu_0^2\hbar^2c^6}{(k_1+k_2)^4}\left((k_1k_4)(k_2k_3) + (k_1k_3)(k_2k_4) + (k_1k_2)m_4m_3c^2/\hbar^2\right)$$

Here two terms, in which the product m_1m_2 of the masses of electrons and positrons are showing up, have been skipped because the terms are negligible as compared to the three other terms in this expression. We re-formulate this expression such, that instead of the four-wavenumbers the energies, masses, and scattering angles of the particles become visible. As the masses of positrons and electrons are equal, and as the masses of antimyons and myons are equal, we get in the center-of-mass system



$$\boldsymbol{k}_2 = -\boldsymbol{k}_1 \quad , \quad \boldsymbol{k}_4 = -\boldsymbol{k}_3$$
$$E_1 = E_2 = E_3 = E_4 \ .$$

As the electron mass is negligible, we furthermore get

$$|\boldsymbol{k}_3| = |\boldsymbol{k}_4| = \sqrt{\left(\frac{E_3}{\hbar c}\right)^2 - \left(\frac{m_\mu c}{\hbar}\right)^2}$$
$$|\boldsymbol{k}_1| = |\boldsymbol{k}_2| = \frac{E_1}{\hbar c}$$
$$k_2k_2 = k_1k_1 = k_1^0k_1^0 - \boldsymbol{k}_1^2 = 0 \ .$$

Thereby the factors in equation (24.78) can be written as follows:

$$k_1k_2 = k_1^0k_2^0 - \boldsymbol{k}_1\boldsymbol{k}_2 = (k_1^0)^2 + |\boldsymbol{k}_1|^2 = 2\left(\frac{E_1}{\hbar c}\right)^2$$

$$
\begin{aligned}
k_1k_3 &= \Big(\frac{E_1}{\hbar c}\Big)^2 - \frac{E_1}{\hbar c}\sqrt{\Big(\frac{E_1}{\hbar c}\Big)^2 - \Big(\frac{m_\mu c}{\hbar}\Big)^2}\,\cos\theta \\
k_1k_4 &= \Big(\frac{E_1}{\hbar c}\Big)^2 + \frac{E_1}{\hbar c}\sqrt{\Big(\frac{E_1}{\hbar c}\Big)^2 - \Big(\frac{m_\mu c}{\hbar}\Big)^2}\,\cos\theta \\
k_2k_3 &= k_1k_4 \quad , \quad k_2k_4 = k_1k_3 \\
k_3k_4 &= \Big(\frac{E_1}{\hbar c}\Big)^2 + \Big(\frac{E_1}{\hbar c}\Big)^2 - \Big(\frac{m_\mu c}{\hbar}\Big)^2 \\
(k_1{+}k_2)^4 &\overset{(24.48)}{=} s^2 = 4(k_1k_2)^2 = 16\Big(\frac{E_1}{\hbar c}\Big)^4
\end{aligned}
\tag{24.79}
$$

Insertion into equation (24.78) results into

$$
\begin{aligned}
\big|\mathcal{M}^{(2)}\big|^2_{\text{SNO}} &= \frac{e^4\mu_0^2\hbar^4c^8}{2E_1^2}\left(\Big(\frac{E_1}{\hbar c} + |\boldsymbol{k}_3|\,\cos\theta\Big)^2 + \right. \\
&\qquad \left. + \Big(\frac{E_1}{\hbar c} - |\boldsymbol{k}_3|\,\cos\theta\Big)^2 + 2\Big(\frac{m_\mu c}{\hbar}\Big)^2\right) \\
&= e^4\mu_0^2\hbar^2c^6\left(1 + \frac{m_\mu^2c^4}{E_1^2} + \Big(1 - \frac{m_\mu^2c^4}{E_1^2}\Big)\cos^2\theta\right) .
\end{aligned}
\tag{24.80}
$$

This again must be inserted into the cross section (24.34). As the electrons and positrons are high-relativistic, $|v_{\text{rel}}| = 2c$. Using $\epsilon_0\mu_0 = c^{-2}$, we get for the annihilation of an electron and a positron into a myon and an antimyon, if the spins are not observed, this cross section:

$$
\begin{aligned}
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\widetilde{\Omega}_3}\Big)_{\text{SNO}} &\overset{(24.34)}{=} \Big(\frac{e^2}{4\pi\epsilon_0\hbar c}\Big)^2\frac{\hbar^2c^2}{16E_1^2}\sqrt{1 - \frac{m_\mu^2c^4}{E_1^2}}\,\cdot \\
&\qquad \cdot\left(1 + \frac{m_\mu^2c^4}{E_1^2} + \Big(1 - \frac{m_\mu^2c^4}{E_1^2}\Big)\cos^2\theta\right)
\end{aligned}
\tag{24.81}
$$

in the center-of-mass system

If the energy of the incoming particles is much larger than the rest mass of the myons, then $E_1 = E_2 \equiv E$ in the cms, and the cross section simplifies to

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\widetilde{\Omega}_3}\right)_{\text{SNO}} = \left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 \cdot \hbar^2 c^2 \cdot \frac{1+\cos^2\theta}{16E^2}$$
$$\text{in the cms, if } m_\mu c^2 \ll E \ . \tag{24.82}$$

We considered the scattering of electrons and myons resp. their antiparticles, to evade the tedious computations of the interference terms between t-channel scattering and s-channel scattering. In 1936 Bhabha[5] published [51] the computation of the scattering with incoming and outgoing leptons and antileptons of same flavor, including the interference-terms. His result is

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\widetilde{\Omega}_3}\right)_{\text{SNO}} = \left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^2 \cdot \frac{\hbar^2 c^2}{8E^2} \cdot$$
$$\cdot \left(\frac{1+\cos^2\theta}{2} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}\right) \tag{24.83}$$
$$\text{in the cms, if } mc^2 \ll E \ .$$

In the first term we discern the s-channel scattering (24.82), and in the third term the t-channel scattering (24.69). The second term, which always is ≤ 0, is caused by interference of both channels.

24.3.4 Crossing Symmetries

In the high-relativistic case

$$k_j k_j = k_j^0 k_j^0 - \boldsymbol{k}_j \boldsymbol{k}_j = |\boldsymbol{k}_j| \cdot |\boldsymbol{k}_j| - \boldsymbol{k}_j \boldsymbol{k}_j = 0 \ , \tag{24.84}$$

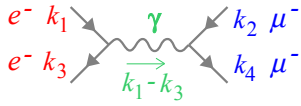
in which the masses of all particles taking part in the scattering event are negligible, the Mandelstam-variables assume the form

$$s \overset{(24.48)}{=} (k_1 + k_2)^2 = +2k_1 k_2 \tag{24.85a}$$
$$t \overset{(24.48)}{=} (k_1 - k_3)^2 = -2k_1 k_3 \tag{24.85b}$$
$$u \overset{(24.48)}{=} (k_1 - k_4)^2 = -2k_1 k_4 \ . \tag{24.85c}$$

[5] Homi Jehangir Bhabha (1909 - 1966)

In this case, the modulus-squares of the scattering matrix in case of not observed spins can be written for the t-channel as follows:

$e^-\mu^- \longrightarrow e^-\mu^-$:

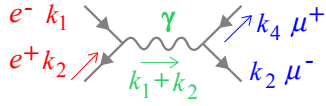


$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} \overset{(24.62)}{=} 8e^4\mu_0^2\hbar^2c^6\,\frac{(k_1k_2)(k_3k_4)+(k_1k_4)(k_2k_3)}{(k_1-k_3)^4}$$

$$\overset{(24.65)}{=} 2e^4\mu_0^2\hbar^2c^6\,\frac{s^2+u^2}{t^4} \tag{24.86a}$$

In the same approximation we find for the s-channel

$e^-e^+ \longrightarrow \mu^-\mu^+$:

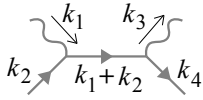


$$\left|\mathcal{M}^{(2)}\right|^2_{\text{SNO}} \overset{(24.78)}{=} 8e^4\mu_0^2\hbar^2c^6\,\frac{(k_1k_4)(k_2k_3)+(k_1k_3)(k_2k_4)}{(k_1+k_2)^4}$$

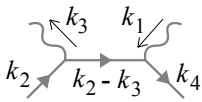
$$\overset{(24.79)}{=} 2e^4\mu_0^2\hbar^2c^6\,\frac{u^2+t^2}{s^4}\ . \tag{24.86b}$$

Thus in case of high-relativistic energies the modulus-squares of the scattering matrices for t-channel scattering and s-channel scattering differ by nothing than the exchange $t \leftrightarrow s$. This close formal relation is called "crossing symmetry". This symmetry does not really come as a surprise, because the two black-painted diagrams (24.86) are identical. Only their interpretations, printed in colors, is different. Again we see, that antiparticles may formally be considered particles, which are moving backwards through time.

24.3.5 Photon-Electron-Scattering



v-channel



w-channel

(24.87)

v-channel scattering and w-channel scattering are contributing to the scattering of a photon at an electron, which is called Compton-scattering[6]. The names v-channel and w-channel are not common in the literature. We defined these names to facilitate referencing. In v-channel scattering, an electron with wavenumber k_2, mass m, and polarization r_2 absorbs a photon with wavenumber k_1 and polarization $\epsilon_{\boldsymbol{k}_1}^{(\alpha_1)}$. Subsequently the electron emits a photon with wavenumber k_3 and polarization $\epsilon_{\boldsymbol{k}_3}^{(\alpha_3)}$, and goes out with wavenumber k_4 and polarization r_4. In w-channel scattering, an electron with wavenumber k_2, mass m, and polarization r_2 emits a photon with wavenumber k_3 and polarization $\epsilon_{\boldsymbol{k}_3}^{(\alpha_3)}$. Subsequently the electron absorbs a photon with wavenumber k_1 and polarization $\epsilon_{\boldsymbol{k}_1}^{(\alpha_1)}$, and goes out with wavenumber k_4 and polarization r_4.

First we consider the signs and symmetry factors of the two diagrams. In both diagrams, we name the left vertex y, the right vertex z. The contractions are

v-channel:

$$\langle 0|\, T\overbrace{\psi(x_4)\, \underbrace{A(x_3)\, \overline{\psi}(z)}\, A(z)}\, \overbrace{\psi(z)\, \overline{\psi}(y)}\, \underbrace{A(y)\, \psi(y)\, \overline{\psi}(x_2)}\, A(x_1)\, |0\rangle_c \tag{24.88a}$$

w-channel:

$$\langle 0|\, T\overbrace{\psi(x_4)\, \underbrace{A(x_3)\, \overline{\psi}(z)}\, \underbrace{A(z)\, \psi(z)\, \overline{\psi}(y)}\, A(y)}\, \overbrace{\psi(y)\, \overline{\psi}(x_2)}\, A(x_1)\, |0\rangle_c \tag{24.88b}$$

In both cases, no permutations of fermion operators are needed. Therefore both diagrams contribute with same signs to the scattering matrix. The symmetry factor of each diagram is 2, because $\overline{\psi}(x_2)$ can be combined to

[6] named in honor of Arthur Holly Compton (1892 - 1962), who investigated in the early twenty-twenties this type of scattering experimentally and theoretically [52].

a propagator with $\psi(y)$ or with $\psi(z)$. After this choice is made (which is equivalent to a permutation of the vertices $y \leftrightarrow z$), only one unique method each is left for the construction of the rest of the two diagrams.

Applying the rules of box 24.1 the following scattering matrix is found:

$$\begin{aligned}\mathcal{M}^{(2)} = 2 \cdot \mathcal{U}_{\mathrm{S}} \frac{1}{2!} \Big(\frac{+ie\gamma^{\nu}}{\hbar}\Big) \Big(\frac{+ie\gamma^{\mu}}{\hbar}\Big) \hbar^2 c^2 \mu_0 \cdot \\ \cdot \Big(\epsilon^{(\alpha_1)\, r_2}_{\boldsymbol{k}_1 \nu} u^{\boldsymbol{k}_2} \frac{i(\gamma^{\sigma}(k_1+k_2)_{\sigma} + mc/\hbar)}{(k_1+k_2)^2 - (mc/\hbar)^2 + i\epsilon'} \epsilon^{(\alpha_3)*\, r_4}_{\boldsymbol{k}_3 \mu} \bar{u}^{\boldsymbol{k}_4} + \\ + \epsilon^{(\alpha_3)*\, r_2}_{\boldsymbol{k}_3 \nu} u^{\boldsymbol{k}_2} \frac{i(\gamma^{\tau}(k_2-k_3)_{\tau} + mc/\hbar)}{(k_2-k_3)^2 - (mc/\hbar)^2 + i\epsilon'} \epsilon^{(\alpha_1)\, r_4}_{\boldsymbol{k}_1 \mu} \bar{u}^{\boldsymbol{k}_4} \Big) \end{aligned} \tag{24.89}$$

As the electron's mass is m and the photon's mass is zero,

$$\begin{aligned}(k_1+k_2)^2 - \frac{m^2c^2}{\hbar^2} &= 0 + 2k_1k_2 + \frac{m^2c^2}{\hbar^2} - \frac{m^2c^2}{\hbar^2} = 2k_1k_2 \\ (k_2-k_3)^2 - \frac{m^2c^2}{\hbar^2} &= \frac{m^2c^2}{\hbar^2} - 2k_2k_3 + 0 - \frac{m^2c^2}{\hbar^2} = -2k_2k_3 \ .\end{aligned}$$

Using this result, and applying the rearrangement-operator $\mathcal{U}_{\mathrm{S}}$ defined in (23.21), the scattering matrix assumes the somewhat simpler form

$$\begin{aligned}\mathcal{M}^{(2)} = -ie^2c^2\mu_0\, {}^{r_4}\bar{u}^{\boldsymbol{k}_4}\, \gamma^{\mu} \Big(\epsilon^{(\alpha_3)*}_{\boldsymbol{k}_3 \mu} \frac{\gamma^{\sigma}(k_1+k_2)_{\sigma} + mc/\hbar}{2k_1k_2 + i\epsilon'} \epsilon^{(\alpha_1)}_{\boldsymbol{k}_1 \nu} - \\ - \epsilon^{(\alpha_1)}_{\boldsymbol{k}_1 \mu} \frac{\gamma^{\tau}(k_2-k_3)_{\tau} + mc/\hbar}{2k_2k_3 - i\epsilon'} \epsilon^{(\alpha_3)*}_{\boldsymbol{k}_3 \nu} \Big) \gamma^{\nu}\, {}^{r_2}u^{\boldsymbol{k}_2} \ . \end{aligned} \tag{24.90}$$

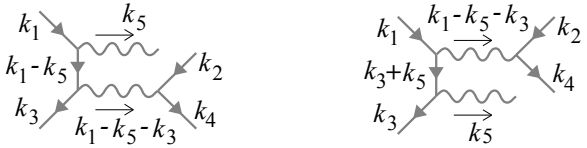
Here a new type of divergence becomes visible: As the photon's rest mass is zero, it's wave number is not bounded from below. In case $k_1 \to 0$ and/or $k_3 \to 0$ the scattering matrix diverges. We encountered such infrared-divergences neither in ψ^s-theory nor in Yukawa-theory, because their bosons are massive. In ψ^s-theory and in Yukawa-theory only UV-divergences are known, which are showing up in graphs with some certain types of loops. We will see, that there are loops with UV-divergences in QED as well. In addition, however, there are IR-divergences in QED which already are showing up in tree-graphs with external photon lines, due to the photon's

vanishing rest mass.

In contrast to UV-divergences, IR-divergences are not treated by renormalization, but by compensation due to other graphs.

24.3.6 Bremsstrahlung and IR-Divergences

Since the end of the 19. century it is known experimentally, that electrically charged particles emit electromagnetic radiation, if they are accelerated. This radiation is called bremsstrahlung. For a scattering event with an incoming electron, an incoming myon, an outgoing electron, an outgoing myon, and an outgoing bremsstrahlungs-photon, in 3. order perturbation computation these two diagrams are found (besides others):

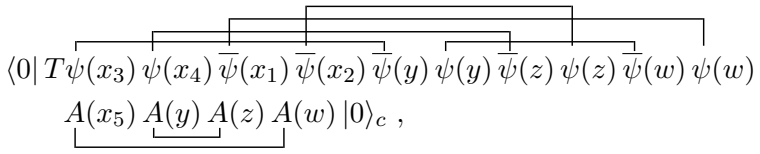


(24.91)

The electron comes in with wavenumber k_1, and goes out with wavenumber k_3. The myon comes in with wavenumber k_2, and goes out with wavenumber k_4. The bremsstrahlungs-photon goes out with wavenumber k_5.

Clearly there will be bremsstrahlung at the myon-current as well. But as the scattering matrix — as we will see immediately — is inversely proportional to the square of the mass of the charged particle, and as the probability of the scattering event again is proportional to the square of the scattering matrix, the probability of bremsstrahlung at the myon current is 10^{-9} times the probability of bremsstrahlung at the electron current, and thus may be neglected. (Remember that the rest mass of a myon is about 207 times as large as the rest mass of an electron.) Therefore, when considering the scattering of electrons by myons, we may concentrate the investigation of bremsstrahlung in good approximation to the two diagrams displayed above.

Permutations of boson operators with arbitrary other operators don't change the matrix element's value. Therefore in the matrix element



$$\langle 0|\, T\psi(x_3)\,\psi(x_4)\,\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\overline{\psi}(y)\,\psi(y)\,\overline{\psi}(z)\,\psi(z)\,\overline{\psi}(w)\,\psi(w)$$
$$A(x_5)\,A(y)\,A(z)\,A(w)\,|0\rangle_c\ , \tag{24.92}$$

the boson operators could be shifted to the right, to make the contraction brackets fit into one line each. Those contractions are indicated, which lead to the left diagram. The same diagram is found after arbitrary permutations of the 3 vertices, thus the symmetry factor of this diagram is $3! = 6$. This is a general rule in QED: The symmetry factor is equal to the faculty of the vertex count. The vertex count again is equal to the order of perturbation computation. Therefore the symmetry factor cancels versus the factor $1/3!$, which comes from the Taylor-expansion of the interaction term. 11 permutations of fermion operators are needed to realize the contraction, resulting into a factor (-1).

To construct the second graph instead of the first from the same matrix element, only boson operators need to be permuted and contracted differently. Therefore the symmetry factors and the signs of both graphs are equal. Applying the rules of box 24.1 the following scattering amplitude is found:

$$\overline{S}^{(3)} = \mathcal{M}^{(3)} \cdot \Big(\frac{1}{2\hbar\Omega}\Big)^{\frac{5}{2}} \sqrt{\frac{1}{\omega_{\boldsymbol{k}_1}\omega_{\boldsymbol{k}_2}\omega_{\boldsymbol{k}_3}\omega_{\boldsymbol{k}_4}\omega_{\boldsymbol{k}_5}}}\,\cdot$$
$$\cdot\ 2\pi\Omega\,\delta\Big(\sum_{j=1,2} k_j^0 - \sum_{j=3,4,5} k_j^0\Big) \cdot \delta_{\sum_{j=1,2}\boldsymbol{k}_j\,,\,\sum_{j=3,4,5}\boldsymbol{k}_j} \tag{24.93a}$$

$$\mathcal{M}^{(3)} = \frac{3!}{3!}\Big(\frac{+ie}{\hbar}\Big)^3 \hbar c\,\sqrt{\mu_0}\,\cdot$$
$$\cdot\,\Big(\Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\nu\,\frac{i(\gamma^\tau(k_1-k_5)_\tau + m_e\frac{c}{\hbar})}{(k_1-k_5)^2 - m_e^2\frac{c^2}{\hbar^2}+i\epsilon'}\,\gamma^{\sigma\,r_1}u^{\boldsymbol{k}_1}\Big)\cdot$$
$$\cdot\,\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma}\,\frac{(-ig_{\nu\mu}\,\mu_0\hbar c)}{(k_1-k_5-k_3)^2+i\epsilon'}\Big({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^{\mu\,r_2}u^{\boldsymbol{k}_2}\Big) +$$
$$+\,\Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\sigma\,\frac{i(\gamma^\tau(k_3+k_5)_\tau + m_e\frac{c}{\hbar})}{(k_3+k_5)^2 - m_e^2\frac{c^2}{\hbar^2}+i\epsilon'}\,\gamma^{\nu\,r_1}u^{\boldsymbol{k}_1}\Big)\cdot$$
$$\cdot\,\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma}\,\frac{(-ig_{\nu\mu}\,\mu_0\hbar c)}{(k_1-k_5-k_3)^2+i\epsilon'}\Big({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^{\mu\,r_2}u^{\boldsymbol{k}_2}\Big)\Big) \qquad (24.93b)$$

The incoming and outgoing particles are on mass-shell. Therefore $k_5^2 = 0$ for the outgoing photon , $k_1^2 = k_3^2 = m_e^2c^2/\hbar^2$ for the electron, and the denominators of the electron propagators can be written in the form

$$(k_1-k_5)^2 - (m_ec/\hbar)^2 + i\epsilon' = -2k_1k_5$$
$$(k_3+k_5)^2 - (m_ec/\hbar)^2 + i\epsilon' = +2k_3k_5\ . \qquad (24.93c)$$

The tiny imaginary term can be skipped, as we are assuming that k_1, k_3, k_5 all are different from zero.

The scattering amplitude (24.93) holds for the emission of bremsstrahlung of arbitrary energy. In the reminder of this section we restrict ourselves to the limit of very low frequency, soft bremsstrahlungs-photons, in which

$$k_1 - k_5 \approx k_1 \qquad , \qquad k_3 + k_5 \approx k_3\ . \qquad (24.94)$$

Then within the first spinor product in (24.93b) there is the factor

$$(\gamma^\tau k_{1\tau} + m_ec/\hbar)\gamma^{\sigma\,r_1}u^{\boldsymbol{k}_1}\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma} =$$
$$\overset{(8.9)}{=} 2g^{\sigma\tau}\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma}k_{1\tau}{}^{r_1}u^{\boldsymbol{k}_1} - \gamma^\sigma\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma}\underbrace{(\gamma^\tau k_{1\tau} - m_ec/\hbar)^{r_1}u^{\boldsymbol{k}_1}}_{0} =$$
$$\overset{(8.60)}{=} 2\epsilon^{(\alpha)\tau*}_{\boldsymbol{k}_5}k_{1\tau}{}^{r_1}u^{\boldsymbol{k}_1}\ . \qquad (24.95)$$

By the same method in the first spinor product of the second term the

factor

$$
\begin{aligned}
&{}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\sigma\,(\gamma^\tau k_{3\tau}+m_e c/\hbar)\,\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma} = \\
&\quad = {}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,2g^{\sigma\tau}k_{3\tau}\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma} - \underbrace{{}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,(\gamma^\tau k_{3\tau}-m_e c/\hbar)}_{0}\,\gamma^\sigma\,\epsilon^{(\alpha)*}_{\boldsymbol{k}_5\sigma} = \\
&\quad = {}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,2\epsilon^{(\alpha)\tau*}_{\boldsymbol{k}_5}k_{3\tau}
\end{aligned}
\tag{24.96}
$$

can be identified. In the photon propagator's denominator, the approximation (24.94) results into

$$
(k_1-k_5-k_3)^2 \approx (k_1-k_3)^2\ . \tag{24.97}
$$

Thus in the approximation of soft bremsstrahlung the scattering matrix becomes

$$
\begin{aligned}
\mathcal{M}^{(3)} = \Big(\frac{+ie}{\hbar}\Big)^3\hbar c\,\sqrt{\mu_0}\,\Big(&\\
&\Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\nu\,\frac{i2\epsilon^{(\alpha)\tau*}_{\boldsymbol{k}_5}k_{1\tau}}{-2k_1k_5}\,{}^{r_1}u^{\boldsymbol{k}_1}\Big)\frac{(-ig_{\nu\mu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\Big({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\mu\,{}^{r_2}u^{\boldsymbol{k}_2}\Big)+\\
&+\Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\frac{i2\epsilon^{(\alpha)\tau*}_{\boldsymbol{k}_5}k_{3\tau}}{2k_3k_5}\,\gamma^\nu\,{}^{r_1}u^{\boldsymbol{k}_1}\Big)\frac{(-ig_{\nu\mu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\Big({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\mu\,{}^{r_2}u^{\boldsymbol{k}_2}\Big)\Big)\\
= -\Big(\frac{+ie}{\hbar}\Big)^2&({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\nu\,{}^{r_1}u^{\boldsymbol{k}_1})\,\frac{(-ig_{\nu\mu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\,({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\mu\,{}^{r_2}u^{\boldsymbol{k}_2})\cdot\\
&\cdot\ ec\,\sqrt{\mu_0}\epsilon^{(\alpha)\tau*}_{\boldsymbol{k}_5}\Big(\frac{k_{3\tau}}{k_3k_5}-\frac{k_{1\tau}}{k_1k_5}\Big)\ .
\end{aligned}
\tag{24.98}
$$

This result differs from the scattering matrix for elastic scattering of electrons by myons in 2. order perturbation theory only by the factor in the last line, compare the first term in (24.47b).

This results holds much more generally, than only for the scattering of an electron by a myon. We consider in arbitrary order n of perturbation theory an arbitrary scattering process, in which an arbitrary fermion (particle or antiparticle) is coming in with wavenumber k_1 and going out with wavenumber k_3. If this fermion emits in this process a bremsstrahlungsphoton with wavenumber k_5, then we call the scattering matrix of this

process $\mathcal{M}^{(n)}_{\text{with}}$. The same process without emission of bremsstrahlung, but identical for the rest, is found in perturbation computation of order $n-1$. We call the scattering matrix of the latter process $\mathcal{M}^{(n-1)}_{\text{without}}$, and state without proof:

$$\mathcal{M}^{(n)}_{\text{with}} = \mathcal{M}^{(n-1)}_{\text{without}} \cdot ec\sqrt{\mu_0}\,\epsilon^{(\alpha)\tau *}_{\boldsymbol{k}_5}\left(\frac{k_{3\tau}}{k_3 k_5} - \frac{k_{1\tau}}{k_1 k_5}\right)$$
$$\text{if } k_1 - k_5 \approx k_1 \text{ and } k_3 + k_5 \approx k_3 \tag{24.99}$$

The dimension of the factor

$$\left[ec\sqrt{\mu_0}\,\epsilon^{(\alpha)\tau *}_{\boldsymbol{k}_5}\left(\frac{k_{3\tau}}{k_3 k_5} - \frac{k_{1\tau}}{k_1 k_5}\right)\right] =$$
$$= \sqrt{\text{energy}\cdot\text{volume}} = \left[\sqrt{2\hbar\omega_{\boldsymbol{k}_5}\Omega}\right]$$

is just compensated by the normalization factor $\sqrt{1/2\hbar\omega_{\boldsymbol{k}_5}\Omega}$, which is (because of the outgoing photon) inserted additionally into the scattering amplitude $S^{(n)}$ according to rule F of box 24.1 .

No photon-detector is able to detect bremsstrahlung of arbitrary low energy. If for example in an experiment the scattering of an electron by a myon is observed, and no photon is thereby detected, then it can not be excluded that still a bremsstrahlungs-photon was emitted which just because of it's low energy escaped detection. If the cross section for any scattering event is computed in QED, therefore always the cross section of the same event with additional emission of unmeasurable low-energy bremsstrahlung must be added. Without that we can not expect that experimental observations and theory will match.

Because of

$$\lim_{|k_5|\to 0} ec\sqrt{\mu_0}\,\epsilon^{(\alpha)\tau *}_{\boldsymbol{k}_5}\left(\frac{k_{3\tau}}{k_3 k_5} - \frac{k_{1\tau}}{k_1 k_5}\right) = \pm\infty \tag{24.100}$$

the adder of scattering events with bremsstrahlung of unmeasurable low energy does diverge. In this book we will encounter many similar formulas with infrared-divergences. The detailed investigation of the IR-divergences of

QED is a most tedious business. The results can be found in a comprehensive publication by Yennie, Frautschi, and Suura [53]. The essential result is quite simple:

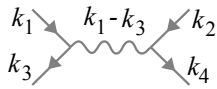
> The infrared-divergences of QED mutually compensate, i. e. they vanish, if *all* graphs are considered, which — with or without bremsstrahlung of unmeasurable low energy — are contributing to an observed phenomenon. (24.101)

We spare ourselves the difficult proof of this fact. Instead whenever we encounter IR-divergent results, we will merely refer to (24.101) and refrain from any further investigation.

24.3.7 The mass of virtual particles

We start our investigations with the t-channel scattering of a fermion, which is coming in with wavenumber k_1, by another fermion, which is coming in with wavenumber k_2:

t-channel:



$$\frac{m_\gamma^2 c^2}{\hbar^2} = (k_3 - k_1)^2 < 0 \tag{24.102}$$

m_γ is the rest-mass of the virtual photon in t-channel scattering. It can not be zero, because then $k_3 = k_1$ would hold, i. e. no scattering would happen at all. It may however be arbitrarily close to zero in case of weak scattering ($k_3 \approx k_1$).

We exploit the fact, that the particles number 1 and number 3 are of same flavor, and consequently have same mass. Furthermore we now assume, that these particles are free, observable particles “on mass-shell”, for which the relation $k_j^2 = (mc/\hbar)^2$ holds. Then we have:

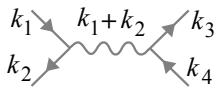
$$\frac{m_\gamma^2 c^2}{\hbar^2} = (k_3 - k_1)^2 \overset{(7.18)}{=} (k_3^0 - k_1^0)^2 - (\boldsymbol{k}_3 - \boldsymbol{k}_1)^2 =$$
$$= \underbrace{(k_3^0)^2 - \boldsymbol{k}_3^2}_{(mc/\hbar)^2} + \underbrace{(k_1^0)^2 - \boldsymbol{k}_1^2}_{(mc/\hbar)^2} - 2\sqrt{(\boldsymbol{k}_3^2 + m^2c^2/\hbar^2)(\boldsymbol{k}_1^2 + m^2c^2/\hbar^2)} + 2\boldsymbol{k}_3\boldsymbol{k}_1 \tag{24.103}$$

We consider the process in the center-of-mass frame of the particles number 1 and number 3. This reference frame is defined by $\boldsymbol{k}_3 = -\boldsymbol{k}_1$. The invariant wavenumber-square of the virtual photon is in this reference frame

$$\frac{m_\gamma^2 c^2}{\hbar^2} = 2\,\frac{m^2c^2}{\hbar^2} - 2\boldsymbol{k}_1^2 - 2\,\frac{m^2c^2}{\hbar^2} - 2\boldsymbol{k}_1^2 = -4\boldsymbol{k}_1^2 < 0\ . \tag{24.104}$$

The invariant wavenumber-square is called invariant, because it is identical in any inertial system. Thus the invariant wavenumber-square of the virtual photon is smaller than zero in t-channel scattering, i. e. it's mass is imaginary. This proves (24.102). Attention: The wavenumber-square of the virtual photon is an invariant, which is independent of the reference system, and which is smaller than zero. But this invariant is not equal to $-4\boldsymbol{k}_1^2$ in any arbitrary reference system, because not $\boldsymbol{k}_1^2$ but $k_1^2 = (k_1^0)^2 - \boldsymbol{k}_1^2$ is invariant.

s-channel:



$$m_\gamma^2 c^2/\hbar^2 = (k_1 + k_2)^2 > 4\,m^2c^2/\hbar^2 \tag{24.105}$$

m_γ is the virtual photon's rest mass in s-channel scattering, m is the rest mass of the fermion particle coming in with wavenumber k_1, and the rest mass of the antiparticle coming in with wavenumber k_2. In s-channel scattering again the invariant wavenumber-square of the virtual photon can not be zero for the following reason: We always have $k_1 \neq -k_2$, because $k_1^0 = +k_2^0$ for the null-components in case $\boldsymbol{k}_1 = -\boldsymbol{k}_2$.

Assuming that the incoming particles are observable particles on mass-shell, the virtual photon's invariant wavenumber-square is

$$\frac{m_\gamma^2 c^2}{\hbar^2} \overset{(24.103)}{=} 2\,\frac{m^2c^2}{\hbar^2} + 2\sqrt{(\boldsymbol{k}_1^2 + m^2c^2/\hbar^2)(\boldsymbol{k}_2^2 + m^2c^2/\hbar^2)} - 2\boldsymbol{k}_1\boldsymbol{k}_2\ .$$

We will do the computation in the center-of-mass frame $\boldsymbol{k}_2 = -\boldsymbol{k}_1$:

$$\frac{m_\gamma^2 c^2}{\hbar^2} = 4\,\frac{m^2c^2}{\hbar^2} + 4\boldsymbol{k}_1^2 > 4\,\frac{m^2c^2}{\hbar^2}$$

This proves (24.105).

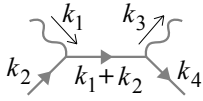
The virtual photon is space-like ($m_\gamma^2 < 0$) in t-channel scattering (the same holds true for u-channel scattering), while the virtual photon is time-like ($m_\gamma^2 > 4m^2 > 0$) in s-channel scattering. In second order perturbation computation there are no virtual photons in QED with masses in the intervall $0 \le m_\gamma \le 2m$. (There are no virtual particles at all in zeroth and first order of perturbation computation in QED.)

Therefore it would be pointless to discuss in t-channel scattering (or in u-channel scattering) whether a particle, which is coming in with wavenumber k_1, first does emit a virtual photon which later is absorbed by another particle coming in with wavenumber k_2, or vice versa. For events with space-like relations, notions like earlier or later are without meaning. In contrast, one may in case of s-channel scattering reasonably state that first a particle and it's antiparticle annihilate into a virtual photon, and that later this virtual photon decays due to pair creation into a particle and it's antiparticle.

On the other hand, in case of t-channel scattering (and u-channel scattering) it is very well possible to determine the locations, where the interaction of the photon with the fermions happens. One can for example let two particle beams cross with well-controlled distance, and measure the (weak) scattering. But it is impossible to determine the relation in space of the locations, where in s-channel scattering the particles are annihilated and where they are created. (In the center-of-mass system they are annihilated and created at the same location, name at the coordinate system's origin. But we clearly are considering a spacelike relation in the laboratory system of the observer.)

The invariant wavenumber-square of the virtual electron in v-channel scattering is

v-channel:



$$\frac{m_{\mathrm{v}}^2c^2}{\hbar^2} = (k_1+k_2)^2 > \frac{m^2c^2}{\hbar^2} \ . \tag{24.106}$$

m_{v} is the virtual fermion's rest mass, m is the observed fermion's rest mass. Assuming that the incoming particles are observed particles, we have

$$\frac{m_{\mathrm{v}}^2c^2}{\hbar^2} \overset{(24.103)}{=} 0 + \frac{m^2c^2}{\hbar^2} + 2\sqrt{\boldsymbol{k}_1^2(\boldsymbol{k}_2^2+m^2c^2/\hbar^2)} - 2\boldsymbol{k}_1\boldsymbol{k}_2 \ .$$

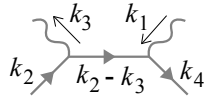
In the center-of-mass system $\boldsymbol{k}_2 = -\boldsymbol{k}_1$ of the incoming particles follows

$$\frac{m_{\mathrm{v}}^2c^2}{\hbar^2} = \frac{m^2c^2}{\hbar^2} + 2\sqrt{\boldsymbol{k}_1^2(\boldsymbol{k}_1^2+m^2c^2/\hbar^2)} + 2\boldsymbol{k}_1^2 > \frac{m^2c^2}{\hbar^2} + 4\boldsymbol{k}_1^2 \ .$$

This proves (24.106).

In w-channel scattering the virtual fermion's invariant wavenumber-square is

w-channel:



$$\frac{m_{\mathrm{w}}^2c^2}{\hbar^2} = (k_2-k_3)^2 < \frac{m^2c^2}{\hbar^2} \ . \tag{24.107}$$

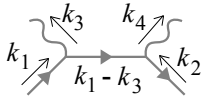
Assuming that the particles number 2 and number 3 are observed, in the center-of-mass system $\boldsymbol{k}_3 = -\boldsymbol{k}_2$

$$\begin{aligned}\frac{m_{\mathrm{w}}^2c^2}{\hbar^2} &\overset{(24.103)}{=} 0 + \frac{m^2c^2}{\hbar^2} - 2\sqrt{\boldsymbol{k}_2^2(\boldsymbol{k}_2^2+m^2c^2/\hbar^2)} - 2\boldsymbol{k}_2^2 \\ &< \frac{m^2c^2}{\hbar^2} - 4\boldsymbol{k}_2^2 < \frac{m^2c^2}{\hbar^2} \ .\end{aligned}$$

This proves (24.107).

As our last example we consider the annihilation of a particle and it's antiparticle into two photons. We will use for this process the (not commonly established) name z-channel scattering:

z-channel:



$$\frac{m_z^2 c^2}{\hbar^2} = (k_2 - k_3)^2 \lessapprox \frac{m^2 c^2}{\hbar^2} \ . \tag{24.108}$$

This symbol is indicating, that the invariant wavenumber of the virtual fermion is only slightly smaller than that of the free particle. Assuming that the particles number 1 and number 3 are observed, in the center-of-mass system holds

$$\frac{m_z^2 c^2}{\hbar^2} \overset{(24.103)}{=} \frac{m^2 c^2}{\hbar^2} + 0 - 2\sqrt{(\boldsymbol{k}_1^2 + m^2 c^2/\hbar^2)\boldsymbol{k}_1^2} + 2\boldsymbol{k}_1^2 \lessapprox \frac{m^2 c^2}{\hbar^2} \ .$$

This proves (24.108).

In second order of perturbation computation there is no gap in the range of possible wavenumber-squares of virtual fermions — in contrast to virtual photons. In v-channel scattering the invariant wavenumber-square of the virtual fermions may assume any arbitrary value $> m^2 c^2/\hbar^2$, in w-channel scattering any arbitrary (even negative) value $< m^2 c^2/\hbar^2$. This difference is caused by the fact, that observed photons may have any arbitrary energy > 0, while the energy of observed fermions always is $\geq mc^2$.

25 Degree of Divergence, and Regularization

25.1 The superficial Degree of Divergence

Any loop in a diagram, whose wavenumber is not fixed by a delta function, leads to an integral with a factor $\mathrm{d}^4k \sim k^4$ in the numerator. If this factor is not compensated due to sufficiently high powers of k in the denominator, then the integral will diverge. In section 20.3 we already observed

$$\text{structure}: \frac{\mathrm{d}^4k}{k^n} \overset{(20.95)}{\Longrightarrow} \begin{cases} n=2 & : \text{ quadratic divergence} \\ n=4 & : \text{ logarithmic divergence} \\ n\geq 6 & : \text{ no divergence .} \end{cases}$$

The loop-exponent $d = 4 - n$ defined by

$$\frac{\mathrm{d}^4k}{k^n} \sim k^{4-n} = k^d \tag{25.1}$$

determines whether and how the loop diverges. $d = 0$ leads to a logarithmic divergence, $d = 1$ to a linear divergence, $d = 2$ to a quadratic divergence, and so on. For $d < 0$ the loop is convergent. The superficial degree of divergence D, which we are going to define immediately, differs from the loop-exponent d. D is an index, which gives a measure for the "tendency to divergence" of a complete diagram with an arbitrary number of lines, loops, and vertices.

When counting the powers of wavenumbers in the denominator, we must discern fermions and bosons, because the fermion-propagator is proportional to k^{-1}, while the boson-propagator is proportional to k^{-2}. Using the notations

$$L \equiv \text{number of loops}$$

$$
\begin{aligned}
V &\equiv \text{number of vertices} \\
B_e &\equiv \text{number of external boson lines} \\
F_e &\equiv \text{number of external fermion lines} \\
B_i &\equiv \text{number of internal boson lines} \\
F_i &\equiv \text{number of internal fermion lines} \ ,
\end{aligned} \tag{25.2}
$$

the superficial degree of divergence is defined by

$$
\boxed{D \equiv 4L - 2B_i - F_i} \ . \tag{25.3}
$$

D is called "superficial", because — being an overall measure for the complete diagram — it can not describe the divergence (or convergence) as clearly and uniquely as the loop-exponent d of a single loop. In any case $4L$ is the power of k showing up in total in the numerators of the loop-integrals. The total number of powers of k in the denominators of all loop-integrals may very well be larger or smaller than $2B_i + 1F_i$, because on the the one hand not all inner lines of a graph necessarily are integrated into loops. If only some part of the inner lines is integrated into loops, then $d_{\max} > D$ will hold for that loop in the graph with the largest loop-index $d_{\max}$. On the other hand inner lines may be parts of several loops at the same time. In that case the superficial degree of divergence gives an overly pessimistic impression. For example, the superficial degree of divergence of the double-loop  of ψ^4-theory is $D = 8 - 6 = 2$, while the largest loop-index is merely $d_{\max} = 0$. Consequently this graph diverges not quadratically, but only logarithmically. Another example for the type of divergence, which is not obvious from the superficial degree of divergence, are the three graphs

$$
L = 2 \qquad V = 4 \qquad B_e = 6 \qquad B_i = 5 \tag{25.4}
$$

of ψ^4-theory. The superficial degree of divergence is the same for all three of these diagrams (namely $D = -2$), and even their values of B_e, B_i, V, L are

equal. We already know from section 20.3, that the left graph is convergent because of $d_{\max} = -2$, while the second graph is diverging logarithmically because of $d_{\max} = 0$, and the right graph is diverging quadratically because of $d_{\max} = +2$. D will be approximately equal to $d_{\max}$ only, if the lines are distributed to the loops such, that $d_{\max}$ assumes the smallest possible value for a given number of lines and vertices.

Thus it's better to evaluate the maximum loop-index $d_{\max}$, if exact informations of the possible divergence of a certain graph are required. The definition of D serves a different purpose. We want to find out, whether in some quantum field theory again and again new types of diverging loops will turn up in again and again higher orders of perturbation computations, which must be fixed by new types of renormalization, or whether a finite number of renormalizations will secure the convergence of the theory in arbitrary orders of perturbation computations. New types of loops will show up — if at all — just in diagrams with minimum $d_{\max}$, and thus can be identified due to $D \geq 0$.

It will turn out useful, to formulate D as a function of B_e, F_e and V. A graph is free of loops (i. e. a tree-graph), if it contains $V = B_i + F_i + 1$ vertices in case of $B_i + F_i$ inner lines. Then $0 = B_i + F_i - V + 1$. If in this graph the numbers B_i or F_i are increased, or if the number V is decreased, then there will be

$$L = B_i + F_i - V + 1 \tag{25.5}$$

loops in that graph. Insertion into (25.3) gives the result

$$D = 2B_i + 3F_i - 4V + 4 \ . \tag{25.6}$$

For the moment being we restrict the investigation to ψ^s-theory. The V vertices are made of $s \cdot V$ field-operators. If all the operators are mutually combined to propagators, a vacuum-bubble with $B_i = sV/2$ inner lines is created. If in contrast an even number B_e of the vertex-operators is combined to propagators with external operators, then only $B_i = (sV - B_e)/2$ inner lines can be constructed, and

$$2B_i = sV - B_e \ . \tag{25.7}$$

This is inserted into (25.6). As we are considering only ψ^s-theory for the moment being, $F_i = 0$:

$$\boxed{\psi^s\text{-theory:} \quad D = (s-4)V - B_e + 4} \tag{25.8}$$

In case of ψ^3-theory, the result is

$$\psi^3\text{-theory:} \quad D = -V - B_e + 4 \ . \tag{25.9}$$

As $-V$ and $-B_e$ are negative in these equations, new types of divergences can turn up only in diagrams with few external lines in low orders of perturbation theory. There exist exactly two:

$$\begin{aligned} &V = 2 \quad B_e = 0 \quad D = 2 \qquad d_{\max} = 2 \\ &V = 2 \quad B_e = 2 \quad D = 0 \qquad d_{\max} = 0 \end{aligned} \tag{25.10}$$

In section 20.3 we ignored the vacuum bubble, because there we were interested only in connected diagrams, and canceled all vacuum bubbles from the LSZ-formula. In higher orders of perturbation theory, diverging graphs may still turn up, but only if the loop-indices within one graph are quite different. For example, in case of two external lines one finds in fourth oder of perturbation computation the two graphs

$$\begin{aligned} &V = 4 \quad B_e = 2 \quad D = -2 \qquad d_{\max} = -2 \\ &V = 4 \quad B_e = 2 \quad D = -2 \qquad d_{\max} = 0 \ . \end{aligned}$$

In spite of $D < 0$, only the first graph is convergent, while the second graph is diverging logarithmically. But this is happening only because in the second graph not all of the 5 inner lines are integrated into the two loops. The type of divergence of the loops of the second graph is already known from second order perturbation computation, and has been cured due to mass renormalization. No divergences of any new type can show up in any

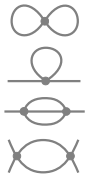
order of perturbation computation of ψ^3-theory, besides the one already known from (25.10).

In ψ^4-theory, the superficial degree of divergence

$$\psi^4\text{-theory:} \quad D = -B_e + 4 \tag{25.11}$$

remarkably is independent of the number V of vertices, and consequently independent of the order $n = V$ of perturbation computation. New types of divergences can show up only in graphs with less than 5 external lines. In total there are four:

$$\begin{array}{llll} V=1 & B_e=0 & D=4 & d_{\max}=2 \\ V=1 & B_e=2 & D=2 & d_{\max}=2 \\ V=2 & B_e=2 & D=2 & d_{\max}=0 \\ V=2 & B_e=4 & D=0 & d_{\max}=0 \end{array} \tag{25.12}$$



Again we see, that the superficial degree of divergence indeed is "superficial", and only gives a quite rough picture of the actual divergences of a graph: The vacuum bubble in the first line is consisting of two tadpoles, each of them diverging quadratically. And the double-loop in the third line actually is diverging logarithmically, but not quadratically. The essential result, however, is that there exist in ψ^4-theory only 3 different types of loops (the vacuum bubble is just a doubled tadpole). Not any new types of loops are possible in any graph. After the diverging graphs in the three bottom lines of (25.12) have been cured due to the renormalizations described in section 22, ψ^4-theory is convergent in all orders of perturbation computations.

$$\psi^5\text{-theory:} \quad D = V - B_e + 4 \tag{25.13}$$

In ψ^5-theory, and in any further ψ^s-theory with $s \geq 5$, D contains the number V of vertices with positive sign. Consequently we must be aware of ever new types of diverging loops in all of the infinitely many orders of perturbation theory, which must be treated due to renormalization of infinitely many parameters.

For historical reasons, such theories are called "not renormalizable". That

notion is not accurate. "Not renormalizable" theories actually are very well renormalizable; just ever new renormalizations are needed in any new order of perturbation computation. In contrast, "renormalizable" theories need renormalizations only in the lowest orders of perturbation computations, but not any more in higher orders.

As long as theorists still hoped to construct quantum field theories, which are correct for arbitrarily huge wave numbers, the difference between "renormalizable" and "not renormalizable" theories seemed to be very important. They changed their point of view, however, since they learned to consider any renormalized quantum field theories as merely effective theories, which are valid only for a limited range of wave numbers (see section 21.3). To compute results for a limited range of wave numbers, only a finite number of parameters need to be renormalized in both "renormalizable" and "not renormalizable" theories. Thus by today "not renormalizable" theories are considered as absolutely acceptable.

In quantum-electrodynamics, each vertex is containing two fermion operators and one photon operator. For the construction of a graph with B_i inner and B_e outer photon lines,

$$V = 2B_i + B_e \tag{25.14a}$$

vertices are needed. For the construction of a graph with F_i inner and F_e outer fermion lines,

$$V = \frac{2F_i + F_e}{2} \tag{25.14b}$$

vertices are needed. Addition of these equation results into

$$(25.14a) + 3 \cdot (25.14b) = 4V = 2B_i + 3F_i + B_e + \frac{3}{2}F_e \ . \tag{25.15}$$

This is inserted into (25.6):

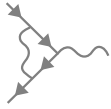
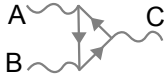
$$\boxed{\text{QED:} \quad D = 4 - B_e - \frac{3}{2}F_e} \tag{25.16}$$

Like in ψ^4-theory, the superficial degree of divergence is in quantum electrodynamics as well independent of the order $n = V$ of perturbation computation, and does depend only on the numbers of external lines. As the sign of these numbers are negative, new types of divergences can show up only in diagrams with few external lines.

A loop can diverge, if the loop-index is

$$d \overset{(25.1)}{=} 4 - f_i - 2b_i \geq 0 \tag{25.17}$$
$$f_i \equiv \text{count of fermion lines in the loop}$$
$$b_i \equiv \text{count of boson lines in the loop .}$$

There exist exactly 5 loops in QED, for which this condition is fulfilled: They are listed in table 25.1. It will be shown in section 26.5, that the graph in the second-last line of the table may be skipped, because is is exactly compensated by another graph. And the value of photon-photon scattering is finite, see section 26.6. There exist only three diverging graphs in QED, namely the three graphs in the first lines of table 25.1. The first and the second diagram are showing up both in propagator-corrections and as parts of larger graphs. The vertex correction exists only as a part of larger diagrams, not as self-contained graph, because all incoming and

(b_i, f_i)	Graph	D	d	Name
$(0,2)$		2	2	vacuum polarization
$(1,1)$		1	1	fermion self-energy
$(1,2)$		0	0	vertex correction
$(0,3)$		1	1	—-
$(0,4)$		0	0	photon-photon scattering

Tab. 25.1 : The 5 possibly diverging loops of QED

outgoing particles of a diagram must fulfill the relation (7.18), i. e. they must be "on mass-shell". That's impossible in a diagram with two external fermion lines and one external photon line.

When we computed Feynman-graphs of ψ^s-theory with diverging loops in chapter 20, we encountered integrals for which we derived the generic formula (20.64), which can be written — using (20.63) — in the form

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{J}{\left(k^2-K^2+i\epsilon'\right)^r}\overset{(20.64)}{=}i(-1)^r\int_0^{+\infty}\frac{\mathrm{d}R}{(2\pi)^4}\,\frac{2\pi^2R^3J_E}{\left(R^2+K^2\right)^r}$$

$$\text{with}\begin{cases}1\le r\in\mathbb{N}\quad,\quad 0\le K^2\in\mathbb{R}\\ J= \quad 1 \quad k^\mu \quad k^2 \quad k^\mu k^\nu \quad (k^2)^2\\ J_E= \quad 1 \quad 0 \quad -R^2 \quad -g^{\mu\nu}R^2/4 \quad R^4\\ r= \quad \ge 1 \quad — \quad \ge 2 \quad \begin{cases}\ge 2 \text{ if } \mu=0\\ \ge 0 \text{ if } \mu\ne 0\end{cases} \quad \ge 3\end{cases} \tag{25.18}$$

On the left side, k is a four-vector in Minkowski-metric, while on the right side R is the radial component of four-dimensional euclidean spherical coordinates. The factor i is resulting from the transformation of the component k^0 in Minkowski-metric into the component ik^0_{E} in euclidean metric. In chapter 26 we will find out, that the diverging loop diagrams of QED result into integrals of the same form. If the loop-index d, which has been defined in (25.1), is larger than zero, then these integrals will diverge.

To cure the divergences, one proceeds in two steps:

$$\underbrace{\int \mathrm{d}x\, A(x)\,B(x)}_{\pm\infty}=\lim_{Y\to\infty}\underbrace{\int \mathrm{d}x\, C(x)\,D(x,Y)}_{\text{finite for } Y<\infty}=$$

$$=\underbrace{\lim_{Y\to\infty}\int \mathrm{d}x\, C(x)\,D_{\text{measured}}(x)}_{\text{finite}} \tag{25.19a}$$

In the first step, called *regularization*, an appropriate parameter D is written as limit $\lim_{Y\to\infty} D(Y)$ of an expression $D(Y)$, which is finite for finite Y.

The second step is the *renormalization* of D, i. e. the replacement of $D(Y)$ by the experimentally determined value D_{measured}.

As an alternative to renormalization, in many computations the existence of a fundamental length according to (21.29) is assumed. Then after regularization, $Y \to \infty$ is replaced by $Y_{\max}$:

$$\underbrace{\int \mathrm{d}x\, A(x)\, B(x)}_{\pm\infty} = \lim_{Y\to\infty} \underbrace{\int \mathrm{d}x\, C(x)\, D(x,Y)}_{\text{finite for } Y<\infty} = \\ = \underbrace{\int \mathrm{d}x\, C(x)\, D(x,Y_{\max})}_{\text{finite}} \tag{25.19b}$$

Whether (25.19a) or (25.19b) is applied, in either case regularization is required as a first step, to get reasonable results. In the following sections we will discuss the most important methods which have been invented for the regularization of diverging loop-integrals.

25.2 Cut-off Regularization

We already regularized diverging integrals by this method in the previous chapters. Cut-off regularizaton does mean, that the integration is cut off at some maximum wavenumber Λ:

$$\int_0^{+\infty} \mathrm{d}R\, \frac{R^3 J_E}{\left(R^2+K^2\right)^r} = \lim_{\Lambda\to\infty} \int_0^{\Lambda} \mathrm{d}R\, \frac{R^3 J_E}{\left(R^2+K^2\right)^r} \tag{25.20}$$

If we assume the existence of a fundamental length according to (21.29), then we must consequently introduce the cut-off wavenumber $\Lambda_{\max} =$ (21.29b). In that case we will not take the limit $\Lambda \to \infty$.

Cut-off regularization is not suitable for quantum electrodynamics, because it does damage the gauge-invariance of the theory. Loss of gauge-invariance would result into a huge amount of new problems. Therefore alternative methods for the regularization of diverging integrals have been invented. In the sequel we will occupy ourselves with the two most important

of them.

25.3 Pauli-Villars Regularization

The regularization method developed by Pauli[1] and Villars[2], which is based on the introduction of counter-terms, is preserving both the gauge-invariance and the Lorentz-invariance of the theory.

Using the step-function

$$\Theta(R-\Lambda_a)=\begin{cases}1 & \text{if } R>\Lambda_a\\ \frac{1}{2} & \text{if } R=\Lambda_a\\ 0 & \text{if } R<\Lambda_a\end{cases},$$

the cut-off regularization may be written in the form

$$\int_0^{+\infty} \mathrm{d}R\,\frac{R^3 J_E}{(R^2+K^2)^r} \overset{(25.20)}{=} \lim_{\Lambda_a\to\infty}\int_0^{\Lambda_a}\mathrm{d}R\,\frac{R^3 J_E}{(R^2+K^2)^r} =$$

$$= \lim_{\Lambda_a\to\infty}\int_0^{\infty}\mathrm{d}R\,R^3 J_E\left(\frac{1}{(R^2+K^2)^r}-\frac{\Theta(R-\Lambda_a)}{(R^2+K^2)^r}\right). \tag{25.21}$$

Now we replace the hard step by the soft transition

$$\int_0^{\infty}\mathrm{d}R\,R^3 J_E\,\frac{\Theta(R-\Lambda_a)}{(R^2+K^2)^r}=\int_0^{\infty}\mathrm{d}R\,R^3 J_E\,\frac{1}{(R^2+\Lambda^2)^r}\,. \tag{25.22}$$

For any $0<\Lambda_a<\infty$ a $0<\Lambda<\infty$ can be found, which is a solution of this equation. Thereby we get instead of (25.21):

[1] Wolfgang Ernst Pauli (1900 – 1958)
[2] Felix Villars (1921 – 2002)

$$\int_0^{+\infty} \mathrm{d}R\, \frac{R^3 J_E}{(R^2+K^2)^r} = \lim_{\Lambda\to\infty} \int_0^{\infty} \mathrm{d}R\, R^3 J_E \left(\frac{1}{(R^2+K^2)^r} - \frac{1}{(R^2+\Lambda^2)^r} \right)$$

This is the most simple form of Pauli-Villars regularization. It can be made more flexible, if instead of only one term an arbitrary number N of counter-terms is added to the integrand:

$$\int_0^{+\infty} \mathrm{d}R\, \frac{R^3 J_E}{(R^2+K^2)^r} = \lim_{\Lambda_j\to\infty} \int_0^{+\infty} \mathrm{d}R \left(\frac{R^3 J_E}{(R^2+K^2)^r} + \sum_{j=1}^{N} C_j \frac{R^3 J_E}{(R^2+\Lambda_j^2)^r} \right) \tag{25.23a}$$

The integral will become convergent due to the added counter-terms, if the constants C_j are chosen such that

$$\sum_{j=1}^{N} C_j = -1\ . \tag{25.23b}$$

Note that the Pauli-Villars method is applicable only if the counter-terms are zero for $\Lambda_j \to \infty$ at finite R, i. e. Λ must be in the denominator but not in the numerator.

As the limits $\Lambda_j \to \infty$ are taken only after the integration (if they are taken at all), all K^2 and all Λ_j^2 are negligible at $R \to \infty$. Consequently the integral (25.23a) is convergent.

25.4 Dimensional Regularization

This method, which has been developed[3] by 't Hooft[4] and Veltman[5], does preserve both the gauge-invariance and the Lorentz-invariance of the theory. Most important, it can be used — different from the Pauli-Villars method

[3] Until early November 2012, this has been the generally accepted point of view regarding the invention of dimensional regularization. A differing, and possibly more correct point of view [54] in contrast is stating, that this method has been developed and published already one year earlier by Carlos Guido Bollini (1926 - 2009) and Juan José Giambiagi (1924 - 1996).

[4] Gerardus (Gerard) 't Hooft (∗ 1946)

[5] Martinus Justinus Godefriedus Veltman (∗ 1931)

— as well in case of the non-Abelian gauge theories of the weak and strong interactions.

The basic idea of dimensional regularization of a diverging integral is to consider it as an analytic function of it's space-time dimension. For that purpose, the integer dimension $n = 1, 2, 3, 4, \ldots$ is replaced by

$$D \equiv n - \epsilon \quad , \quad \epsilon \in \mathbb{C} \; , \tag{25.24}$$

with an arbitrary complex ϵ. With the real part of D sufficiently small, almost any integral will become convergent. For example in case of the generic four-dimensional loop-integral

$$(25.18) = i(-1)^r \int_0^{+\infty} \frac{\mathrm{d}R}{(2\pi)^4} \, \frac{2\pi^2 R^3 J_E}{\left(R^2 + K^2\right)^r} \tag{25.25a}$$

we get

$$(25.18) = \lim_{D \to 4} i(-1)^r \int_0^{\infty} \frac{\mathrm{d}R}{(2\pi)^D} \, \frac{S_D J_E}{\left(R^2 + K^2\right)^r} \quad . \tag{25.25b}$$

Note that the integral over the radial wavenumber R is a simple, one-dimensional integral. All problems with D-dimensional integration have been absorbed into the factor S_D, which is representing the surface of a D-dimensional Euclidean sphere with radius R, and is replacing the surface $S_4 \overset{(20.62)}{=} 2\pi^2 R^3$ of the four-dimensional sphere in (25.25a). How can we find S_D for non-integer dimensional D?

From the recursion formulas (20.62) we can derive the following formula for the surface S_n of an n-dimensional Euclidean sphere with radius R:

$$2 \leq n \in \mathbb{N} :$$

$$S_n = \left(\frac{2\pi}{n}\right)^{(n-2)/2} \pi n R^{n-1} \quad \text{if } n \text{ is even} \tag{25.26a}$$

$$S_n = \left(\frac{2\pi}{n}\right)^{(n-3)/2} \frac{4\pi}{3} \, n R^{n-1} \quad \text{if } n \text{ is odd} \tag{25.26b}$$

The volume of a one-dimensional sphere with radius R is the length $V_1 = 2R$, and consequently

$$S_1 = \frac{\mathrm{d}V_1}{\mathrm{d}R} = 2 \tag{25.26c}$$

is it's surface. Now we employ the gamma-function

$$\Gamma(z) \equiv \lim_{\nu\to\infty} \frac{\nu!\,\nu^z}{z(z+1)(z+2)\dots(z+\nu)} \quad , \quad \nu \in \mathbb{N}\ ,\ z \in \mathbb{C}\ , \tag{25.27}$$

which is defined for arbitrary $z \in \mathbb{C}$, except for the poles at $z = 0, -1, -2, \dots$. A graph of the gamma-function with real arguments is displayed in fig. 25.1 on page 576. The basic property of the gamma-function is

$$\Gamma(z+1) = z\,\Gamma(z) \quad , \quad z \in \mathbb{C}\ . \tag{25.28}$$

Of particular importance for our purpose is the gamma-function with integer and half-integer real arguments:

$$\Gamma(n) = (n-1)! \quad , \quad 1 \le n \in \mathbb{N} \tag{25.29a}$$

$$\Gamma(n+1/2) = \frac{(2n)!}{n!\,4^n}\,\sqrt{\pi} \quad , \quad 0 \le n \in \mathbb{N} \tag{25.29b}$$

$$\Gamma(-n+1/2) = \frac{n!\,(-4)^n}{(2n)!}\,\sqrt{\pi} \quad , \quad 1 \le n \in \mathbb{N} \tag{25.29c}$$

By explicit computation, it's easy to check that

$$S_n \overset{(25.26)}{=} \frac{2\pi^{n/2}}{\Gamma(n/2)} R^{n-1} \quad \text{for } n = 1, 2, 3, 4, 5\ . \tag{25.30}$$

Note that this relation does not hold for $n \ge 6$. Thats no grave restriction, as we almost exclusively need to solve integrals in dimensions $n = 1, 2, 3, 4$. Thus the

$$\text{Definition: } S_D \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)}\, R^{D-1} \equiv \text{surface of a } D\text{-dimensional sphere with radius } R \quad , \quad 0.5 < \mathrm{Re}(D) < 5.5 \tag{25.31}$$

certainly is reasonable and will lead to consistent results, as in any case we need eventually to consider the limit $\lim_{D\to n}$ with $n = 1, 2, 3, 4, 5$. With this surface factor, our standard loop-integral gets this form:

$$(25.25) = \lim_{D\to 4} \frac{i(-1)^r}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \int_0^\infty \mathrm{d}R\, \frac{R^{D-1} J_E}{\left(R^2+K^2\right)^r} \tag{25.32}$$

We evaluate the most simple integral with $J_E \equiv 1$:

$$I_D \equiv \int_0^\infty \mathrm{d}R\, \frac{R^{D-1}}{\left(R^2+K^2\right)^r} \overset{(20.63)}{=} \frac{1}{2}\int_0^\infty \mathrm{d}R^2\, \frac{(R^2)^{D/2-1}}{\left(R^2+K^2\right)^r} \tag{25.33}$$

We assume

$$K^2 > 0 \tag{25.34a}$$

and substitute

$$X \equiv \frac{K^2}{R^2+K^2} \quad , \quad \mathrm{d}R^2 = -\frac{K^2}{X^2}\,\mathrm{d}X \quad , \quad R^2 = K^2\,\frac{1-X}{X}\,. \tag{25.34b}$$

Thereby (25.33) becomes

$$I_D = -\frac{1}{2(K^2)^{r-D/2}} \int_1^0 \mathrm{d}X\, X^{r-D/2-1}(1-X)^{D/2-1}\,. \tag{25.35}$$

Using the beta-function

$$\mathrm{Beta}(\alpha,\beta) \equiv \int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,, \tag{25.36}$$

we get

$$I_D = \int_0^\infty \mathrm{d}R \, \frac{R^{D-1}}{\left(R^2+K^2\right)^r} = + \frac{\Gamma(r-D/2)\,\Gamma(D/2)}{2(K^2)^{r-D/2}\,\Gamma(r)} \ ,$$

and by insertion into (25.32)

$$\lim_{D\to 4} \int_{-\infty}^{+\infty} \frac{\mathrm{d}^D k}{(2\pi)^D} \, \frac{1}{\left(k^2-K^2+i\epsilon'\right)^r} = \lim_{D\to 4} \frac{i(-1)^r\,\Gamma(r-D/2)}{2^D \pi^{D/2}\,\Gamma(r)\,(K^2)^{r-D/2}} \ . \quad (25.37)$$

We won't compute the integrals with $J \neq 1$ by ourselves, but take them from the literature[6]:

$$\frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \, \frac{J}{\left(k^2-K^2+i\epsilon'\right)^r} \overset{(25.18)}{=} i(-1)^r \int_0^{+\infty} \frac{\mathrm{d}R}{(2\pi)^4} \, \frac{2\pi^2 R^3 J_E}{\left(R^2+K^2\right)^r} = \quad (25.38a)$$

$$= \lim_{D\to 4} \frac{i(-1)^\eta}{(4\pi)^{D/2}} \, \frac{J_D\,\Gamma(\eta-\frac{D}{2})}{\Gamma(r)} \Big(\frac{1}{K^2}\Big)^{\eta-\frac{D}{2}} \quad (25.38b)$$

with

$$\begin{cases} 1 \le r \in \mathbb{N} \quad , \quad 0 \le K^2 \in \mathbb{R} \\ \begin{array}{lccccc} J = & 1 & k^\mu & k^2 & k^\mu k^\nu & (k^2)^2 \\ J_E = & 1 & 0 & -R^2 & -g^{\mu\nu}R^2/4 & R^4 \\ r = & \ge 1 & — & \ge 2 & \begin{cases} \ge 2 \text{ if } \mu = 0 \\ \ge 0 \text{ if } \mu \neq 0 \end{cases} & \ge 3 \\ J_D = & 1 & — & D/2 & g^{\mu\nu}/2 & D(D+2)/4 \\ \eta = & r & — & r-1 & r-1 & r-1 \end{array} \end{cases}$$

These functions have poles at $(\eta - 2) = 0, -1, -2, \ldots$ due to the factor $\Gamma(\eta - D/2) \overset{(25.24)}{=} \Gamma(\eta - 2 + \epsilon/2)$. See the graph of the gamma-function on page 576. By means of the expansion

[6] These and many many further useful formulas related to dimensional regularization can be found in the textbook of Peskin and Schroeder [3, section 7.5 and appendix A.4].

$$\Gamma(z) = \Big[z \cdot e^{\gamma z} \cdot \prod_{\nu=1}^{\infty} \Big(1 + \frac{z}{\nu}\Big) e^{-z/\nu} \Big]^{-1} \tag{25.39}$$

$$\gamma = \text{Euler-Mascheroni constant} = \lim_{\nu\to\infty} \Big(-\ln \nu + \sum_{j=1}^{\nu} \frac{1}{j} \Big) \approx 0.5772$$

the gamma-function can be approximated nearby the pole $\Gamma(0)$:

$$\begin{aligned} \Gamma(\epsilon/2) &= \Big[\frac{\epsilon}{2} \cdot \Big(1 + \frac{\gamma\epsilon}{2} + \mathcal{O}(\epsilon^2)\Big) \cdot \Big(1 + \mathcal{O}(\epsilon^2)\Big) \Big]^{-1} = \\ &= \frac{2}{\epsilon} \cdot \Big(1 - \frac{\gamma\epsilon}{2} + \mathcal{O}(\epsilon^2)\Big) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon^2) \end{aligned} \tag{25.40a}$$

$\Gamma(\epsilon/2 - n)$ with $n = 1, 2, 3, \ldots$ can by means of the basic property (25.28) of the gamma-function

$$\begin{aligned} \Gamma(\epsilon/2) &= (\epsilon/2 - 1)\,\Gamma(\epsilon/2 - 1) = (\epsilon/2 - 1)(\epsilon/2 - 2)\,\Gamma(\epsilon/2 - 2) = \\ &= (\epsilon/2 - 1)(\epsilon/2 - 2)(\epsilon/2 - 3)\,\Gamma(\epsilon/2 - 3) = \ldots \\ \Gamma(\epsilon/2 - n) &= \Gamma(\epsilon/2) \cdot \prod_{\nu=1}^{n} \frac{1}{\epsilon/2 - \nu} \quad , \quad n = 1, 2, 3, \ldots \end{aligned} \tag{25.40b}$$

be reduced to $\Gamma(\epsilon/2)$. Note that

$$\lim_{\epsilon\to 0} \Gamma(\epsilon/2) = \begin{cases} +\infty & \text{if } \mathrm{Re}(\epsilon) > 0 \\ -\infty & \text{if } \mathrm{Re}(\epsilon) < 0 \\ \text{undefined} & \text{if } \mathrm{Re}(\epsilon) = 0 \,, \end{cases} \tag{25.41}$$

see the graph of the gamma-function in fig. 25.1 on the next page. Furthermore the unphysical minus-sign for the limit with $\mathrm{Re}(\epsilon) < 0$, and the unphysical minus-sign resulting from application of (25.40b) with odd n must be avoided. These considerations motivate the following rules:

$$\begin{aligned} &* \text{ Choose a path in the complex plane with } \mathrm{Re}(\epsilon) \neq 0 \\ &\quad \text{when taking the limit } \lim_{\epsilon\to 0} \Gamma(\epsilon/2) \,. \end{aligned} \tag{25.42a}$$

$*$ Insert a factor $1 \cdot \operatorname{sign}\Big(\operatorname{Re}(\epsilon)\Big)$

when taking the limit $\lim_{\epsilon\to 0}\Gamma(\epsilon/2)$. (25.42b)

$*$ Insert a factor $(-1)^n$ when applying (25.40b) . (25.42c)

In the sequel we will continue to evaluate the regularized integral (25.38b) for simplicity with $J_D = 1$ and hence $\eta = r$. In view of (25.42) we decide for $0 < \epsilon \in \mathbb{R}$. Furthermore we insert (25.24):

$$(25.38b) = \lim_{\epsilon\to 0} \frac{i(-1)^r}{(4\pi)^{2-\epsilon/2}} \frac{\Gamma(r-2+\epsilon/2)}{\Gamma(r)} \Big(\frac{1}{K^2}\Big)^{r-2+\epsilon/2} \tag{25.43}$$

With $r \geq 3$ there is no divergence, and there would be no reason to apply any regularization in the first place. Therefore we will consider only the cases $r = 1$ and $r = 2$, starting with $r = 1$.

With $\Gamma(1) = 1$ and

$$\Gamma(\epsilon/2 - 1) \overset{(25.40b)}{=} -\frac{2}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon) \tag{25.44}$$

and the factor (-1) according to (25.42c) we get

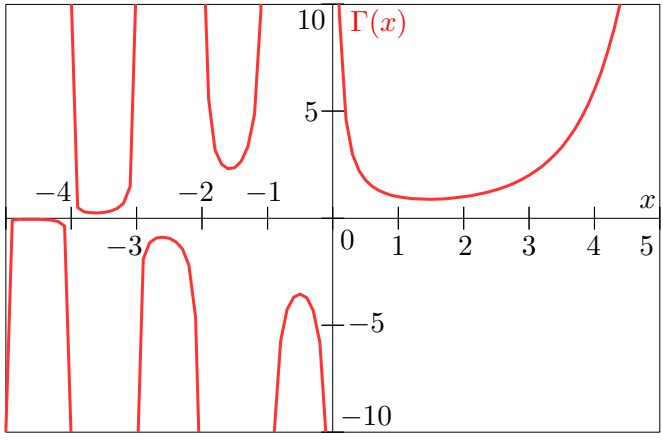


Fig. 25.1 : The gamma-function with $x \in \mathbb{R}$

$$(25.38b) \overset{r=1}{=} -i \lim_{\epsilon\to 0} \Big(\frac{K}{4\pi}\Big)^2 \Big(\frac{2}{\epsilon} - \gamma + 1\Big)\Big(\frac{4\pi}{K^2}\Big)^{\epsilon/2} + \mathcal{O}(\epsilon) \ . \tag{25.45}$$

Now we make use of

$$a^\epsilon = \exp\{\epsilon \ln a\} = \sum_{n=0}^{\infty} \frac{1}{n!}(\epsilon \ln a)^n = 1 + \epsilon \ln a + \mathcal{O}(\epsilon^2) \tag{25.46}$$

and define an undetermined constant κ with dimension $[\kappa]$ = wavenumber, to get the logarithm dimension-less:

$$\begin{aligned}(25.38b) &\overset{r=1}{=} -i \lim_{\epsilon\to 0} \Big(\frac{K}{4\pi}\Big)^2 \Big(\frac{1}{\kappa^2}\Big)^{\epsilon/2} \Big(\frac{2}{\epsilon} - \gamma + 1\Big)\Big[1 + \frac{\epsilon}{2} \ln\Big(\frac{4\pi\kappa^2}{K^2}\Big)\Big] \\ &= -i \lim_{\epsilon\to 0} \Big(\frac{K}{4\pi}\Big)^2 \Big[\frac{2}{\epsilon} - \gamma + 1 + \ln\Big(\frac{4\pi\kappa^2}{K^2}\Big)\Big] \end{aligned} \tag{25.47}$$

We have skipped — considering $\lim_{\epsilon\to 0}$ — all terms $\mathcal{O}(\epsilon)$ in the last line.

Now we turn to the case $r = 2$. With $\Gamma(\epsilon/2) \overset{(25.40a)}{=} 2/\epsilon - \gamma$ and $\Gamma(2) = 1$ we get

$$\begin{aligned}(25.38b) &\overset{r=2}{=} \lim_{\epsilon\to 0} \frac{i}{(4\pi)^2}\Big(\frac{2}{\epsilon} - \gamma\Big)\Big(\frac{4\pi}{K^2}\Big)^{\epsilon/2} \\ &\overset{(25.46)}{=} \lim_{\epsilon\to 0} \frac{i}{(4\pi)^2}\Big(\frac{1}{\kappa^2}\Big)^{\epsilon/2}\Big(\frac{2}{\epsilon} - \gamma\Big)\Big[1 + \frac{\epsilon}{2} \ln\Big(\frac{4\pi\kappa^2}{K^2}\Big)\Big] \\ &= \lim_{\epsilon\to 0} \frac{i}{(4\pi)^2}\Big[\frac{2}{\epsilon} - \gamma + \ln\Big(\frac{4\pi\kappa^2}{K^2}\Big)\Big] \ . \end{aligned} \tag{25.48}$$

It's convenient, in a measure called

$$\text{MS} \equiv \text{minimal subtraction} \ ,$$

to condense the diverging term $\lim_{\epsilon\to 0} 2/\epsilon$ and the completely undetermined factor κ to a new variable w:

$$\lim_{\epsilon\to 0} \frac{2}{\epsilon} + \ln\Big(\frac{4\pi\kappa^2}{K^2}\Big) \quad \xrightarrow{\text{MS}} \quad \lim_{w\to\infty} \ln\Big(\frac{4\pi w^2}{K^2}\Big) \tag{25.49a}$$

In an alternative method, called

$$\overline{\text{MS}} \equiv \text{modified minimal subtraction} \ ,$$

furthermore γ and 4π are absorbed into w:

$$\lim_{\epsilon\to 0} \frac{2}{\epsilon} - \gamma + \ln\left(\frac{4\pi\kappa^2}{K^2}\right) \quad \xrightarrow{\overline{\text{MS}}} \quad \lim_{w\to\infty} \ln\left(\frac{w^2}{K^2}\right) \tag{25.49b}$$

Due to $\overline{\text{MS}}$ the regularized integral becomes

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{\left(k^2 - K^2 + i\epsilon'\right)^r} \begin{cases} \overset{r=1}{=} -i\left(\frac{K}{4\pi}\right)^2 \left[1 + \lim_{w\to\infty} \ln\left(\frac{w^2}{K^2}\right)\right] & \text{(25.50a)} \\ \overset{r=2}{=} +\lim_{w\to\infty} \frac{i}{(4\pi)^2} \ln\left(\frac{w^2}{K^2}\right) . & \text{(25.50b)} \end{cases}$$

Whatever may be the powers of k in (25.38), due to (25.40b) the divergence with dimensional regularization always is reduced to

$$\lim_{\epsilon\to 0} 2/\epsilon \overset{(25.49)}{\sim} \lim_{w\to\infty} \ln(w^2/K^2) .$$

With cut-off regularization and with Pauli-Villars regularization we considered a divergence $\sim \lim_{\Lambda\to\infty} \ln(\Lambda/K)$ less severe than a divergence $\sim \lim_{\Lambda\to\infty} \Lambda/K$, and this again less severe than a divergence $\sim \lim_{\Lambda\to\infty} \Lambda^2/K^2$. Of course all these expressions are diverging, but we had in mind that there might exist some (yet unknown) very small fundamental length, and hence some maximum wavenumber Λ_{max}, replacing the diverging limits by $(\Lambda_{\text{max}}/K)^2 \gg (\Lambda_{\text{max}}/K) \gg \ln(\Lambda_{\text{max}}/K)$. No such physical argument is known to stop the limit $\epsilon \to 0$ resp. $w \to \infty$ at some finite $\epsilon_{\text{min}} > 0$ resp. $w_{\text{max}} < \infty$. Diverging integrals are still strictly infinite after dimensional regularization, and the regularization does nothing else than to prepare them appropriately for renormalization of some parameter. Thus with dimensional regularization it would be meaningless to discuss the severity of divergences.

To eliminate the divergence from the theory, in a following step one of the parameters coupling-constant, or mass, or field-amplitude must be renormalized such that it absorbs the diverging parameter $\lim_{w\to\infty} \ln(w^2/K^2)$.

26 Renormalization of QED

In table 25.1 on page 566 the five loop-diagrams are listed, which possibly might diverge. In this chapter, we will compute these five diagrams, and treat them — if needed — by renormalization.

26.1 Vacuum-Polarization

At the beginning of section 20.3 we explained, why the LSZ-formula can not be applied for the computation of propagator-corrections. The same clearly holds for the propagator-corrections of QED. Instead we must revert to the generic formula for matrix elements

$$\begin{aligned}\langle 0|\, T\psi_{(W)}(x_r)\dots\overline{\psi}_{(W)}(x_1)\,|0\rangle &\overset{(23.11)}{=} \\ &= \frac{\langle 0|\, T\psi(x_r)\dots\overline{\psi}(x_1)\sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^n|0\rangle}{\langle 0|\, T\sum_{m=0}^{\infty}\frac{1}{m!}\Big(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}\tau\, H(\tau)\Big)^m|0\rangle}\,. \end{aligned} \tag{26.1}$$

As stipulated in (19.21), the operators with index $_{(W)}$ are the complete operators of the theory with interactions. Operators without this index are operators in the interaction-picture. The vacuum bubbles showing up in the numerator, as usually will be canceled versus the denominator. Thus only the numerators of connected diagrams need to be computed. As we do not want to change our notation

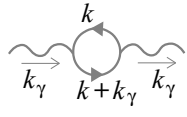
$$S^{(0)}(x_2-x_1)\equiv S(x_2-x_1)\overset{(16.31)}{=}\langle 0|\, T\psi(x_2)\psi(x_1)\,|0\rangle$$

$$D^{(0)}_{\mu\nu}(x_2-x_1) \equiv D_{\mu\nu}(x_2-x_1) \overset{(17.85)}{=} \langle 0|\, T A_\mu(x_2) A_\nu(x_1)\, |0\rangle$$

for Feynman-propagators, we use for propagators with self-interaction the notation

$$\begin{aligned} S^{(W)}(x_2-x_1) &= \sum_{n=0}^{\infty} S^{(n)}(x_2-x_1) \;= \\ &= \langle 0|\, T \psi_{(W)}(x_1)\, \psi_{(W)}(x_2)\, |0\rangle \\ D^{(W)}_{\mu\nu}(x_2-x_1) &= \sum_{n=0}^{\infty} D^{(n)}_{\mu\nu}(x_2-x_1) \;= \\ &= \langle 0|\, T A_{(W)\mu}(x_1)\, A_{(W)\nu}(x_2)\, |0\rangle \;. \end{aligned}$$

From comparison of boxes 20.4 and 20.2 of ψ^s-theory it is obvious, that the Feynman-rules for the computation of propagator-corrections merely differ by some modifications from the rules for the computation of S-matrices. To have the rules for the computation of propagator-corrections of QED conveniently at hand, the accordingly modified rules have been extracted from box 24.1 and compiled in box 26.1 on the next page. According to these rules, the first graph in (25.1)

$$\widetilde{D}^{(2)}_{\mu\nu}(k_\gamma) \mathrel{\widehat{=}} 2 \cdot$$ 

$$, \qquad (26.2)$$

which often is called self-energy of the photon, gets a negative sign and the symmetry-factor 2, because according to rule K both inner fermion lines must be interpreted as particles (not antiparticles):

$$\langle 0|\, T A(x_1)\, A(x_2)\, \overline{\psi}(y)\, \gamma^\mu A_\mu(y)\, \psi(y)\, \overline{\psi}(z)\, \gamma^\nu A_\nu(z)\, \psi(z)\, |0\rangle \qquad (26.3a)$$

Consequently the operator-pair $\overline{\psi}(y)\, \psi(z)$ must be reversed, to shift the creation operator to the right of the annihilation operator. This results into a factor (-1). We already have pointed out at the end of section 23.1.4, that this negative sign does always show up for pure fermion loops, no matter

Box 26.1: **Feynman-rules in energy-momentum space** for the computation of propagator corrections of n^{th} order in quantum electrodynamics

A The propagator correction equals the sum of all connected graphs with the structure

$$\text{propagator}^{(n)} = \text{propagator}^{(0)} \cdot F^{(n)} \cdot \text{propagator}^{(0)} \ .$$

B The symmetry factor is equal to the number of alternatives for the pairwise combination of the operators, which are building up the graph, to the graph's propagators. Only one of the equivalent graphs is inserted into $F^{(n)}$, and multiplied by the symmetry factor.

B′ When combining the operators to propagators, any permutation of two fermion operators gives a factor (-1).

C The structure of any vertex is  .

D Include for the n vertices a factor

$$\frac{1}{n!}\left(\frac{-iq\gamma^{\nu}}{\hbar}\right)^{n} \ .$$

G Include for each inner photon line with wavenumber k a factor

$$\widetilde{D}_{\nu\mu}(k) \overset{(17.86)}{=} \frac{-ig_{\nu\mu}\,\mu_0\hbar c}{k^2 + i\epsilon'}$$

and for each inner fermion line with wavenumber k a factor

$$\widetilde{S}(k) \overset{(12.24)}{=} \frac{i(\gamma^{\nu}k_{\nu} + m\frac{c}{\hbar})}{k^2 - m^2\frac{c^2}{\hbar^2} + i\epsilon'} \ .$$

H Take the sum and integral over the wavenumber k of an inner line due to

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \ ,$$

unless k is fixed by conservation of energy and momentum.

J Apply the rearrangement-operator $\mathcal{U}_{\text{S}} = (23.21)$.

K Inner fermion lines must *always* be interpreted as particles (not as antiparticles).

how many lines are contained in the loop.

As in all diagrams of QED (and of Yukawa-theory) the symmetry factor $n!$ (in this case 2!) results from the permutation of vertices, which allows for this second possible contraction:

$$\langle 0|\, T A(x_1)\, A(x_2)\, \overline{\psi}(y)\,\gamma^\mu A_\mu(y)\, \psi(y)\,\overline{\psi}(z)\,\gamma^\nu A_\nu(z)\,\psi(z)\,|0\rangle \tag{26.3b}$$

Thus the correction of the photon propagator becomes in second order perturbation computation:

$$\begin{aligned}
\widetilde{D}^{(2)}_{\mu\nu}(k_\gamma) &= \widetilde{D}_{\mu\alpha}(k_\gamma)\cdot F_{\mathrm{P}}^{\alpha\beta}\cdot \widetilde{D}_{\beta\nu}(k_\gamma) \\
F_{\mathrm{P}}^{\alpha\beta} &= -2\cdot\mathcal{U}_{\mathrm{S}}\,\frac{1}{2!}\,\frac{(-iq\gamma^\alpha)}{\hbar}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi} \\
&\quad\cdot\frac{i\Big(\gamma^\sigma(k_\gamma+k)_\sigma + m\tfrac{c}{\hbar}\Big)}{\Big((k_\gamma+k)^2 - m^2\tfrac{c^2}{\hbar^2}+i\epsilon'\Big)}\,\frac{(-iq\gamma^\beta)}{\hbar}\,\frac{i\Big(\gamma^\tau k_\tau + m\tfrac{c}{\hbar}\Big)}{\Big(k^2 - m^2\tfrac{c^2}{\hbar^2}+i\epsilon'\Big)} \\
&= -\frac{q^2}{\hbar^2}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{\mathrm{tr}\Big\{\gamma^\alpha\Big(\gamma^\sigma(k_\gamma+k)_\sigma + m\tfrac{c}{\hbar}\Big)\gamma^\beta\Big(\gamma^\tau k_\tau + m\tfrac{c}{\hbar}\Big)\Big\}}{\Big((k_\gamma+k)^2 - m^2\tfrac{c^2}{\hbar^2}+i\epsilon'\Big)\Big(k^2 - m^2\tfrac{c^2}{\hbar^2}+i\epsilon'\Big)}
\end{aligned} \tag{26.4}$$

In the numerator of the last line, the spinor factors have been linked up to the trace according to the definition (23.21) of the rearrangement-operator $\mathcal{U}_{\mathrm{S}}$. Counting the powers of k, we find the loop-index

$$k^d \overset{(25.1)}{\sim} \mathrm{d}^4k\,\frac{k^2}{k^4} \sim k^2 \implies d = 2\ .$$

Consequently a quadratic divergence of this graph is to be expected. Closer examination will show, however, that the divergence actually is "merely" logarithmic. To preserve the gauge-invariance of QED, we will not apply the cut-off method for this integral's regularization, but the method of Pauli and Villars, which has been described in section 25.3. This means that we

add N additional terms to the integrand, which contain constants M_j and C_j:

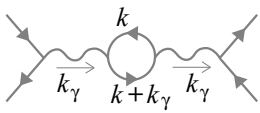
$$F_{\mathrm{P}}^{\alpha\beta} = -\lim_{M_j\to\infty} \frac{q^2}{\hbar^2}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\Bigg(\frac{\mathrm{tr}\Big\{\gamma^\alpha\Big(\gamma^\sigma(k_\gamma+k)_\sigma + mc/\hbar\Big)\gamma^\beta\Big(\gamma^\tau k_\tau + mc/\hbar\Big)\Big\}}{\Big((k_\gamma+k)^2-(mc/\hbar)^2+i\epsilon'\Big)\Big(k^2-(mc/\hbar)^2+i\epsilon'\Big)} + \\ + \sum_{j=1}^{N} C_j\,\frac{\mathrm{tr}\Big\{\gamma^\alpha\Big(\gamma^\sigma(k_\gamma+k)_\sigma + M_jc/\hbar\Big)\gamma^\beta\Big(\gamma^\tau k_\tau + M_jc/\hbar\Big)\Big\}}{\Big((k_\gamma+k)^2-(M_jc/\hbar)^2+i\epsilon'\Big)\Big(k^2-(M_jc/\hbar)^2+i\epsilon'\Big)}\Bigg) \tag{26.5}$$

For $|k|\to\infty$, all masses m and M_j are negligibly small. If the constants are chosen such that

$$\sum_{j=1}^{N} C_j = -1\ , \tag{26.6}$$

then the integral's divergence is suppressed due to the added counter-terms.

When computing (26.5), we often will encounter the incoming photon's invariant wavenumber-square k_γ^2. For an observed photon, $k_\gamma^2 = 0$. Consequently we could skip all terms, in which the factor k_γ^2 is contained. We will not do that, however, because we want to do the computation of vacuum polarization from the outset such, that it covers as well the case that the propagator correction is an internal part within a larger diagram. For example, in fourth order perturbation computation of the scattering of a fermion by it's antiparticle one encounters besides others this graph:



(26.7)

We know from the evaluations in section 24.3.7, that the invariant wavenumber-square of the virtual photon in this graph is $k_\gamma^2 < 0$ (t-channel scattering) or $k_\gamma^2 > 4m^2c^2/\hbar^2$ (s-channel scattering, with m being the incoming fermion's

mass). Furthermore we will allow for $0 < k_\gamma^2 \leq 4m^2c^2/\hbar^2$, which is possible only if the outer fermion lines of the graph (26.7) actually are inner lines of an even larger diagram.

The computation of $F_{\mathrm{P}}^{\alpha\beta} =$ (26.5) is demonstrated in A.24 in all details. Thereby the following facts turn up:

* Different from our first expectation (based on the loop index $d = 2$), $F_{\mathrm{P}}^{\alpha\beta}$ would diverge without regularization due to the counterterms "merely" logarithmically, but not quadratically.
* One single counterterm is sufficient to regularize the integral (26.5) by means of the Pauli-Villars method. Consequently we fix

$$N = 1 \qquad C_1 = -1 \qquad M \equiv M_1 \ . \tag{26.8}$$

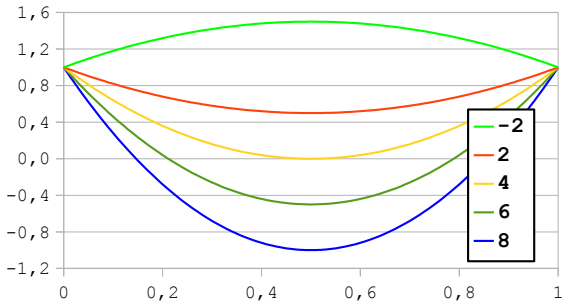
The computations in appendix A.24 give the following result:

$$F_{\mathrm{P}}^{\alpha\beta} \overset{\text{(A.182)}}{=} (k_\gamma^\alpha k_\gamma^\beta - k_\gamma^2 g^{\alpha\beta}) \cdot \Pi \tag{26.9a}$$

$$\Pi \equiv \lim_{M\to\infty} \int_0^1 \mathrm{d}\xi \, \frac{-q^2(\xi - \xi^2)}{2\pi^2\hbar^2} \cdot V \cdot \cdot \left(-\ln\left(1 - \frac{(\xi - \xi^2)k_\gamma^2\hbar^2}{m^2c^2}\right) + \ln\left(\frac{M^2}{m^2}\right)\right) \tag{26.9b}$$

$$V \equiv \begin{cases} 1 & \text{if } (\xi - \xi^2)k_\gamma^2 < m^2c^2/\hbar^2 \\ 0 & \text{if } (\xi - \xi^2)k_\gamma^2 \geq m^2c^2/\hbar^2 \end{cases} \tag{26.9c}$$

$F_{\mathrm{P}}^{\alpha\beta}$ is called polarization tensor, Π is called polarization function. m is the mass of the fermions constituting the bubble (26.2), M is the mass in the Pauli-Villars counterterm. From (26.9b) it is obvious, that the electron — being the lightest electrically charged fermion — gives the dominating contribution to Π. Therefore in the sequel we will consider $m = m_e$ as mass and $q = -e$ as charge of the electron. This is a relatively rough approximation, because in particular the contributions of the light quarks (see table 28.1 on page 642) should be included in a more precise computation.



The parameters in the legend indicate the ratio $k_\gamma^2\hbar^2/(mc)^2$.

Fig. 26.2: $1-(\xi-\xi^2)k_\gamma^2\hbar^2/(mc)^2$ as function of ξ

The dimension-less integration parameter ξ assumes values in the intervall $0\ldots1$. The factor $(\xi-\xi^2)$ thus varies inbetween 0 (at $\xi=0$ and $\xi=1$) and 0.25 (at $\xi=0.5$). Consequently the argument of the first logarithm in (26.9b) will become zero or negative at $\xi=0.5$ if $k_\gamma^2 \geq 4m^2c^2/\hbar^2$. This is just the threshold at which the virtual photon's energy becomes sufficient for the creation of two real fermions. This competing alternative sucks probability amplitude off from the graph (26.2). The function $V=$ (26.9c) blanks this part of the integral out, because it does not contribute to $F_{\mathrm{P}}^{\alpha\beta}$. At the same time V will prevent the explosion of the polarization factor Π caused by the mathematically senseless logarithm with negative argument.

The probability amplitude of the diagram (26.2) will never be zero, not even for arbitrary large k_γ^2, because in any case $(\xi-\xi^2)k_\gamma^2$ is smaller than $m^2c^2/\hbar^2$ in the neighborhood of $\xi=0$ and $\xi=1$, or rather $1-(\xi-\xi^2)k_\gamma^2\hbar^2/(mc)^2$ is larger than zero, and therefore the function V assumes the value 1. See[1] figure 26.2. We use the notations η and $1-\eta$ for those values of ξ, at which $1-(\xi-\xi^2)k_\gamma^2\hbar^2/(mc)^2$ is zero, and devide (26.9b) into three sections:

[1] Note that on the axes of fig. 26.2 and fig. 26.3 and fig. 26.4 the comma instead of the point is used as decimal marker.

$$\underbrace{\int_0^1 \mathrm{d}\xi \ldots}_{\Pi} = \underbrace{\int_0^\eta \mathrm{d}\xi \ldots}_{\Pi_a} + \underbrace{\int_\eta^{1-\eta} \mathrm{d}\xi \ldots}_{\Pi_b} + \underbrace{\int_{1-\eta}^1 \mathrm{d}\xi \ldots}_{\Pi_c} \tag{26.10}$$

While Π_a and Π_c are different from zero, Π_b is clamped to zero due to the function V.

To get simple formulas, we restrict our investigation for the rest of this section to the case $k_\gamma^2 < 4m^2c^2/\hbar^2$, for which V always is 1, independent of the value of ξ. We emphasize, however, that everything which will be said in the sequel applies analogously for the case $k_\gamma^2 \geq 4m^2c^2/\hbar^2$, whereby the integral over ξ must be sub-divided according to (26.10), and only the two integrals Π_a and Π_c will contribute to the result.

Due to insertion of the fine-structure constant

$$\alpha_q = \frac{q^2\mu_0 c}{4\pi\hbar} = \frac{q^2}{4\pi\epsilon_0\hbar c} \tag{26.11a}$$

$$\alpha \equiv \alpha_e = \frac{e^2\mu_0 c}{4\pi\hbar} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \tag{26.11b}$$

into the formulas, (26.9) simplifies to

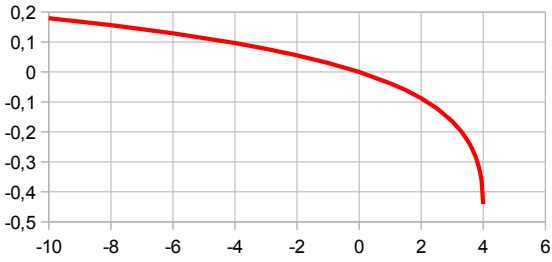
$$F_{\mathrm{P}}^{\alpha\beta} \overset{(26.9)}{=} (k_\gamma^\alpha k_\gamma^\beta - k_\gamma^2 g^{\alpha\beta})\,\Pi \tag{26.12a}$$

$$\Pi = \lim_{M\to\infty} \frac{2\alpha}{\pi\hbar\mu_0 c}\left(I - \frac{1}{6}\ln\left(\frac{M^2}{m^2}\right)\right) \tag{26.12b}$$

$$I \equiv \int_0^1 \mathrm{d}\xi\,(\xi - \xi^2)\,\ln\left(1 - \frac{(\xi-\xi^2)k_\gamma^2\hbar^2}{m^2c^2}\right) \tag{26.12c}$$

$$\text{if } k_\gamma^2 < 4m^2c^2/\hbar^2$$

We spare ourselves the tedious analytical computation of the integral I, and instead let the PC do the work. The result of the numerical integration is displayed[1] in figure 26.3 on the next page. Because of $\lim_{\xi\to 0}\xi\ln(\xi) = 0$ it does not diverge at $k_\gamma^2 = 4m^2c^2/\hbar^2$, but assumes the finite value -0.44. The integral's value raises continuously, is 0 at the invariant wavenumber-square

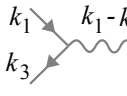


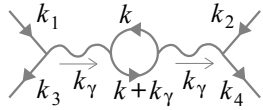
further values:
$I(-1E12) = 4.33$
$I(-1E9) = 3.18$
$I(-1E6) = 2.02$
$I(-1E3) = 0.87$
$I(-1E1) = 0.18$

Fig. 26.3: $I =$ (26.12c) as function of $k_\gamma^2\hbar^2/(mc)^2$

$k_\gamma^2 = 0$ of a free photon, and becomes approximately 4 at the wavenumber-square $k_\gamma^2 = -10^{12}m^2c^2/\hbar^2$ of the space-like virtual photon. Inserting the mass $m \approx 0.5\,\mathrm{MeV}$ of an electron, $k_\gamma^2 = -10^{12}m^2c^2/\hbar^2$ corresponds to the invariant mass $\hbar c|k_\gamma| \approx 500\,\mathrm{GeV}/c^2$ of the virtual photon. I is finite, because k_γ^2 is finite. In all cases of practical relevance it's value is $-0.44 < I < 5$.

Now we are going to consider the modifications of the scattering matrix $\mathcal{M} =$ (24.47b), which we have computed in section 24.3.1 in second order perturbation computation for the t-channel scattering of an electron by a muon, due to vacuum polarization, which here is turning up as an effect of fourth order.

$$\mathcal{M}^{(2)} + \mathcal{M}_a^{(4)} \mathrel{\widehat{=}} 2 \cdot$$ 

$$+\,24 \cdot$$ 

(26.13)

As always in QED, the symmetry factors are equal to the faculty of the number of vertices. That implies at the same time, that they cancel versus the factor $1/n!$ resulting from the Taylor expansion of the interaction term in the matrix element. In (26.4) we inserted $2/2! = 1$, because there we computed the second-order propagator correction. This factor must be replaced by $24/4! = 1$ in the fourth-order graph, i. e. the change of symmetry factors will not affect at all the following formulas. The index a has been inserted, because there are several further corrections versus the second-order t-channel scattering matrix, which we will discuss only later. We compose the scattering matrices from equations (24.47b), (26.4),

and (26.12). As the flavors of the incoming fermions are different, the green colored factors in (24.47b) (which belong to u-channel scattering) must be set to zero. Furthermore $k_\gamma^2 = (k_1 - k_3)^2 < 0$, because the four outer fermion lines are considered to be really outer lines.

$$\begin{aligned}
\mathcal{M}^{(2)} + \mathcal{M}_a^{(4)} &= -\Big(\frac{+ie}{\hbar}\Big)^2 ({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\mu\,{}^{r_1}u^{\boldsymbol{k}_1})\,\frac{(-ig_{\mu\alpha}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\cdot \\
&\quad\cdot\Big(g^\alpha{}_\nu + (-iF_{\mathrm{P}}^{\alpha\beta})\frac{(-ig_{\beta\nu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\Big)({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\nu\,{}^{r_2}u^{\boldsymbol{k}_2}) \\
&= \lim_{M\to\infty}\frac{e^2}{\hbar^2}({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\mu\,{}^{r_1}u^{\boldsymbol{k}_1})\,\frac{(-ig_{\mu\alpha}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\cdot \\
&\quad\cdot\Bigg(g^\alpha{}_\nu - i\Big((k_1-k_3)^\alpha(k_1-k_3)^\beta - (k_1-k_3)^2 g^{\alpha\beta}\Big)\frac{2\alpha}{\pi\hbar\mu_0 c}\cdot \\
&\quad\cdot\Big(I - \frac{1}{6}\,\ln(\frac{M^2}{m^2})\Big)\,\frac{(-ig_{\beta\nu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\Bigg)({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\nu\,{}^{r_2}u^{\boldsymbol{k}_2})
\end{aligned} \tag{26.14}$$

One term can be skipped in this expression, because the spinors are solutions of the free Dirac equation:

$$\begin{aligned}
&{}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\mu\,{}^{r_1}u^{\boldsymbol{k}_1}g_{\mu\alpha}(k_1-k_3)^\alpha = {}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,(\not{k}_1 - \not{k}_3)^{r_1}u^{\boldsymbol{k}_1} = \\
&= \underbrace{{}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,(-\not{k}_3 + mc/\hbar)}_{0} + \underbrace{\not{k}_1 - mc/\hbar)^{r_1}u^{\boldsymbol{k}_1}}_{0} = 0
\end{aligned}$$

Furthermore we can simplify

$$g_{\mu\alpha}(X g^\alpha{}_\nu + Y g^{\alpha\beta}g_{\beta\nu}) = g_{\mu\nu}(X+Y)$$

and cancel one factor $(k_1-k_3)^2$. (One term $i\epsilon'$ thereby becomes dispensable.)

$$\begin{aligned}
\mathcal{M}^{(2)} + \mathcal{M}_a^{(4)} &= \frac{e^2}{\hbar^2}({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^\mu\,{}^{r_1}u^{\boldsymbol{k}_1})\,\frac{(-ig_{\mu\nu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\cdot \\
&\cdot\Big(1 + \frac{2\alpha}{\pi}\Big(I - \frac{1}{6}\lim_{M\to\infty}\ln\Big\{\frac{M^2}{m^2}\Big\}\Big)\Big)({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^\nu\,{}^{r_2}u^{\boldsymbol{k}_2})
\end{aligned} \tag{26.15}$$

Because of $e^2 \sim \alpha$, we may write $e^2 \cdot (\ )$ in the form

$$e^2\Big(\ \Big) = e^2\Big(1 + \frac{2\alpha}{\pi}\, I\Big) \cdot Z_3 + \mathcal{O}(\alpha^3)$$

$$Z_3 \equiv 1 - \frac{\alpha}{3\pi} \lim_{M\to\infty} \ln\Big\{\frac{M^2}{m^2}\Big\} \tag{26.16}$$

We require that all results shall be correct up to order $\mathcal{O}(\alpha^2)$. Therefore the square $\mathcal{O}(\alpha^3)$ could be completed.

The factor Z_3 will be eliminated due to renormalization of the electron's charge. That means: The charge e, which turned up in the above equations, is considered to be the "bare", not observable charge. For the bare charge we use the notation e_0 with an index zero, while e denotes the charge which can be observed in experiments. For the fine-structure constant, we use as well the notation α_0, if it contains the product e_0^2, but the notation α if it contains the product e^2. The following relation between e_0 and e is postulated:

$$e_0^2 \cdot Z_3 = e_0^2\Big(1 - \frac{\alpha_0}{3\pi} \lim_{M\to\infty} \ln\Big\{\frac{M^2}{m^2}\Big\}\Big) \xrightarrow{\text{renormalization}} e^2 \qquad \text{measured at } k_\gamma^2 \to 0 \tag{26.17}$$

The renormalized value of e^2, which shall be inserted here, must be fixed experimentally, for example due to the scattering of electrons by muons. The measurements shall be done with momentum transfer k_γ^2 as small as possible (that is called forward-scattering), to avoid the complicated non-linear correction due to the factor I. For $k_\gamma^2 \approx 0$ according to figure 26.3 $I \approx 0$.

The renormalized scattering matrix for t-channel scattering with arbitrarily high momentum transfer is

$$\mathcal{M}^{(2)} + \mathcal{M}_a^{(4)} = \frac{e^2}{\hbar^2}\Big(1 + \frac{2}{\pi}\,\alpha\, I\Big)\cdot$$
$$\cdot\ ({}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\gamma^{\mu}\,{}^{r_1}u^{\boldsymbol{k}_1})\,\frac{(-ig_{\mu\nu}\,\mu_0\hbar c)}{(k_1-k_3)^2+i\epsilon'}\,({}^{r_4}\bar{u}^{\boldsymbol{k}_4}\,\gamma^{\nu}\,{}^{r_2}u^{\boldsymbol{k}_2}) =$$
$$\overset{(24.47b)}{=} \mathcal{M}^{(2)}_{\text{not renormalized}}\cdot\frac{e^2}{e_0^2}\Big(1+\frac{2}{\pi}\,\alpha\, I\Big)\ . \qquad (26.18)$$

For t-channel scattering the factor I is positive due to $k_\gamma^2 = (k_3 - k_1)^2 < 0$. Therefore the experimentally observed effective charge $e^2(1 + 2\alpha I/\pi)$ increases at increasing momentum transfer. In contrast, it decreases for s-channel scattering at increasing momentum transfer due to $k_\gamma^2 > 0$. Thus there exists no obvious pictorial interpretation of the factor I. The effect of I must not be confused with the "running coupling constant", which will be discussed in (26.21)ff.

For a free (not virtual) photon $(k_\gamma)^2 = 0$ and $I = 0$. Using the not renormalized fine-structure constant α_0 and

$$\frac{\alpha_0}{3\pi}\lim_{M\to\infty}\ln\Big(\frac{M^2}{m^2}\Big) \overset{(26.17)}{=} 1 - Z_3\ ,$$

the propagator correction in second-order perturbation computation becomes

$$\widetilde{D}^{(2)}_{\mu\nu'}(k_\gamma) \overset{(26.4)}{=} \widetilde{D}_{\mu\alpha}(k_\gamma)\cdot F_{\mathrm{P}}^{\alpha\beta}\cdot\widetilde{D}_{\beta\nu}(k_\gamma)$$
$$\overset{(26.14)}{=} \frac{(-ig_{\mu\alpha}\,\mu_0\hbar c)}{(k_\gamma)^2+i\epsilon'}\,\frac{k_\gamma^\alpha k_\gamma^\beta\,\alpha_0}{3\pi\hbar\mu_0 c}\lim_{M\to\infty}\ln\Big(\frac{M^2}{m^2}\Big)\,\frac{(-ig_{\beta\nu}\,\mu_0\hbar c)}{(k_\gamma)^2+i\epsilon'}$$
$$\overset{?}{=} \frac{-g_{\mu\alpha}k_\gamma^\alpha k_\gamma^\beta g_{\beta\nu}\,\mu_0\hbar c(1-Z_3)}{(k_\gamma)^4+i\epsilon'} \qquad \text{preliminary only!}$$

This expression is useless, because Z_3 is unknown. The factor e_0^2 is missing, with which Z_3 would combine to the known product $Z_3 e_0^2 = e^2$. The problem is caused by the concept of a free photon, which is physically questionable. We can get informations on the existence of a photon only because it once has been emitted by a charged fermion, or because it will be absorbed by a fermion. Thus in principle any photon line should be considered as an inner line. From (26.18) one can conclude for the product

($e_0^2 \cdot$ propagator with self-interaction) of an

$$\text{inner photon line:} \qquad e_0^2 \widetilde{D}^{(W)}_{\mu\nu}(k_\gamma) = e_0^2 \widetilde{D}^{(0)}_{\mu\nu}(k_\gamma) + e_0^2 \widetilde{D}^{(2)}_{\mu\nu}(k_\gamma) =$$

$$= \underbrace{\frac{(-ig_{\mu\nu}\,\mu_0\hbar c)}{(k_\gamma)^2 + i\epsilon'} \left(1 + \frac{2}{\pi}\,\alpha_0 I\right) Z_3 e_0^2}_{\text{not renormalized}} = \underbrace{\frac{(-ig_{\mu\nu}\,\mu_0\hbar c)}{(k_\gamma)^2 + i\epsilon'} \left(1 + \frac{2}{\pi}\,\alpha I\right) e^2}_{\text{renormalized}} \tag{26.19}$$

Translated back to the matrix element (24.45) in time-position space this means

$$\langle 0|\, T \ldots A^\mu_{(W)}(y)\, A^\nu_{(W)}(z) \ldots |0\rangle_c =$$
$$= \langle 0|\, T \ldots \sqrt{Z_3} A^\mu(y)\, \sqrt{Z_3} A^\nu(z) \ldots |0\rangle_c\,. \tag{26.20}$$

The field-operator of the gauge field gets an additional normalization factor $\sqrt{Z_3}$ due to the self-interaction. Whenever the field docks to a vertex with charge-factor q_0, the product $q_0\sqrt{Z_3} = q$ is formed. The self-interaction graph (26.2) disappears from the theory (we should better say: it becomes invisible), if firstly the charge parameters q_0 are replaced by experimentally determined charge parameters q, and if secondly for each inner photon line an additional factor $(1 + 2\alpha I\pi)$ with $I =$ (26.12c) is inserted.

Multiplying (26.17) by $1/(4\pi\epsilon_0\hbar c)$, one gets an equation for the coupling constant:

$$\alpha_0 - \frac{\alpha_0^2}{3\pi} \lim_{M\to\infty} \ln\left\{\frac{M^2}{m^2}\right\} = \alpha \tag{26.21}$$

Obviously α_0 must diverge in a well-defined manner, such that the measured coupling constant $\alpha \approx 1/137$ results. Interesting conclusions can be drawn from this equation, if a fundamental length r according to (21.29) is assumed, and therefore the integrals over the wavenumbers must be cut off at $\Lambda \approx \pi/r$. If Pauli-Villars regularization is applied, then this means that for the mass of the counter term the finite value

$$\frac{Mc}{\hbar} = \Lambda = \frac{\pi}{r} \implies M = \frac{\pi\hbar}{rc} \tag{26.22a}$$

must be inserted. Thereby the coupling constant α_0 in (26.21) becomes α_r:

$$\alpha_r\Big[1-\frac{\alpha_r}{3\pi}\,\ln\Big\{\frac{\pi^2\hbar^2}{r^2c^2m^2}\Big\}\Big]=\alpha$$

$$\alpha_r\approx\frac{\alpha}{1-\dfrac{\alpha}{3\pi}\,\ln\Big\{\dfrac{\pi^2\hbar^2}{r^2c^2m^2}\Big\}}=\frac{\alpha}{1-\dfrac{\alpha}{3\pi}\,\ln\Big\{\dfrac{\Lambda^2\hbar^2}{m^2c^2}\Big\}}\approx\alpha_\Lambda \tag{26.22b}$$

Clearly this is an acceptable approximation only if the modulus of the second term is small versus 1. The measured coupling constant $\alpha=1/137$ for $r\to\infty$ (i. e. very small momentum transfer between the scattering fermions resp. very small $|k_\gamma|$ of the virtual photon) is known, and the electron's Compton-wavelength $2\pi\hbar/(mc)=2.4\cdot10^{-12}$m as well. Inserting these values, one finds

$$\begin{aligned}
&\alpha_r\approx1/126 \quad\text{for}\quad r=1,6\cdot10^{-35}\text{m}=l_{\text{Planck}}\\
&\alpha_r\approx1/128 \quad\text{for}\quad r=10^{-30}\text{m}\\
&\alpha_r\approx1/134 \quad\text{for}\quad r=10^{-18}\text{m}\\
&\alpha_r\approx1/137 \quad\text{for}\quad r=10^{-12}\text{m}\approx l_{\text{Compton}}\ .
\end{aligned} \tag{26.22c}$$

This is qualitatively, but not quantitatively, in accordance with the measured values (21.20a). To compute the running coupling constant more precisely, obviously terms $\mathcal{O}(\alpha^3)$ must be added, and most of all the contributions of virtual quarks to the vacuum-polarization must be considered, see the remark below (26.9). But as a rough approximation we may state, that the relative change α_0/α of the QED coupling constant upon renormalization should be much less than a factor 2, if the fundamental length r (provided it exists at all) is not much smaller than the Planck-length. In (26.17) the relative change of the coupling constant is diverging, because the fundamental length $r=0$ is assumed.

The running coupling constant is illustrated sometimes in the literature by quite suggestive pictures, in which the dielectric effect of the electron-bubble is interpreted as the displacement of virtual electrons versus virtual positrons. Such pictures can be easily misunderstood, to say the least. According to theorem (23.39), virtual fermions must never be interpreted

in QED as virtual anti-particles, but always as virtual particles. Thus both inner fermion lines in the right graph of (26.13) must be interpreted as electrons with charge $-e$. The dielectric screening effect is brought about by the factor (-1), which is resulting from the anti-symmetry of the matrix elements (26.3) under permutation of two fermion operators. Again we notice that quantum field theory is much less pictorial than the classical theory with regard to attraction or repulsion between charged particles.

26.2 Fermion Self-Energy

The propagator of a fermion with self-interaction

$$\widetilde{S}^{(W)}(k_1) = \sum_{n=0}^{\infty} \widetilde{S}^{(n)}(k_1) \tag{26.23a}$$

is in zeroth order perturbation computation equal to the propagator

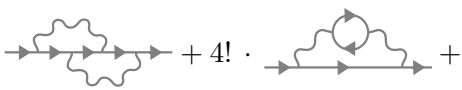
$$\widetilde{S}^{(0)}(k_1) \equiv \widetilde{S}(k_1) \mathrel{\widehat{=}} \tag{26.23b}$$

of that fermion in the theory without interactions. The second-order diagram

$$\widetilde{S}^{(2)}(k_1) \mathrel{\widehat{=}} 2! \cdot \tag{26.23c}$$

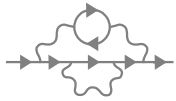
is just the second diagram in table (25.1) of the possibly diverging loop-diagrams of QED. We will compute it immediately. In fourth order there are (besides others) the following diagrams:

$$\widetilde{S}^{(4)}(k_1) \mathrel{\widehat{=}} 4! \cdot + 4! \cdot + \tag{26.23d}$$



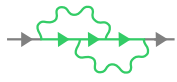
$$+ 4! \cdot + 4! \cdot \ldots$$

The third diagram is one-particle-reducibel, see the definition below (20.82) on page 443. The two other diagrams are 1PI = one-particle-irreducibel, because they can not be divided into two diagrams by the intersection of one single line. In sixth order one finds (besides others) the diagrams

$$\widetilde{S}^{(6)}(k_1) \mathrel{\widehat{=}} 6! \cdot$$  $$+\, 6! \cdot$$  $$+$$

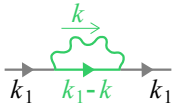
$$+\, 6! \cdot$$  $$+ \ldots \qquad (26.23e)$$

The first diagram is 1PI, the two others are not. We define the sum of all 1PI diagrams with the exception of $\widetilde{S}^{(0)}(k_1)$ as

$$\widetilde{S}(k_1)\,\Sigma(k_1)\,\widetilde{S}(k_1) \mathrel{\widehat{=}} 2! \cdot$$  $$+\, 4! \cdot$$  $$+$$

$$+\, 4! \cdot$$  $$+\, 4! \cdot \ldots +\, 6! \cdot$$  $$+ \ldots \,. \qquad (26.24)$$

Note that Σ is not the summation-symbol in this context, but an infinite series named Sigma. According to this definition, the incoming and outgoing propagators in these diagrams are not parts of Σ, but only the green-painted stuff inbetween them. In particular the term $\mathcal{O}(q^2)$ in in this series is equal to



$$\mathrel{\widehat{=}}\, \widetilde{S}^{(2)}(k_1) = \widetilde{S}(k_1) \cdot \Sigma^{(2)}(k_1) \cdot \widetilde{S}(k_1)\ . \qquad (26.25)$$

Using the definition (26.24), the propagator of a fermion with self-interaction can be written as a geometric series. We use again the notation with the Feynman-dagger $\not{k} \equiv \gamma^\nu k_\nu$:

$$\boxed{\begin{aligned} \widetilde{S}^{(W)} &= \widetilde{S} + \widetilde{S}\Sigma\widetilde{S} + \widetilde{S}\Sigma\widetilde{S}\Sigma\widetilde{S} + \ldots \overset{(12.23),(12.24)}{=} \\ &= \frac{\dfrac{i}{\not{k}_1 - m_0 c/\hbar}}{1 - \dfrac{i\Sigma}{\not{k}_1 - m_0 c/\hbar}} = \frac{i}{\not{k}_1 - m_0 c/\hbar - i\Sigma} \end{aligned}} \qquad (26.26)$$

In a simplifying wording, the second-order diagram (26.25) often is called fermion self-energy, even though it is merely the lowest order term of the complete propagator correction (26.26). The bare fermion mass m_0 is not directly measurable and exists only as a parameter in the theory. What

can be measured is the effective mass m, which causes the pole in the renormalized propagator

$$\widetilde{S}^{(W)} = \frac{i}{\not{k}_1 - mc/\hbar} \ .$$

We must not, however, simply consider $i\Sigma\hbar/c$ as mass correction, because $\Sigma(k_1)$ is not a constant, but does depend in some rather complicated manner on k_1. We therefore expand $\Sigma(k_1)$ in a Taylor series around $\not{k}_1 = m_0 c/\hbar$:

$$\begin{aligned}\Sigma(k_1) &= \sum_{n=0}^{\infty} (\not{k}_1 - m_0 c/\hbar)^n W_n \ = \\ &= \underbrace{A + (\not{k}_1 - m_0 c/\hbar)B}_{\mathcal{O}(q^2)} + \underbrace{\sum_{n=2}^{\infty} (\not{k}_1 - m_0 c/\hbar)^n W_n}_{\mathcal{O}(q^4)} \end{aligned} \tag{26.27}$$

We need to explain why all terms with $(\not{k}_1 - m_0 c/\hbar)^n, n \geq 2$ are considered to be of higher than second order in perturbation theory. For that purpose we again have a look at the diagrams (24.106), (24.107), and (24.108), and at the virtual fermion masses estimated there. We see that these masses differ only slightly from the masses of the free fermions, if realistic values are assumed for the momenta of the incoming and outgoing particles. This is caused by the fact that the mass of the incoming and outgoing photons is zero, because they are free particles. Only in diagrams of higher order, which contain both virtual photons and virtual fermions, the masses of virtual fermions may deviate significantly from the masses of free fermions.

Therefore terms $(\not{k}_1 - m_0 c/\hbar)^n, n \geq 2$ may be neglected in the Taylor-expansion (26.27), if we content ourselves with the accuracy $\mathcal{O}(q^2)$ in the correction of the fermion propagator. This accuracy of the propagator is equivalent to an accuracy $\mathcal{O}(q^4)$ in the scattering diagrams with two incoming and two outgoing particles, because the computations of the graphs (24.106), (24.107), and (24.108) can be extended to fourth order of perturbation computation, i. e. $\mathcal{O}(q^4)$, by attaching to the virtual fermion line the second-order correction (26.25), which we will compute in the sequel. Thus one gets a result which is correct up to (and including) $\mathcal{O}(q^2)$, if the

coefficients A and B in the Taylor-expansion (26.27) are included, while all further coefficients W_n with $n \geq 2$ are ignored.

The two coefficients $A, B \in \mathbb{R}$ are of order of magnitude $\mathcal{O}(q^2)$. Therefore the denominator of the propagator (26.26) may be written in the form

$$\frac{\not{k}_1 - m_0 c/\hbar - iA}{1 + iB} = \frac{\not{k}_1 - m_0 c/\hbar - iA - \not{k}_1 iB + iB m_0 c/\hbar - AB}{(1+iB)(1-iB)} =$$

$$= \not{k}_1 - m_0 c/\hbar \underbrace{-iA - (\not{k}_1 - m_0 c/\hbar) iB}_{-i\Sigma^{(2)}} + \mathcal{O}(q^4) \ . \quad (26.28)$$

Thus the propagator with self-interaction assumes this form:

$$\widetilde{S}^{(W)} \overset{(26.26)}{=} \frac{i\,(1+iB)}{\not{k}_1 - m_0 c/\hbar - iA} + \mathcal{O}(q^4) = \frac{i\,Z_2}{\not{k}_1 - mc/\hbar} + \mathcal{O}(q^4)$$
$$\text{with } Z_2 \equiv 1 + iB \quad \text{and} \quad m \equiv m_0 + iA\hbar/c \quad (26.29)$$

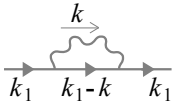
As we content ourselves with an accuracy $\mathcal{O}(q^2)$, the coefficients

$$A \overset{(26.27)}{=} \Sigma^{(2)}\Big|_{\not{k}_1 = m_0 c/\hbar} \quad \text{and} \quad B \overset{(26.27)}{=} \frac{\partial \Sigma^{(2)}}{\partial \not{k}_1}\Big|_{\not{k}_1 = m_0 c/\hbar} \quad (26.30)$$

can be found, once the algebraic form of $\Sigma^{(2)}$ is known. This we now need to compute.

We will find a logarithmic divergence of the diagram (26.25). We will cure that divergence due to the renormalization of the fermion's mass and due to another renormalization of it's charge. Only the mass renormalization will survive in the end, however, because the second charge renormalization (remember that we just performed a first charge renormalization to cure the divergence of vacuum polarization) will be exactly compensated by a third charge renormalization, which will be applied in the next section to handle the vertex correction. Therefore we will distinguish in the sequel between the fermion's not measurable "bare" mass m_0 and it's renormalized mass m. The computation of the diagram is done with the not renormalized mass m_0, but with the (in the previous section) renormalized charge. According

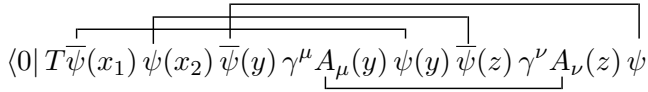
to the rules of box 26.1, this is the value of diagram (26.25):



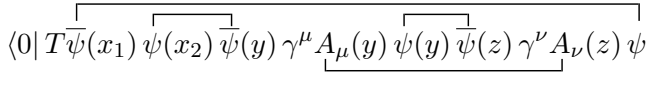
$$\widehat{=}\ \widetilde{S}^{(2)}(k_1) = \widetilde{S}(k_1)\cdot\Sigma^{(2)}(k_1)\cdot\widetilde{S}(k_1) \tag{26.31}$$

$$\begin{aligned}\Sigma^{(2)} &= -2\cdot\mathcal{U}_{\mathrm{S}}\,\frac{1}{2!}\,\frac{(-iq\gamma^{\alpha})}{\hbar}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\\ &\quad\cdot\,\frac{i(\gamma^{\nu}(k_1-k)_{\nu}+m_0c/\hbar)}{(k_1-k)^2-(m_0c/\hbar)^2+i\epsilon'}\,\frac{(-i)g_{\alpha\beta}\,\mu_0\hbar c}{k^2+i\epsilon'}\,\frac{(-iq\gamma^{\beta})}{\hbar}\\ &= \frac{q^2\mu_0 c}{\hbar}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{\gamma^{\alpha}\Big(\gamma^{\nu}(k_1-k)_{\nu}+m_0c/\hbar\Big)\gamma_{\alpha}}{\Big((k_1-k)^2-m_0^2c^2/\hbar^2\Big)k^2+i\epsilon}\end{aligned} \tag{26.32}$$

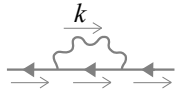
Here we inserted the symmetry-factor 2, which — as always in QED — is identical to the faculty of the number of vertices, and consequently cancels versus the factor 1/2! from the Taylor-expansion of the interaction term. The self-energy matrix element of a fermion particle (not antiparticle), which is moving from x_1 to y, there emits a photon with wavenumber k, then at z absorbs that photon again, and eventually moves to x_2, has the contraction



5 permutations of fermion operators are needed to construct the propagators. This gives a factor (-1). If the incoming particle instead moves from x_1 to z, and eventually goes out from y to x_2, then the matrix element has the contraction



1 permutation of fermion operators is needed to construct the propagators. Again this gives a factor (-1).



If we instead consider the graph of an antiparticle, which is moving from x_1 to y or to z, there emits a photon with wavenumber k, then absorbs that photon again at z or at y, and eventually goes out to x_2, then we get a matrix element with these two alternative contractions:

$$\langle 0|\, T\psi(x_1)\,\overline{\psi}(x_2)\,\overline{\psi}(y)\,\gamma^\mu A_\mu(y)\,\psi(y)\,\overline{\psi}(z)\,\gamma^\nu A_\nu(z)\,\psi(z)\,|0\rangle$$

$$\langle 0|\, T\psi(x_1)\,\overline{\psi}(x_2)\,\overline{\psi}(y)\,\gamma^\mu A_\mu(y)\,\psi(y)\,\overline{\psi}(z)\,\gamma^\nu A_\nu(z)\,\psi(z)\,|0\rangle$$

The symmetry-factor again is 2 in this case. We get a negative sign, because both contractions require 3 permutations of fermion operators for the constructions of the propagators. Consequently the self-energy correction of the antiparticle is identical to the correction (26.32) of the particle. Therefore no separate computation is needed.

Without the small term $i\epsilon$ the integral (26.32) would have poles at $(k_1-k)^2 = m_0^2\frac{c^2}{\hbar^2}$ and at $k^2 = 0$. Besides the logarithmic UV-divergence at $|k| \to \infty$ we will find a logarithmic IR-divergence at $|k| \to 0$. The latter will however be clearly visible only in (26.44). To keep the IR-divergence in check, we assign to the photon formally a very small, but finite rest-mass m_γ, i. e. we shift the pole of the photon propagator towards $k^2 = m_\gamma^2 c^2/\hbar^2$. This formal measure is acceptable, because eventually the IR-divergence will be compensated — as stated in section 24.3.6 — by graphs with bremsstrahlung-photons with immeasurably small energy. This means that there wouldn't be an IR-divergence at all if we would consider *all* relevant graphs.

Now we combine the two factors in the denominator by means of method (20.72):

$$\frac{1}{\left((k_1-k)^2 - m_0^2\frac{c^2}{\hbar^2}\right)\left(k^2 - m_\gamma^2\frac{c^2}{\hbar^2}\right)} =$$

$$= \int_0^1 \frac{\mathrm{d}\xi}{\Big(\xi\Big((k_1-k)^2 - m_0^2\tfrac{c^2}{\hbar^2}\Big) + (1-\xi)\Big(k^2 - m_\gamma^2\tfrac{c^2}{\hbar^2}\Big)\Big)^2} =$$

$$= \int_0^1 \frac{\mathrm{d}\xi}{\Big((k-\xi k_1)^2 + \xi(1-\xi)k_1^2 - (1-\xi)m_\gamma^2\tfrac{c^2}{\hbar^2} - \xi m_0^2\tfrac{c^2}{\hbar^2}\Big)^2} \tag{26.33}$$

Using the defining relation (8.9) we compute some contractions of γ-matrices:

$$\gamma^\alpha\gamma_\alpha = 4 \tag{26.34a}$$

$$\gamma^\alpha\gamma^\tau\gamma_\alpha = (2g^{\alpha\tau} - \gamma^\tau\gamma^\alpha)\gamma_\alpha = 2\gamma^\tau - \gamma^\tau\cdot 4 = -2\gamma^\tau \tag{26.34b}$$

$$\begin{aligned}\gamma^\alpha\gamma^\sigma\gamma^\tau\gamma_\alpha &= (2g^{\alpha\sigma} - \gamma^\sigma\gamma^\alpha)\gamma^\tau\gamma_\alpha = \\ &= 2\gamma^\tau\gamma^\sigma + 2\gamma^\sigma\gamma^\tau = 4g^{\sigma\tau}\end{aligned} \tag{26.34c}$$

$$\begin{aligned}\gamma^\alpha\gamma^\rho\gamma^\sigma\gamma^\tau\gamma_\alpha &= (2g^{\alpha\rho} - \gamma^\rho\gamma^\alpha)\gamma^\sigma\gamma^\tau\gamma_\alpha = 2\gamma^\sigma\gamma^\tau\gamma^\rho - 4g^{\sigma\tau}\gamma^\rho = \\ &= 4g^{\sigma\tau}\gamma^\rho - 2\gamma^\tau\gamma^\sigma\gamma^\rho - 4g^{\sigma\tau}\gamma^\rho = -2\gamma^\tau\gamma^\sigma\gamma^\rho\end{aligned} \tag{26.34d}$$

We make use of (26.34a) and (26.34b), substitute the variable k of summation and integration by $k \to \kappa = k - \xi k_1$, and eventually rename κ into k. Thus we get the integral

$$\Sigma^{(2)} \overset{(26.32)}{=} -\int_0^1 \mathrm{d}\xi\, \frac{2q^2\mu_0 c}{\hbar}\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\cdot$$
$$\cdot\,\frac{(1-\xi)\not{k}_1 - \not{k} - 2m_0c/\hbar}{\Big(k^2 + \xi(1-\xi)k_1^2 - (1-\xi)m_\gamma^2c^2/\hbar^2 - \xi m_0^2c^2\hbar^2 + i\epsilon\Big)^2}\,. \tag{26.35}$$

Here again the Feynman-dagger $\not{k} \equiv \gamma^\nu k_\nu$ has been used. The term $\not{k}$ in the numerator may be skipped, because it is antisymmetric in $\pm k$, while the rest of the integrand is symmetric in $\pm k$. Therefore the integral over this term is zero. The loop-index of the remaining integral is

$$k^d \sim \mathrm{d}^4k\, k^{-4} \implies d = 0\,.$$

Consequently for $|k| \to \infty$ a logarithmic divergence is to be expected.

$\Sigma^{(2)}$ can be written in the form

$$\Sigma^{(2)} = \frac{2q^2\mu_0 c}{\hbar} \int_0^1 \mathrm{d}\xi\, V\left(2\,\frac{m_0 c}{\hbar} - (1-\xi)\not{k}_1\right) \cdot \cdot \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{1}{\left(k^2 - K^2 + i\epsilon\right)^2} \tag{26.36a}$$

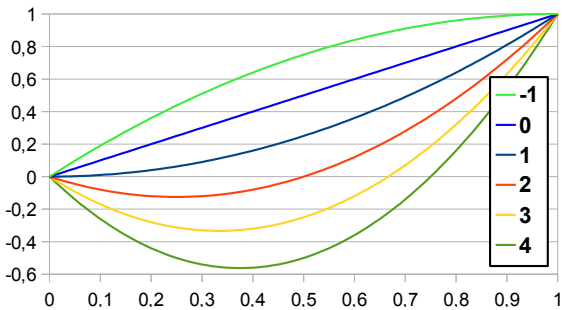
$$K^2 \equiv -(\xi-\xi^2)k_1^2 + (1-\xi)m_\gamma^2 c^2/\hbar^2 + \xi m_0^2 c^2/\hbar^2 \tag{26.36b}$$

$$V \equiv \begin{cases} 1 & \text{if } K^2 \geq 0 \\ 0 & \text{if } K^2 < 0\ . \end{cases} \tag{26.36c}$$

$K\hbar/c$ here is acting as an effective mass, because $\Sigma^{(2)}$ has poles at $k = \pm K$. Neglecting the very small m_γ^2 we have

$$\frac{K^2\hbar^2}{m_0^2 c^2} = \xi - (\xi - \xi^2)\frac{k_1^2\hbar^2}{m_0^2 c^2}\ .$$

This function is displayed[1] in figure 26.4. For a free particle with $k_1^2 = m_0^2 c^2/\hbar^2$, K^2 is always ≥ 0 at arbitrary ξ. The same holds true for a virtual particle with $k_1^2 < m_0^2 c^2/\hbar^2$. In contrast, for a virtual particle with $k_1^2 > m_0^2 c^2/\hbar^2$ there exists a range of small values of ξ, for which $K^2 < 0$, i. e. the effective mass is imaginary. The imaginary mass is the formal indicator



The parameters in the legend indicate the ratio $k_1^2\hbar^2/(m_0 c)^2$.

Fig. 26.4: $K^2\hbar^2/(m_0 c)^2$ as a function of ξ

of a dissipative process, in which energy is converted into heat and thus disappears from the evaluated system. In this case dissipation means that a channel becomes accessible, which is competing with the diagram (26.32): At $k_1^2 > m_0^2 \frac{c^2}{\hbar^2}$ the energy of the system is sufficient to convert the virtual photon with wavenumber k into a real photon with wavenumber k, and convert the virtual fermion with wavenumber $k_1 - k$ into a real fermion with wavenumber $k_1 - k$. Thus (26.32) is replaced by a bremsstrahlung-process of the type (24.91).

In the integral over ξ, the dissipative range with $K^2 < 0$ is masked by the function V. Therefore we may apply the generic integral-formula

$$\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{1}{\left(k^2-K^2+i\epsilon'\right)^r} \overset{(20.64)}{=} \frac{i(-1)^r}{16\pi^2}\int_0^{+\infty}\mathrm{d}R^2\,\frac{R^2}{\left(R^2+K^2\right)^r}$$
$$r\in\mathbb{R}, r>\frac{1}{2}\ ,\ K^2\in\mathbb{R}, K^2\geq 0\ . \tag{26.37}$$

Thereby we get

$$\Sigma^{(2)} = \int_0^1 \mathrm{d}\xi\, V\,\frac{iq^2\mu_0 c\left(2m_0c/\hbar-(1-\xi)\not{k}_1\right)}{8\pi^2\hbar}\int_0^{+\infty}\mathrm{d}R^2\,\frac{R^2}{(R^2+K^2)^2}\ . \tag{26.38}$$

The integral over R is diverging logarithmically for $R\to\infty$. In ψ^s-theory we regularized it by means of a cut-off parameter Λ. We do not want to repeat that method here, because it would spoil the gauge-invariance of QED. Instead we regularize the integral by the method of Pauli and Villars, introducing counterterms:

$$\Sigma^{(2)} = \lim_{M\to\infty}\int_0^1 \mathrm{d}\xi\, V\,\frac{iq^2\mu_0 c\left(2m_0c/\hbar-(1-\xi)\not{k}_1\right)}{8\pi^2\hbar}\cdot$$
$$\cdot\int_0^{+\infty}\mathrm{d}R^2\left(\frac{R^2}{(R^2+K^2)^2}-\frac{R^2}{(R^2+K_M^2)^2}\right) \tag{26.39a}$$

$$K^2 \overset{(26.36b)}{=} -(\xi-\xi^2)k_1^2 + (1-\xi)m_\gamma^2 c^2/\hbar^2 + \xi m_0^2 c^2/\hbar^2 \tag{26.39b}$$

$$\begin{aligned} K_M^2 &\equiv -(\xi-\xi^2)k_1^2 + (1-\xi)m_\gamma^2 c^2/\hbar^2 + \xi M^2 c^2/\hbar^2 \approx \\ &\approx \xi M^2 c^2/\hbar^2 \quad \text{because of } M^2 \gg m_0^2 \end{aligned} \tag{26.39c}$$

M, and consequently K_M, are chosen large but finite. Thus for $R \to \infty$ both K^2 and K_M^2 will become negligible versus R^2, and the two terms in (26.39a) will mutually compensate to zero.

$$\begin{aligned} \Sigma^{(2)} &\overset{(26.39a)}{=} \lim_{M\to\infty} \int_0^1 \mathrm{d}\xi\, V \frac{iq^2\mu_0 c\left(2m_0 c/\hbar - (1-\xi)\not{k}_1\right)}{8\pi^2\hbar} \cdot \\ &\quad \cdot \left(\frac{K^2}{R^2+K^2} + \ln\left(\frac{R^2+K^2}{R^2+K_M^2}\right) - \frac{K_M^2}{R^2+K_M^2} \right)\Bigg|_0^\infty \\ &= \frac{i\alpha_q}{2\pi} \int_0^1 \mathrm{d}\xi\, V \left(2\frac{m_0 c}{\hbar} - (1-\xi)\not{k}_1\right) \lim_{M\to\infty} \ln\left(\frac{K_M^2}{K^2}\right) . \end{aligned} \tag{26.40}$$

The finestructure-constant α_q has been defined in (26.11).

Observing

$$(\not{k}_1)^2 = \tfrac{1}{2}k_\alpha(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha)k_\beta \overset{(8.9)}{=} \tfrac{1}{2}k_\alpha 2g^{\alpha\beta}k_\beta = (k)^2 \tag{26.41}$$

we now are going to compute the coefficients A and B:

$$A = \Sigma^{(2)}\Big|_{\not{k}_1 = m_0 c/\hbar} \overset{(26.40)}{=} \frac{i\alpha_q m_0 c}{2\pi\hbar} \int_0^1 \mathrm{d}\xi\, V(1+\xi) \lim_{M\to\infty} \ln\left(\frac{k_M^2}{K^2}\right)$$

The function $V =$ (26.36c) is prohibiting a general solution of the integral over ξ. We therefore restrain the evaluation for the rest of this section to the case $k_1^2 \le m_0^2 c^2/\hbar^2$, i. e. to free fermions or space-like virtual fermions. Under this assumption, K^2 is ≥ 0 for arbitrary ξ, and consequently $V = 1$ always holds. Using

$$\ln\Big(\frac{k_M^2}{K^2}\Big)\Big|_{\not{k}_1=m_0c/\hbar} = \ln\Big(\frac{\xi M^2}{(\xi^2-\xi)m_0^2+(1-\xi)m_\gamma^2+\xi m_0^2}\Big) =$$

$$= \ln(\xi) + \ln\Big(\frac{M^2}{m_0^2}\Big) - \underbrace{\ln\Big(\xi^2+(1-\xi)\frac{m_\gamma^2}{m_0^2}\Big)}_{\approx 2\ln(\xi)}$$

$$\int_0^1 \mathrm{d}\xi\,(1+\xi)\ln(\xi) = -\frac{5}{4} \quad , \quad \int_0^1 \mathrm{d}\xi\,(1+\xi) = \frac{3}{2}$$

we then can compute A:

$$A = \frac{i\alpha_q m_0 c}{2\pi\hbar}\Big(\frac{3}{2}\lim_{M\to\infty}\ln\Big(\frac{M^2}{m_0^2}\Big)+\frac{5}{4}\Big) \tag{26.42}$$

Now M can be eliminated due to renormalization of the fermion's bare mass m_0 to the experimentally determined fermion mass m:

$$\boxed{m \overset{(26.29)}{=} m_0 + i\,\frac{A\hbar}{c} \overset{(26.42)}{=} m_0 - \frac{\alpha_q m_0}{2\pi}\Big(\frac{3}{2}\lim_{M\to\infty}\ln\Big(\frac{M^2}{m_0^2}\Big)+\frac{5}{4}\Big)} \tag{26.43}$$

It's instructive to compare this mass renormalization to the coupling constant renormalization

$$\alpha \overset{(26.17)}{=} \alpha_0 - \frac{\alpha_0^2}{3\pi}\lim_{M\to\infty}\ln\Big\{\frac{M^2}{m^2}\Big\}$$

executed in the previous section. The structure of both renormalization-equations is identical. Therefore the renormalized mass is as well a "running constant", whose value according to (26.22) does depend on the scale of length r, at which the theory is submitted to experimental test. And same as in case of the coupling constant we can roughly estimate that the relative change m_0/m of the mass due to renormalization would be less than a factor 2, if the fundamental length — provided it does exist at all — is not significantly smaller than the Planck-length. In (26.43) the relative change of the mass diverges because the fundamental length $r = 0$ is assumed.

Next we compute the normalization factor Z_2 in the propagator (26.29).

Again we constrain the evaluation to the case $k_1^2 \leq m_0^2c^2/\hbar^2$, in which V is 1 for arbitrary ξ:

$$\begin{aligned} B = i(1-Z_2) = \frac{\partial\Sigma^{(2)}}{\partial \not{k}_1}\Big|_{\not{k}_1=m_0c/\hbar} &\overset{(26.40)}{=} \frac{i\alpha_q}{2\pi} \lim_{m_\gamma\to 0} \int_0^1 \mathrm{d}\xi \Big((-1+\xi) \\ &\cdot \lim_{M\to\infty} \ln\Big(\frac{\xi M^2c^2/\hbar^2}{(\xi^2-\xi)k_1^2 + (1-\xi)m_\gamma^2c^2/\hbar^2 + \xi m_0^2c^2/\hbar^2}\Big) + \\ &+ \Big(2\frac{m_0c}{\hbar} - (1-\xi)\not{k}_1\Big)\cdot \\ &\cdot \frac{(-2)(\xi^2-\xi)\not{k}_1}{(\xi^2-\xi)k_1^2 + (1-\xi)m_\gamma^2c^2/\hbar^2 + \xi m_0^2c^2/\hbar^2}\Big)\Big|_{\not{k}_1=m_0c/\hbar} \\ &= \frac{i\alpha_q}{2\pi}\Big(\frac{3}{4} - \frac{1}{2}\lim_{M\to\infty}\ln\Big(\frac{M^2}{m_0^2}\Big) - C\Big) \end{aligned} \tag{26.44}$$

$$C \equiv \int_0^1 \mathrm{d}\xi \Big(\frac{2(\xi^3-\xi)}{\xi^2 + (1-\xi)\frac{m_\gamma^2}{m_0^2}} + (\xi-1)\underbrace{\ln\Big(\xi^2 + (1-\xi)\frac{m_\gamma^2}{m_0^2}\Big)}_{\approx 2\ln(\xi)}\Big)$$

Before we continue, we replace m_0 everywhere in B by the just renormalized parameter m. $m_\gamma \neq 0$ is needed to avoid the IR-divergence in the linear term (not in the logarithmic term) in C. We can approximate C due to replacing the term $(1-\xi)m_\gamma^2/m^2$ by a shift of the lower integration limit:

$$\begin{aligned} C &\approx \int_{m_\gamma/m}^1 \mathrm{d}\xi \Big(\frac{2(\xi^3-\xi)}{\xi^2} + (\xi-1)\ln(\xi^2)\Big) = 1 - \frac{m_\gamma^2}{m^2} + \ln\Big(\frac{m_\gamma^2}{m^2}\Big) - \\ &\quad - \frac{1}{2} - \frac{1}{2}\frac{m_\gamma^2}{m^2}\ln\Big(\frac{m_\gamma^2}{m^2}\Big) + \frac{1}{2}\frac{m_\gamma}{m} + 2 + \frac{m_\gamma}{m}\ln\Big(\frac{m_\gamma^2}{m^2}\Big) - 2\frac{m_\gamma}{m} \\ C &\approx -\ln\Big(\frac{m^2}{m_\gamma^2}\Big) + \frac{5}{2} \quad \text{because of } m_\gamma \ll m \end{aligned}$$

Thereby we find

$$Z_2 \overset{(26.29)}{=} 1 + iB = 1 + \frac{\alpha_q}{2\pi}\left(\frac{1}{2}\lim_{M\to\infty}\ln\left(\frac{M^2}{m^2}\right) + \frac{7}{4} - \ln\left(\frac{m^2}{m_\gamma^2}\right)\right) . \quad (26.45)$$

The term with m_γ, in which for $m_\gamma \to 0$ an IR-divergence turns up, will be compensated by diagrams with immeasurably low-energy bremsstrahlung, as stated in section 24.3.6. Thus we may ignore that divergence. The factor M is treated by renormalization: The term Z_2 is contained in the numerator of the propagator (26.29), and therefore can be considered to be an additional normalization factor of the field operators in the propagator

$$\langle 0|\, T\psi^{(W)}(x_2)\,\overline{\psi}^{(W)}(x_1)\,|0\rangle = \langle 0|\, T\,\sqrt{Z_2}\psi_0(x_2)\;\sqrt{Z_2}\,\overline{\psi}_0(x_1)\,|0\rangle \ .$$

The vertex z of a graph is constructed by the operator product $\sqrt{Z_2}\,\psi_0(z)$ $\sqrt{Z_2}\,\overline{\psi}_0(z)\,A(z)$. Thus the vertex-factor — which is $-iq\gamma^\mu/\hbar$ according to Rule D of box 24.1 — is multiplied by $\sqrt{Z_2}\,\sqrt{Z_2} = Z_2$. Consequently another renormalization of the fermion's bare charge q_0 becomes necessary due to the fermion's self-interaction:

$$\boxed{Z_2 \cdot q_0 \xrightarrow{\text{renormalization}} q} \quad (26.46)$$

A first renormalization of the fermion-charge by means of the renormalization-constant Z_3 has been executed already because of vacuum polarization, see (26.17). And the evaluation of the vertex correction, which will be done in the next section, will show that a third renormalization of the fermion-charge due to a renormalization-constant $1/Z_1$ is necessary. The overall result will be

$$\sqrt{Z_3}\,\frac{Z_2}{Z_1}\cdot q_0 \xrightarrow{\text{renormalization}} q \ . \quad (26.47)$$

But we will also find out $Z_2 = Z_1$, i. e. in total the renormalized charge will have just that value, which we determined by means of Z_3 in (26.17).

As a result of the renormalizations (26.43) and (26.46), all self-energy corrections indicated in (26.24) are to be replaced in all graphs by the simple propagator-line ⟶. This leads to graphs, which have been considered already in lower order of perturbation computation. The effect is, that any graph with one of the self-energy corrections (26.24) shall be discarded.

Thus these graphs become “invisible”. But they do not really disappear from the theory: Implicitly they are considered in the renormalized fermion mass m and in the renormalized fermion charge q.

26.3 Vertex Correction

According to rule D of box 24.1 on page 510 a factor

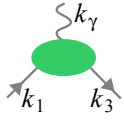
$$\widehat{=} -\frac{iq\gamma^\nu}{\hbar} \tag{26.48a}$$

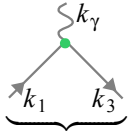
shall be inserted for each vertex. Furthermore a graph with n vertices gets a factor $1/n!$. Thus in total the rule is

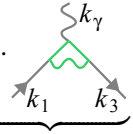
$n \cdot$ 

$$\widehat{=} \frac{1}{n!}\left(\frac{-iq\gamma^\nu}{\hbar}\right)^n \quad . \tag{26.48b}$$

Here the vertices have been marked by green dots, to emphasize that these factors are referring only to the contact points of the incoming and outgoing propagators, but not to the propagators themselves. We define a corrected vertex function $\Gamma(k_1, k_3)$ by

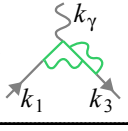
$-\frac{iq\Gamma^\nu(k_1,k_3)}{\hbar} \widehat{=}$ 

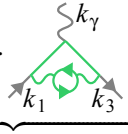
$\equiv$ 

$+ 3! \cdot$ 

$+$

$\Gamma^{(0)\nu}$ $\Gamma^{(2)\nu}$

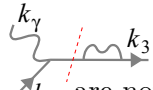
$+ 5! \cdot$ 

$+ 5! \cdot$ 

$+ 5! \cdot \ldots + \ldots \,.$

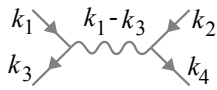
$\Gamma^{(4)\nu}$ (26.48c)

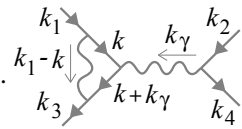
As always in QED, the symmetry factors are equal to the faculty of the number of vertices. Note: The index (n) of $\Gamma^{(n)\nu}$ is indicating the number of vertices which in addition to (26.48a) exist in the graph. Note: Only 1PI-



diagrams are considered in Γ^ν. Diagrams like e. g. are not considered, because this diagram can be divided into two diagrams by the section of one single line (as indicated by the dashed red line). As energy and momenta of the incoming and outgoing particles must be conserved, k_γ is uniquely determined by k_1 and k_3 and does not need to be explicitly specified in the definition of $\Gamma^\nu(k_1, k_3)$.

Clearly $\Gamma^{(0)\nu}$ is equal to γ^ν. In the sequel we will compute Γ^ν only up to (and including) order $\Gamma^{(2)\nu}$, i. e. only the lowest-order vertex correction. We will do this for the example of t-channel scattering of an electron by an muon. We have computed this process in second order of perturbation computation in section 24.3.1. In section 26.1 we have already evaluated the vacuum polarization as a fourth-order correction, see graph (26.13). As a further fourth-order correction we will now consider the lowest order vertex correction, which has been listed in table (25.1) as the third possibly diverging diagram of QED.

$$\mathcal{M}^{(2)} + \mathcal{M}_b^{(4)} \mathrel{\widehat{=}} 2 \cdot$$ 

$$+ 24 \cdot$$ 

(26.49)

k_1 is symbolizing the incoming electron, k_2 the incoming muon. The photon wavenumber is $k_\gamma = k_2 - k_4 = k_3 - k_1$. We will see immediately, that the vertex correction is proportional to m^{-4}, with m being the fermion mass. As the muon is about 207 times as heavy as the electron, the vertex correction of the muon current is by more than 10^{-9} times smaller than the vertex correction of the electron current, and may be neglected.

From the second-order matrix element of t-channel scattering

$$(24.45) = \langle 0| T\psi_3\psi_4\overline{\psi}_1\overline{\psi}_2\overline{\psi}_y \not{A}_y\psi_y\overline{\psi}_z \not{A}_z\psi_z|0\rangle \tag{26.50}$$

we conclude that there is a factor (-1), because an odd number of permutations of fermion-operators is required for the contraction of the matrix element to the graph's propagators. Furthermore we conclude, that the

second alternative of contraction, in which the vertices y and z are permuted, again gets a factor (-1), because it requires an additional even number of permutations of fermion operators. Obviously this is a generic rule of QED: As an even number (i. e. two) of fermion operators belong to each vertex, the permutation of vertices does not result in a change of sign. All contractions which are leading to the same graph, have the same sign.

The matrix element $\mathcal{M}_b^{(4)}$ is constructed from

$$\langle 0|\, T\psi_3\psi_4\overline{\psi}_1\overline{\psi}_2\overline{\psi}_v \not{A}_v\psi_v\overline{\psi}_w \not{A}_w\psi_w\overline{\psi}_y \not{A}_y\psi_y\overline{\psi}_z \not{A}_z\psi_z|0\rangle \ . \tag{26.51}$$

Again only one of the $4! = 24$ alternative contractions has been indicated in the graph (26.49). All of them have the same (namely negative) sign, because there are 6 crossovers of the contraction brackets, and in addition 3 pairs of fermion operators must be permuted, in order that all creation operators will be right and all annihilation operators will be left.

As always in QED, the symmetry-factor of vertex correction is identical to the faculty of the number of vertices. Therefore this factor is always canceled by the factor $1/n!$, which is resulting from the Taylor expansion of the interaction term in the matrix element. By means of the rules compiled in box 24.1, the term with the vertex correction in the scattering matrix can be computed:

$$\begin{aligned}\mathcal{M}_b^{(4)} = -\Big(\frac{ie}{\hbar}\Big)^4 \frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\ \frac{{}^{r_3}\overline{u}^{\boldsymbol{k}_3}\,\gamma^\sigma\, i(\not{k}+\not{k}_\gamma + mc/\hbar)}{(k+k_\gamma)^2 - m^2c^2/\hbar^2 + i\epsilon'} \\ \cdot\ \frac{\gamma^\mu\, i(\not{k}+mc/\hbar)\,\gamma^\tau\,{}^{r_1}u^{\boldsymbol{k}_1}}{k^2 - m^2c^2/\hbar^2 + i\epsilon'}\ \cdot\ \frac{(-ig_{\sigma\tau}\,\mu_0\hbar c)}{(k_1-k)^2 - m_\varphi^2c^2/\hbar^2 + i\epsilon'} \\ \cdot\ \frac{(-ig_{\mu\nu}\,\mu_0\hbar c)}{k_\gamma^2 - m_\gamma^2c^2/\hbar^2 + i\epsilon'}\,({}^{r_4}\overline{u}^{\boldsymbol{k}_4}\,\gamma^\nu\,{}^{r_2}u^{\boldsymbol{k}_2})\end{aligned} \tag{26.52}$$

Again we have inserted small masses for the virtual photons, to prevent IR-divergences. m_γ is the mass of the photon with wavenumber k_γ. m_φ is the mass of the photon with wavenumber $(k_1 - k)$. Care has been taken

to arrange the sequence of spinor products correctly according to (23.21). With (24.47b) in mind (where the green factors must be set to zero, because we now are evaluating t-channel scattering) the scattering matrix may be written as follows:

$$\mathcal{M}^{(2)} + \mathcal{M}_b^{(4)} = -\Big(\frac{ie}{\hbar}\Big)^2 \Big({}^{r_3}\bar{u}^{\boldsymbol{k}_3} \, (\underbrace{\Gamma^{(0)\mu}}_{\gamma^\mu} + \Gamma^{(2)\mu}) \, {}^{r_1}u^{\boldsymbol{k}_1} \Big) \cdot$$
$$\cdot \, \frac{(-ig_{\mu\nu}\,\mu_0\hbar c)}{k_\gamma^2 - m_\gamma^2 c^2/\hbar^2 + i\epsilon'} \, ({}^{r_4}\bar{u}^{\boldsymbol{k}_4} \, \gamma^\nu \, {}^{r_2}u^{\boldsymbol{k}_2}) \tag{26.53a}$$

$$\Gamma^{(2)\mu} = \Big(\frac{ie}{\hbar}\Big)^2 \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \, \frac{\gamma^\sigma \, i(\not{k} + \not{k}_\gamma + mc/\hbar)}{(k+k_\gamma)^2 - m^2c^2/\hbar^2 + i\epsilon'} \cdot$$
$$\cdot \, \frac{\gamma^\mu \, i(\not{k} + mc/\hbar) \, \gamma^\tau}{k^2 - m^2c^2/\hbar^2 + i\epsilon'} \cdot \frac{(-ig_{\sigma\tau}\,\mu_0\hbar c)}{(k_1 - k)^2 - m_\varphi^2 c^2/\hbar^2 + i\epsilon'} \tag{26.53b}$$

$$\Gamma^\mu = \Gamma^{(0)\mu} + \Gamma^{(2)\mu} + \ldots = \gamma^\mu F_1 + [\gamma^\nu, \gamma^\mu] \, \frac{(k_\gamma)_\nu \hbar}{4mc} \, F_2 \tag{26.53c}$$

It's not obvious that Γ^μ can be broken down into two parts with "form-factors" F_1 and F_2. And it's also not obvious that F_1 and F_2 only depend on $k_\gamma^2 = (k_3 - k_1)^2$, i. e. on the square of the four-momentum transfer inbetween electron and muon in the scattering experiment, but not on the modulus of k_1 and k_3. In the sequel we will re-shape $\Gamma^{(2)\mu}$ such that this structure will become clearly visible.

In second order perturbation computation (i. e. $\Gamma^\mu = \gamma^\mu$) $F_1 = 1$ and $F_2 = 0$. The values of the two form-factors in fourth order of perturbation computation (i. e. $\Gamma^\mu = \gamma^\mu + \Gamma^{(2)\mu}$) will be evaluated immediately. With regard to the physical meaning of the form-factors we remark without proof (the derivation can be found e. g. in [3, section 6.2]): In scattering experiments with very small momentum transfer $k_\gamma^2 \to 0$ (this is called forward-scattering), the effective charge of the electron is

$$e_{\text{effective}} = e \cdot F_1(k_\gamma^2 \approx 0) \; , \tag{26.54a}$$

and it's magnetic moment is

$$\boldsymbol{\mu}_{\text{effective}} = g\,\frac{e}{2m}\boldsymbol{S} = 2\Big(F_1(k_\gamma^2 \approx 0) + F_2(k_\gamma^2 \approx 0)\Big)\frac{e}{2m}\boldsymbol{S}\ , \tag{26.54b}$$

with $\boldsymbol{S}$ being the electron's spin, and g being the Landé-factor. Thus the computation of the form-factors in fourth order (and even higher orders) will result in radiation corrections of e and g, which can be compared to the results of precision experiments.

In the spinor product

$$\begin{aligned} Y &\equiv \gamma^\sigma\,(\not{k} + \not{k}_\gamma + mc/\hbar)\,\gamma^\mu\,(\not{k} + mc/\hbar)\,\gamma^\tau g_{\sigma\tau} \;= \\ &= \gamma^\sigma\,\Big((\not{k} + \not{k}_\gamma)\gamma^\mu\not{k} + (\not{k} + \not{k}_\gamma)\gamma^\mu mc/\hbar + \gamma^\mu\not{k}mc/\hbar + \gamma^\mu m^2c^2/\hbar^2\Big)\,\gamma_\sigma\ , \end{aligned}$$

which is a part in the numerator of $\Gamma^{(2)\mu}$, there are several pairs of contracted matrix products $\gamma^\sigma\gamma_\sigma$ with 3, 2, or 1 further γ-matrices in-between them. Note that also in each product $\not{k} \equiv \gamma^\alpha k_\alpha$ with the Feynman-dagger a γ-matrix is enclosed. Using (26.34) the contractions can be performed:

$$Y = 2\Big(-\not{k}\gamma^\mu(\not{k} + \not{k}_\gamma) + 2(2k + k_\gamma)^\mu mc/\hbar - \gamma^\mu m^2c^2/\hbar^2\Big) \tag{26.55}$$

Using the formula

$$\frac{1}{A_1A_2A_3} \overset{(20.73b)}{=} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\,\frac{2\,\delta(x+y+z-1)}{(X)^3} \tag{26.56a}$$

$$X \equiv xA_1 + yA_2 + zA_3\ , \tag{26.56b}$$

the three denominators, in which k is contained, can be combined:

$$\begin{aligned} X &= x\Big((k+k_\gamma)^2 - m^2c^2/\hbar^2\Big) + y(k^2 - m^2c^2/\hbar^2) + \\ &\quad + z\Big((k_1-k)^2 - m_\varphi^2c^2/\hbar^2\Big) + (x+y+z)i\epsilon' \;= \\ &= (k + xk_\gamma - zk_1)^2 - (xk_\gamma - zk_1)^2 + xk_\gamma^2 + zk_1^2 - \\ &\quad - (1-z)m^2c^2/\hbar^2 - zm_\varphi^2c^2/\hbar^2 + i\epsilon' \end{aligned} \tag{26.57}$$

Here we made use of $x + y + z = 1$, and we will do this again repeatedly

in the following steps. Now we restrict the evaluation to the case that the incoming and outgoing electrons are free particles, for which the relation

$$k_1^2 = k_3^2 = m^2c^2/\hbar^2 = (k_3 - k_1 + k_1)^2 = (k_\gamma + k_1)^2 = \\ = m^2c^2/\hbar^2 + 2k_1k_\gamma + k_\gamma^2 \quad \Longrightarrow \quad 2k_1k_\gamma = -k_\gamma^2$$

holds. Thereby the denominator can be further simplified:

$$X = (k + xk_\gamma - zk_1)^2 - (-xyk_\gamma^2 + (1-z)^2m^2c^2/\hbar^2 + zm_\varphi^2c^2/\hbar^2 + i\epsilon'$$

We insert X into $\Gamma^{(2)\mu} = (26.53b)$, then substitute $k \to \kappa = k + xk_\gamma - zk_1$, and eventually rename κ into k:

$$\Gamma^{(2)\mu} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, 2\,\delta(x+y+z-1) \left(\frac{ie}{\hbar}\right)^2 i\,\mu_0\hbar c \cdot \\ \cdot \frac{1}{\Omega} \sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \frac{Y}{(k^2 - K^2 + i\epsilon')^3} \tag{26.58a}$$

$$K^2 \equiv -xyk_\gamma^2 + (1-z)^2m^2c^2/\hbar^2 + zm_\varphi^2c^2/\hbar^2 \tag{26.58b}$$

$$Y \overset{(26.55)}{=} 2\Big(-(\not{k} - x\not{k}_\gamma + z\not{k}_1)\gamma^\mu\Big(\not{k} + (1-x)\not{k}_\gamma + z\not{k}_1\Big) + \\ + \Big(4k + 4zk_1 + 2(1-2x)k_\gamma\Big)^\mu mc/\hbar - \gamma^\mu m^2c^2/\hbar^2\Big) \tag{26.58c}$$

We now are going to simplify those terms in Y, in which $\not{k}$, $\not{k}_\gamma$, or $\not{k}_1$ is contained, and start with $\not{k}$. From (26.58a) we know that the numerator of the integral over k is symmetric in $\pm k$. Therefore upon integration all terms in Y, which are linear in k, will give a null result and thus may be skipped from the outset. In particular the term $-\not{k}\gamma^\mu\not{k}$ is a composition of 16 sub-terms, of which only 4 are quadratic in k^α and consequently give a non-vanishing contribution upon integration:

$$-\int \mathrm{d}^4k\,\frac{\not k\gamma^\mu\not k}{X^3} = -\int \mathrm{d}^4k\,\frac{k_\alpha\gamma^\alpha\gamma^\mu\gamma_\beta k^\beta}{X^3} = \\ = -\frac{1}{4}\int \mathrm{d}^4k\,\frac{k_\sigma\gamma^\alpha\gamma^\mu\gamma_\alpha k^\sigma}{X^3} \overset{(26.34)}{=} \frac{1}{2}\int \mathrm{d}^4k\,\frac{k^2\gamma^\mu}{X^3}$$

Therefore Y can be simplified:

$$Y = 2\Big(\tfrac{1}{2}k^2\gamma^\mu + (x\not k_\gamma - z\not k_1)\gamma^\mu\Big((1-x)\not k_\gamma + z\not k_1\Big) + \\ + \Big(4zk_1 + 2(1-2x)k_\gamma\Big)^\mu c/\hbar - \gamma^\mu m^2c^2/\hbar^2\Big) \tag{26.59}$$

In the sequel we make use of $k_\gamma = k_3 - k_1 \implies \not k_\gamma = \not k_3 - \not k_1$. This relation can be concluded from the right diagram (26.49), because the vertex correction does not change the fact that in the end the balance of energies and momenta must be correct. By means of the commutator-relations

$$\not k_j\gamma^\mu = \gamma^\alpha\gamma^\mu(k_j)_\alpha = (2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha)(k_j)_\alpha = 2(k_j)^\mu - \gamma^\mu\not k_j \\ \not k_i\not k_j = (k_j)_\beta(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha)(k_i)_\alpha = 2k_ik_j - \not k_j\not k_i$$

we furthermore re-arrange all products, which contain a Feynman-dagger, such that $\not k_3$ comes to the very left and $\not k_1$ to the very right of the product:

$$Y = 2\Big(\tfrac{1}{2}k^2\gamma^\mu + 2x(1-x)\not k_3k_3^\mu - x(1-x)\not k_3\not k_3\gamma^\mu + \\ + z\not k_3\gamma^\mu\not k_1 - (2x+2z)(1-x)\not k_3k_1^\mu + 2(x-x^2+z-xz)k_1k_3\gamma^\mu - \\ - 2(x-x^2+z-xz)k_3^\mu\not k_1 + (2x-2x^2+2z-4xz-2z^2)k_1^\mu\not k_1 + \\ + (-x+x^2+2xz-z+z^2)\gamma^\mu\not k_1\not k_1 + \\ + (4z-2+4x)k_1^\mu mc/\hbar + (2-4x)k_3^\mu mc/\hbar - \gamma^\mu m^2c^2/\hbar^2\Big) \tag{26.60}$$

As the incoming and the outgoing electron are observed, the spinors ${}^{r_1}u^{\boldsymbol{k}_1}$ and ${}^{r_3}\bar u^{\boldsymbol{k}_3}$ are solutions of the Dirac equation

$$
\begin{aligned}
(\not{k}_1 - mc/\hbar)\,{}^{r_1}u^{\boldsymbol{k}_1} &= (\gamma^\nu k_\nu - mc/\hbar)\,{}^{r_1}u^{\boldsymbol{k}_1} = 0\\
\not{k}_1\not{k}_1\,{}^{r_1}u^{\boldsymbol{k}_1} &= {}^{r_1}u^{\boldsymbol{k}_1}\, m^2c^2/\hbar^2\\
{}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,(\not{k}_3 - mc/\hbar) &= 0
\end{aligned}
$$

$$
{}^{r_3}\bar{u}^{\boldsymbol{k}_3}\,\not{k}_3\not{k}_3 = {}^{r_3}\bar{u}^{\boldsymbol{k}_3}\, m^2c^2/\hbar^2\ .
$$

As $\Gamma^{(2)\mu}$, and consequently Y, stands in (26.53a) right of the factor ${}^{r_3}\bar{u}^{\boldsymbol{k}_3}$ and left of the factor ${}^{r_1}u^{\boldsymbol{k}_1}$, we find

$$
\begin{aligned}
Y = 2\Big(&k^2\gamma^\mu/2 + (2x^2 - 2x + 2xz + z^2 - 1)\gamma^\mu m^2c^2/\hbar^2 +\\
&+ (2x - 2x^2 + 2z - 2xz)k_1k_3\gamma^\mu + 2(-z + xz + 1 - 2x)k_3^\mu mc/\hbar +\\
&+ 2(2x - xz - z^2 + 2z - 1)k_1^\mu mc/\hbar\Big)\\
= 2\Big(&k^2\gamma^\mu/2 + (-1 + 2z + z^2)\gamma^\mu m^2c^2/\hbar^2 +\\
&+ (-2xy - 2z)\gamma^\mu m^2c^2/\hbar^2 + (2xy + 2z)k_1k_3\gamma^\mu + 2xzk_3^\mu mc/\hbar +\\
&+ 2yzk_1^\mu mc/\hbar + (xz - yz - 2x + 2y)k_\gamma^\mu mc/\hbar\Big)\ .
\end{aligned}
\tag{26.61}
$$

Repeatedly we have made use of $x+y+z=1$. The denominator of (26.58a) is symmetric under permutation of $x \leftrightarrow y$ because of $1-z = x+y$. For that reason x and y may be exchanged in the terms of the numerator. This holds for each term separately, independent of the other terms. Therefore the last term is zero. Furthermore we insert again $k_\gamma = k_3 - k_1$. Now we find, because x and y may be exchanged,

$$
\begin{aligned}
&(-yz - 2x + 2y + 3xz)k_3^\mu mc/\hbar + (-xz - 2y + 2x + 3yz)k_1^\mu mc/\hbar\ =\\
&= 2xz(k_3^\mu + k_1^\mu)mc/\hbar = (x+y)z(k_3^\mu + k_1^\mu)mc/\hbar\ =\\
&= (1-z)z(k_3^\mu + k_1^\mu)mc/\hbar\ .
\end{aligned}
$$

Using $m^2c^2/\hbar^2 = k_1^2 = k_3^2$ we can complete the squares:

$$Y = 2\Big(k^2\gamma^\mu/2 + (-1+2z+z^2)\gamma^\mu m^2c^2/\hbar^2 + \\ - (xy+z)k_\gamma^2\gamma^\mu + (1-z)z(k_3^\mu + k_1^\mu)mc/\hbar\Big) \tag{26.62}$$

The last term can be transformed by means of the

Gordon-identity:

$$\bar{u}^{\boldsymbol{k}}\gamma^\mu\, u^{\boldsymbol{q}} = \bar{u}^{\boldsymbol{k}}\Big(\frac{(k+q)^\mu}{2mc/\hbar} + \frac{(k-q)_\nu}{4mc/\hbar}\cdot[\gamma^\nu,\gamma^\mu]\Big)\, u^{\boldsymbol{q}}\ , \tag{26.63}$$

which we apply without proof. After these transformations we have

$$\Gamma^{(2)\mu} \overset{(26.58a)}{=} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, 2\,\delta(x+y+z-1)\left(\frac{ie}{\hbar}\right)^2 i\,\mu_0\hbar c\cdot \\ \cdot\ \frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\,\frac{Y}{(k^2-K^2+i\epsilon')^3} \tag{26.64a}$$

$$Y = 2\Big(k^2/2 + (-1+4z-z^2)m^2c^2/\hbar^2 - (xy+z)k_\gamma^2\Big)\gamma^\mu - \\ - 4(1-z)z\,\frac{m^2c^2}{\hbar^2}\,\frac{(k_\gamma)_\nu}{4mc/\hbar}\,[\gamma^\nu,\gamma^\mu] \tag{26.64b}$$

$$K^2 \overset{(26.58b)}{=} -xyk_\gamma^2 + (1-z)^2m^2c^2/\hbar^2 + zm_\varphi^2c^2/\hbar^2\ . \tag{26.64c}$$

Now we have arrived at the structure (26.53c), in which $(\gamma^\mu + \Gamma^{(2)\mu})$ is broken down into a form-factor F_1, which is proportional to γ^μ, and a form-factor F_2, which is proportional to $[\gamma^\nu,\gamma^\mu](k_\gamma)_\nu/(4mc/\hbar)$.

Next we must perform the summation and integration over k. Counting the powers of k gives $\mathrm{d}^4k\,k^2/k^6$. Thus a logarithmic divergence is to be expected for $|k|\to\infty$. In section 24.3.7 we found that in t-channel scattering, with which we are dealing here, $k_\gamma^2 < 0$. Therefore $K^2 \geq 0$ and K are real, and the generic loop-integral formula (20.64) is applicable. Thereby we get

$$\Gamma^{(2)\mu} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, 2\,\delta(x+y+z-1)\left(\frac{ie}{\hbar}\right)^2 \mu_0 \hbar c \cdot \cdot \int_0^{+\infty} \mathrm{d}R^2\, \frac{R^2}{16\pi^2}\, \frac{(R^2\gamma^\mu + J^\mu)}{(R^2+K^2)^3} \tag{26.65a}$$

$$J^\mu \equiv 2\Big((-1+4z-z^2)m^2c^2/\hbar^2 - (xy+z)k_\gamma^2\Big)\gamma^\mu - - 4(1-z)z\,\frac{m^2c^2}{\hbar^2}\,\frac{(k_\gamma)_\nu \hbar}{4mc}\,[\gamma^\nu, \gamma^\mu] \tag{26.65b}$$

$$K^2 \overset{(26.58b)}{=} -xyk_\gamma^2 + (1-z)^2 m^2c^2/\hbar^2 + zm_\varphi^2 c^2/\hbar^2\ . \tag{26.65c}$$

The integral

$$\int_0^{+\infty} \mathrm{d}R^2\, \frac{R^2}{(R^2+K^2)^3} = \left(-\frac{1}{R^2+K^2} + \frac{K^2}{2(R^2+K^2)^2}\right)\Big|_0^\infty = \frac{1}{K^2} - \frac{K^2}{2K^4} = \frac{1}{2K^2}$$

does converge. In contrast, the integral

$$\int_0^{+\infty} \mathrm{d}R^2\, \frac{R^4}{(R^2+K^2)^3} = \left(\ln(R^2+K^2) + \frac{2K^2}{R^2+K^2} - \frac{K^4}{2(R^2+K^2)^2}\right)\Big|_0^\infty$$

is diverging logarithmically, as expected. To avoid the divergence, we regularize the integral according to the Pauli-Villars method by insertion of a counter-term:

$$L^2 \equiv (1-z)^2 M^2 c^2/\hbar^2 \qquad , \qquad \int_0^{+\infty} \mathrm{d}R^2\, \frac{R^4}{(R^2+K^2)^3} \ =$$

$$= \lim_{M\to\infty} \int_0^{+\infty} \mathrm{d}R^2 \left(\frac{R^4}{(R^2+K^2)^3} - \frac{R^4}{(R^2+L^2)^3} \right) =$$

$$= \lim_{M\to\infty} \left(-\ln\left(\frac{K^2}{L^2}\right) - \frac{2K^2}{K^2} + \frac{K^4}{2K^4} + \frac{2L^2}{L^2} - \frac{L^4}{2L^4} \right) = \lim_{M\to\infty} \ln\left(\frac{L^2}{K^2}\right)$$

Before the limit is taken, M is a large but finite mass. K^2 and L^2 become negligible for $R \to \infty$. Thus the integral's convergence is enforced. The result with the counterterm is

$$\Gamma^{(2)\mu} = -\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, \delta(x+y+z-1)\, \frac{\alpha}{2\pi} \left(\gamma^\mu \lim_{M\to\infty} \ln\left(\frac{L^2}{K^2}\right) + \frac{J^\mu}{2K^2} \right) . \tag{26.66}$$

Here the finestructure-constant $\alpha = e^2\mu_0 c/(4\pi\hbar) \approx 1/137$ has been inserted. In (26.53c) we defined the two form-factors

$$\Gamma^\mu \overset{(26.53c)}{=} \gamma^\mu F_1 + [\gamma^\nu, \gamma^\mu] \frac{(k_\gamma)_\nu \hbar}{4mc} F_2 .$$

Using

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, \delta(x+y+z-1)\, f(x,y,z) = \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y\, f(1-y-z,y,z)$$

we now can state the form-factors explicitly in the approximation $\Gamma^\mu \approx \gamma^\mu + \Gamma^{(2)\mu}$:

$$F_1 = 1 - \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \, \frac{\alpha}{2\pi} \left(\ln\left((1-z)^2\right) + \lim_{M\to\infty} \ln\left(\frac{M^2}{m^2}\right) - \right.$$
$$- \ln\left(-(1-y-z)y \, \frac{k_\gamma^2 \hbar^2}{m^2 c^2} + (1-z)^2 + z\frac{m_\varphi^2}{m^2} \right) +$$
$$\left. + \frac{(-1+4z-z^2)m^2c^2/\hbar^2 - ((1-y-z)y+z)k_\gamma^2}{-(1-y-z)yk_\gamma^2 + (1-z)^2 m^2 c^2/\hbar^2 + z m_\varphi^2 c^2/\hbar^2} \right) \tag{26.67a}$$

$$F_2 = \frac{\alpha}{2\pi} \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \, \left(\frac{2(1-z)z m^2 c^2/\hbar^2}{-(1-y-z)yk_\gamma^2 + (1-z)^2 m^2 c^2/\hbar^2 + z m_\varphi^2 c^2/\hbar^2} \right) \tag{26.67b}$$

F_1 would have an IR-divergence at $z \to 1$ for forward-scattering, i. e. for very weak scattering with $k_\gamma^2 \approx 0$, unless we had prevented that divergence by means of the small photon mass m_φ. Even though we continue to concentrate our evaluation to the vertex correction in t-channel scattering, in which $k_\gamma^2 = (k_3 - k_1)^2 < 0$ holds for the virtual photon, we must not simply ignore the possible value $k_\gamma^2 \approx 0$, because no detector can measure with arbitrary accuracy. Therefore it is reasonable to keep in mind the limit $k_\gamma^2 \approx 0$ (in the sense of immeasurably small k_γ^2) even in case of t-channel scattering. Note: The IR-divergence is not prevented by the fictive mass m_γ of that photon, which is transferring the momentum $-\boldsymbol{k}_\gamma$ to the muon, but by the fictive mass m_φ of the photon with momentum $\boldsymbol{k}_1 - \boldsymbol{k}$ in the graph (26.49):

$$F_1(k_\gamma^2 \approx 0) \overset{(26.67)}{=} 1 - \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \, \frac{\alpha}{2\pi} \left(\ln\left((1-z)^2\right) + \right.$$
$$\left. + \lim_{M\to\infty} \ln\left(\frac{M^2}{m^2}\right) - \ln\left((1-z)^2\right) + \frac{(-1+4z-z^2)m^2}{(1-z)^2 m^2 + z m_\varphi^2} \right) =$$
$$= 1 - \frac{\alpha}{2\pi} \left(\frac{1}{2} \ln\left(\frac{M^2}{m^2}\right) + \int_0^1 \mathrm{d}z \, (1-z) \frac{(-1+4z-z^2)}{(1-z)^2 + z m_\varphi^2/m^2} \right)$$

The small term with m_φ^2/m^2 has been neglected in the logarithm. A

good approximate estimation of the integral's value can be found, if the divergence-preventing term in the denominator is replaced by an appropriate modification of the upper limit in the integral over z. As z and m_φ^2/m^2 both are showing up in the integrand's denominator quadratically, we replace the integration limit 1 by $1 - m_\varphi/m$ with linear m_φ/m:

$$\int_0^{1-\frac{m_\varphi}{m}} \mathrm{d}z\, \frac{(1-z)(-1+4z-z^2)}{(1-z)^2} = \int_0^{1-\frac{m_\varphi}{m}} \mathrm{d}z \left(-1+z+\frac{2z}{1-z}\right) =$$
$$= -\left(1-\frac{m_\varphi}{m}\right) + \frac{1}{2}\left(1-\frac{m_\varphi}{m}\right)^2 - 2\left(1-\frac{m_\varphi}{m}\right) - 2\ln\left(\frac{m_\varphi}{m}\right)$$

Because of $m_\varphi/m \ll 1$ we thus get

$$\frac{1}{Z_1} \equiv F_1(k_\gamma^2 \approx 0) \overset{(26.67)}{=} 1 - \frac{\alpha}{2\pi}\left(\frac{1}{2}\lim_{M\to\infty} \ln\left(\frac{M^2}{m^2}\right) - \frac{5}{2} + \ln\left(\frac{m^2}{m_\varphi^2}\right)\right) . \tag{26.68}$$

We may ignore the IR-divergence at $m_\varphi \to 0$, because it is compensated — as stated in section 24.3.6 — by a graph with bremsstrahlung of immeasurably low frequency. And by a further charge renormalization the parameter $\lim_{M\to\infty} \ln(M^2/m^2)$ can be eliminated. According to rule D of box 24.1 a factor $-iq\gamma^\mu/\hbar$ shall be inserted for each vertex. We just have computed in fourth order

$$\Gamma^\mu = \gamma^\mu + \Gamma^{(2)\mu} + \mathcal{O}(q^5) \overset{(26.53c)}{=} \gamma^\mu F_1 + [\gamma^\nu, \gamma^\mu]\, \frac{(k_\gamma)_\nu \hbar}{4mc}\, F_2 \ . \tag{26.69}$$

γ^μ is multiplied by $F_1(k_\gamma^2 \approx 0) = 1/Z_1$. Thus we can eliminate the factor $\lim_{M\to\infty} \ln(M^2/m^2)$ from our formulas due to

$$\boxed{\frac{q_0}{Z_1} \xrightarrow{\text{renormalization}} q} \tag{26.70}$$

with q_0 being the charge before and q being the charge after renormalization. In the previous section we performed in (26.46) the charge renormalization

$$Z_2 \cdot q_0 \xrightarrow{\text{renormalization}} q$$
$$Z_2 \overset{(26.45)}{=} 1 + \frac{\alpha_q}{2\pi}\left(\frac{1}{2}\lim_{U\to\infty} \ln\left(\frac{U^2}{m^2}\right) + \frac{7}{4} - \ln\left(\frac{m^2}{m_\gamma^2}\right)\right) ,$$

to eliminate the large parameter U, which is showing up in the electron's self-energy graph. (To make the parameters discernible, which have been inserted for regularization, we have renamed here the M of (26.45) to U.) We compute the product of Z_2 and (26.68):

$$\begin{aligned}\frac{Z_2}{Z_1} &= \left[1 + \frac{\alpha_q}{2\pi}\left(\frac{1}{2}\lim_{U\to\infty} \ln\left(\frac{U^2}{m^2}\right) + \frac{7}{4} - \ln\left(\frac{m^2}{m_\gamma^2}\right)\right)\right] \cdot \\ &\cdot \left[1 - \frac{\alpha}{2\pi}\left(\frac{1}{2}\lim_{M\to\infty} \ln\left(\frac{M^2}{m^2}\right) - \frac{5}{2} + \ln\left(\frac{m^2}{m_\varphi^2}\right)\right)\right] = \\ &= 1 + \mathcal{O}(\alpha^3) \quad \text{because of} \quad \frac{\lim_{U\to\infty} \ln(U/m) + 7/4}{\lim_{M\to\infty} \ln(M/m) - 5/2} = 1 \end{aligned} \tag{26.71}$$

The neglected term is $\mathcal{O}(\alpha^3)$, because our computations of Z_2 and Z_1 already are $\mathcal{O}(\alpha^2)$. The two IR-divergent terms have been ignored because of (24.101). Thus the two renormalizations of charge, which we have performed because of the fermion self-energy and because of the vertex correction, mutually compensate, and the ratio of renormalized and bare charge is determined by vacuum polarization alone:

$$\sqrt{Z_3} \cdot \frac{Z_2}{Z_1} \cdot q_0 = \sqrt{Z_3} \cdot q_0 \xrightarrow{\text{renormalization}} q \tag{26.72}$$

Due to the renormalization (26.70) (and the neglect of the IR-divergent term) $F_1(k_\gamma^2 \approx 0)$ assumes the simple value

$$F_1(k_\gamma^2 \approx 0) = 1 + \mathcal{O}(\alpha^3) \; . \tag{26.73a}$$

Different from F_1, the form-factor F_2 neither contains the regularization-parameter M, nor an IR-divergent term. We compute F_2 as well for the case of forward scattering:

$$F_2(k_\gamma^2 \approx 0) = \frac{\alpha}{2\pi} \int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}y \, \frac{2z}{(1-z)} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^3) \tag{26.73b}$$

Now we can compute the electron's Landé-factor:

$$\begin{aligned} g &\overset{(26.54b)}{=} 2\Big(F_1(k_\gamma^2 \approx 0) + F_2(k_\gamma^2 \approx 0)\Big) = \\ &= 2\Big(1 + \frac{\alpha}{2\pi}\Big) = 2 \cdot 1.001\,161\,4 + \mathcal{O}(\alpha^3) \end{aligned} \tag{26.74}$$

The value found experimentally [55] is

$$g = 2 \cdot 1.001\,159\,652\,180\,73(28) \ .$$

The last two digits are to be interpreted as 73 ± 28. Obviously our computation is quite good, but still not good enough. Higher-order perturbation computations are needed, to check the correctness of the theory by comparison with the measured data.

After $F_1(k_\gamma^2 \approx 0)$ has been renormalized, there are no further diverging factors $\ln(M/m)$ and no further IR-divergences in $\Gamma^{(2)\mu}$, because $\Gamma^{(2)\mu}$ can be expanded in a Taylor series around $k_\gamma^2 \approx 0$ for any k_γ^2-value of measurable size:

$$\Gamma^{(2)\mu} = \sum_{n=0}^{\infty} (k_\gamma^2)^n W_n = \Gamma_0^{(2)\mu} + \sum_{n=1}^{\infty} (k_\gamma^2)^n W_n = \Gamma_0^{(2)\mu} + \Gamma_{\text{rest}}^{(2)\mu} \tag{26.75}$$

The term with M has disappeared from $\Gamma_{\text{Rest}}^{(2)\mu}$ due to subtraction. It is embodied only in $\Gamma_0^{(2)\mu}$. And k_γ^2 is measurably different from zero in $\Gamma_{\text{Rest}}^{(2)\mu}$. Thus there won't be any IR-divergence. All "problematic" terms have been concentrated in $\Gamma_0^{(2)\mu}$ — to be more precise: in $F_1(k_\gamma^2 \approx 0)$ — , and have either been eliminated due to renormalization, or have been ignored with reference to (24.101).

26.4 Ward-Takahashi Identity

In the previous sections we derived the two normalization factors

$$Z_2 = (26.45) \qquad \text{and} \qquad 1/Z_1 = (26.68)\ , \tag{26.76a}$$

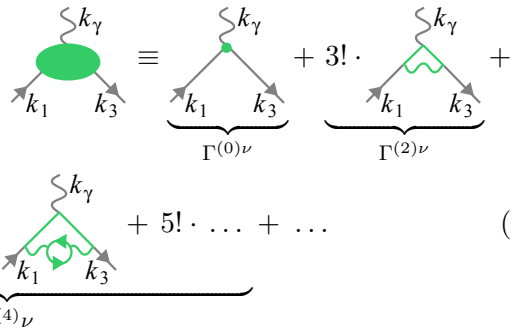
and detected the relation

$$\frac{Z_2}{Z_1} \overset{(26.71)}{=} 1 + \mathcal{O}(\alpha^2)\ . \tag{26.76b}$$

In (26.71) we have indicated $\mathcal{O}(\alpha^3)$, because there we considered the muon current as part of our computation. But in this section we will confine the evaluation to the vertex correction and to the self-energy of fermions, which we have both computed in the previous sections up to (and including) order $\mathcal{O}(q^2) = \mathcal{O}(\alpha)$.

The relation (26.76b) actually is but a special case of a relation between the vertex function

$$-\frac{iq\Gamma^\nu(k_1,k_3)}{\hbar} \overset{(26.48c)}{\widehat{=}}$$

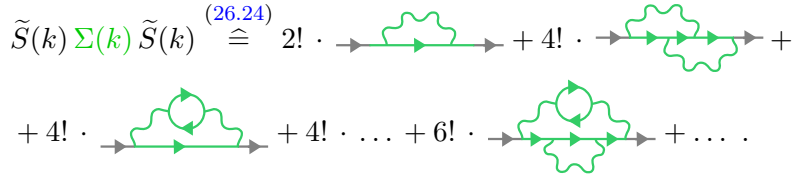


(26.77)

and the fermion self-energy

$$\widetilde{S}^{(W)}(k) \overset{(26.26)}{=} \frac{i}{\not{k} - m_0\frac{c}{\hbar} - i\Sigma}\ , \tag{26.78}$$

$$\widetilde{S}(k)\,\Sigma(k)\,\widetilde{S}(k) \overset{(26.24)}{\widehat{=}}$$



That relation has been detected by Ward[2] and has been generalized by

[2] John Clive Ward (1924 – 2000)

Takahashi[3]. To improve clarity, here those parts of the graphs, which are representing $-iq\Gamma^\nu/\hbar$ resp. Σ, have been highlighted by green color. The green part of the first diagram in the last equation is $\Sigma^{(2)}(k)$, the green parts of the two next diagrams belong to $\Sigma^{(4)}(k)$, and the green part of the last diagram belongs to $\Sigma^{(6)}(k)$.

Takahashi [56] proved in 1957 the relation

$$\boxed{(k_3-k_1)_\mu\Gamma^\mu(k_3,k_1)=\frac{i}{\widetilde{S}^{(W)}(k_3)}-\frac{i}{\widetilde{S}^{(W)}(k_1)}}\ , \tag{26.79a}$$

which is also (but not only) valid, if the fermions coming in and going out at the vertex with k_1 and k_3 are not on mass-shell, but are inner lines of a more extensive graph.

We insert the fermion propagator $\widetilde{S}^{(W)}(k) =$ (26.78) into the relation (26.79a):

$$(k_3-k_1)_\mu\Gamma^\mu(k_3,k_1)=\not{k}_3-\not{k}_1-i\Sigma(k_3)+i\Sigma(k_1)$$

Because of $(k_3-k_1)_\mu\Gamma^{(0)\mu}(k_3,k_1)\overset{(26.77)}{=}(k_3-k_1)_\mu\gamma^\mu=\not{k}_3-\not{k}_1$ the relation may be written in the form

$$\sum_n(k_3-k_1)_\mu\Gamma^{(n)\mu}(k_3,k_1)=-i\sum_n\left(\Sigma^{(n)}(k_3)-\Sigma^{(n)}(k_1)\right)$$
$$\text{with } n=2,4,6,\ldots,\infty\ . \tag{26.79b}$$

In the limit

$$k_3\to k_1\quad\Longleftrightarrow\quad k_\gamma^2=(k_3-k_1)^2\approx 0\ ,$$

(26.79a) can be interpreted as a differential quotient:

$$\lim_{k_3\to k_1}\Gamma^\mu(k_3,k_1)=\lim_{k_3\to k_1}\frac{i/\widetilde{S}^{(W)}(k_3)-i/\widetilde{S}^{(W)}(k_1)}{(k_3-k_1)_\mu}$$
$$\Gamma^\mu(k_\gamma\approx 0)=\frac{\mathrm{d}}{\mathrm{d}\,k_\mu}\frac{i}{\widetilde{S}^{(W)}(k)} \tag{26.80a}$$

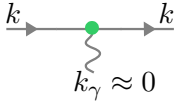
[3] Takahashi Yasushi (1924 - 2013)

Because of $\Gamma^{(0)\mu}(k_\gamma \approx 0) \overset{(26.77)}{=} \gamma^\mu = \mathrm{d}\not{k}/\mathrm{d}k_\mu$, this can be written as

$$\sum_n \Gamma^{(n)\mu}(k_\gamma \approx 0) \overset{(26.26)}{=} -i \sum_n \frac{\mathrm{d}\Sigma^{(n)}(k)}{\mathrm{d}\,k_\mu}$$
$$\text{with } n = 2, 4, 6, \ldots, \infty \; . \tag{26.80b}$$

(26.80) is the form, in which Ward by 1950 published [57] the relation between the vertex function Γ and the fermion propagator resp. the function Σ of fermion self-energy. (26.80) can be visualized by means of Feynman-graphs. In zeroth order

$$-\frac{iq\Gamma^{(0)\mu}}{\hbar} = -\frac{iq\gamma^\mu}{\hbar} \overset{(26.80a)}{=} \frac{q}{\hbar}\frac{\mathrm{d}}{\mathrm{d}\,k_\mu}\frac{1}{\widetilde{S}^{(0)}(k)} = \frac{q}{\hbar}\frac{\mathrm{d}}{\mathrm{d}\,k_\mu}\frac{1}{\widetilde{S}(k)} \mathrel{\widehat{=}}$$

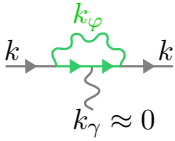
$\mathrel{\widehat{=}}$ 

$= \frac{q}{\hbar}\frac{\mathrm{d}}{\mathrm{d}\,k_\mu}\Big($ 

$\Big)^{-1}$.

In second order we find

$$-\frac{iq\Gamma^{(2)\mu}(k_\gamma \approx 0)}{\hbar} \overset{(26.80b)}{=} \frac{q}{\hbar}\frac{\mathrm{d}\Sigma^{(2)}(k)}{\mathrm{d}\,k_\mu} \mathrel{\widehat{=}}$$

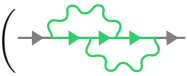
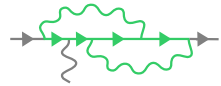
$\mathrel{\widehat{=}}$ 

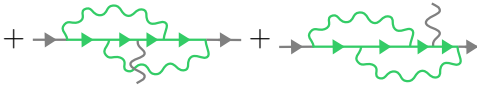
$= \frac{q}{\hbar}\frac{\mathrm{d}}{\mathrm{d}\,k_\mu}$ 

.

In fourth order we illustrate only one of several graphs belonging to $\Sigma^{(4)}$:

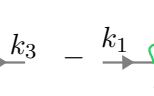
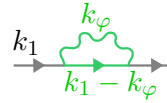
$$\frac{q}{\hbar}\frac{\mathrm{d}\Sigma^{(4)}(k)}{\mathrm{d}\,k_\mu} \overset{(26.80b)}{=} -\frac{iq\Gamma^{(4)\mu}(k_\gamma \approx 0)}{\hbar} \mathrel{\widehat{=}}$$

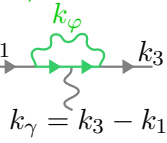
$\mathrel{\widehat{=}} \frac{q}{\hbar}\frac{\mathrm{d}}{\mathrm{d}\,k_\mu}\Big($  $+\ldots\Big) =$  $+$



Thus the derivative of Σ with respect to k_μ does mean, that into one fermion propagator of Σ each a vertex with a photon line $k_\gamma \approx 0$ shall be inserted, and then a summation over all graphs shall be performed. Takahashi's relation (26.79) can be illustrated by graphs as well. For example we get in second order

$$\frac{q}{\hbar}\Big(\Sigma^{(2)}(k_3) - \Sigma^{(2)}(k_1)\Big) \overset{(26.79b)}{=} -\frac{iq\Gamma^{(2)\mu}(k_3, k_1)}{\hbar} \cdot (k_3 - k_1)_\mu \mathrel{\widehat{=}}$$

$\mathrel{\widehat{=}} \frac{q}{\hbar}\Big($  $-$  $\Big) =$

$=$  $\cdot\,(k_3 - k_1)_\mu\,.$

We will not prove the relation (26.79). Instead we will apply it to prove (26.76b). Inserting the Taylor expansion

$$\Sigma(k) \overset{(26.27)}{=} \underbrace{\Sigma^{(2)}\Big|_{\not{k}=m_0c/\hbar}}_{A} + (\not{k} - m_0c/\hbar)\underbrace{\frac{\partial\Sigma^{(2)}}{\partial\not{k}}\Big|_{\not{k}=m_0c/\hbar}}_{B} + \mathcal{O}(q^4)$$

into

$$\Gamma^\mu(k_\gamma \approx 0) - \gamma^\mu \overset{(26.80b)}{=} -i\,\frac{\mathrm{d}\Sigma^{(2)}(k)}{\mathrm{d}\,k_\mu} + \mathcal{O}(q^4)$$

$$\Gamma^\mu(k_\gamma \approx 0) = \gamma^\mu(1 - iB) + \mathcal{O}(q^4) \ ,$$

we find by means of

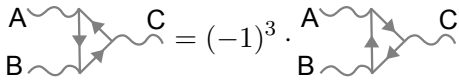
$$\Gamma^\mu(k_\gamma \approx 0) \overset{(26.53c)}{=} \gamma^\mu F_1(k_\gamma^2 \approx 0) \overset{(26.68)}{=} \frac{\gamma^\mu}{Z_1}$$
$$Z_2 \overset{(26.29)}{=} 1 + iB \overset{(26.28)}{=} \frac{1}{1 - iB} + \mathcal{O}(q^4)$$

the same result as in (26.76b):

$$\Gamma^\mu(k_\gamma^2 \approx 0) = \frac{\gamma^\mu}{Z_1} = \frac{\gamma^\mu}{Z_2} + \mathcal{O}(q^4) \tag{26.81}$$

26.5 Fermion Loops

The fourth entry in the listing (25.1) of the five possibly divergent QED graphs is a loop of three fermion lines. We now are going to prove, that any diagram containing this structure may be discarded from all computations from the outset.



(26.82)

If a matrix element can be contracted to the graph displayed left, then it can be contracted as well to the graph shown on the right side. Thereby it doesn't matter whether (some of) the items symbolized by A,B,C are further parts of the graph, or whether (some of) the three photon lines are external lines. The right graph differs from the left graph by nothing than the reversed arrows on the fermion lines. These both graphs must (besides other contractions) be added coherently to the overall result. We will prove in the sequel that the right graph differs from the left graph by a factor $(-1)^3$. Thus the sum of these both graphs is zero. Generally the

Theorem: The value of a graph, which is containing a loop of n fermion lines and zero boson lines, changes by the factor $(-1)^n$ if the arrows of the fermion-lines are reversed. (26.83)

holds, which is known as "Furry's theorem". Therefore all graphs which are containing a pure fermion loop consisting of an odd number of lines may be discarded upfront from all computations. To prove this theorem we note, that in the computation of a loop, which is consisting of n fermion lines and zero boson lines, according to the rules of the boxes 24.1 and 26.1 besides other the factors

$$\underbrace{\cdots \frac{ie\gamma^\alpha}{\hbar} \cdot \frac{i(\gamma^\sigma k_{1\sigma} + mc/\hbar)}{k_1^2 - m^2c^2/\hbar^2 + i\epsilon'} \cdots \frac{ie\gamma^\nu}{\hbar} \cdot \frac{i(\gamma^\tau k_{n\tau} + mc/\hbar)}{k_n^2 - m^2c^2/\hbar^2 + i\epsilon'} \cdots}_{n \text{ vertexfactors} \cdot n \text{ propagators}}$$

will show up. The factors in the denominator commute, but not the factors in the numerator. Besides not interesting constants, the numerators have the form

$$\begin{aligned} &\gamma^\alpha_{ab}(\gamma^\sigma_{bc} k_{1\sigma} + \mathbb{1}_{bc} mc/\hbar) \ldots \gamma^\nu_{vw}(\gamma^\tau_{wa} k_{n\tau} + \mathbb{1}_{wa} mc/\hbar) = \\ &= \mathrm{tr}\Big\{\gamma^\alpha(\gamma^\sigma k_{1\sigma} + mc/\hbar) \ldots \gamma^\nu(\gamma^\tau k_{n\tau} + mc/\hbar)\Big\} \,. \end{aligned} \tag{26.84a}$$

In the first line the spinor-indices have been indicated explicitly. The sequence of factors is determined by the rearrangement-operator $\mathcal{U}_\mathrm{S} =$ (23.21). As we are considering a closed loop of fermion lines with no defined starting point, the last and the first spinor index of the product must be identical, i. e. the trace of the product must be computed.

If the arrows on the fermion lines are reversed, then we get the trace

$$\mathrm{tr}\Big\{(\gamma^\tau k_{n\tau} + mc/\hbar)\gamma^\nu \ldots (\gamma^\sigma k_{1\sigma} + mc/\hbar)\gamma^\alpha\Big\} \,, \tag{26.84b}$$

in which the sequence of factors is reversed. To compare this expression with (26.84a), we insert behind each factor in (26.84b) a factor

$$1 = D_C^{-1} D_C \equiv (\gamma^0\gamma^2)^{-1}\gamma^0\gamma^2 \,.$$

Remark: D_C is the representation of charge conjugation on the basis of spinors, which we have investigated in section 11.2.3. For the following proof it is irrelevant whether or not charge conjugation is a symmetry of the evaluated theory, because it goes without saying that we may at any time insert into any product of any theory the factors $1 = (\gamma^0\gamma^2)^{-1}\gamma^0\gamma^2$.

D_C does neither change the Lorentz-vectors k_j nor the constants $m\frac{c}{\hbar}$. But the γ-matrices are transformed by D_C according to

$$-D_C^{-1}\gamma^{\nu\sim}D_C \overset{(11.23)}{=} \gamma^\nu \quad \Longrightarrow \quad D_C^{-1}\gamma^\nu D_C = -\gamma^{\nu\sim} \ .$$

Here $\sim$ is indicating the transposed matrix. The factors may be cyclically permuted under the trace. Therefore

$$\begin{aligned}
(26.84b) &= \mathrm{tr}\Big\{(\gamma^\tau k_{n\tau} + mc/\hbar)D_C^{-1}D_C\gamma^\nu D_C^{-1}D_C \ldots \\
&\qquad \ldots(\gamma^\sigma k_{1\sigma} + mc/\hbar)D_C^{-1}D_C\gamma^\alpha D_C^{-1}D_C\Big\} = \\
&= \mathrm{tr}\Big\{(D_C\gamma^\tau D_C^{-1}k_{n\tau} + mc/\hbar)D_C\gamma^\nu D_C^{-1} \ldots \\
&\qquad \ldots(D_C\gamma^\sigma D_C^{-1}k_{1\sigma} + mc/\hbar)D_C\gamma^\alpha D_C^{-1}\Big\} = \\
&= \Big((-1)^4\Big)^n(-1)^n\,\mathrm{tr}\Big\{(\gamma^{\tau\sim}k_{n\tau} + mc/\hbar)\gamma^{\nu\sim}\ldots(\gamma^{\sigma\sim}k_{1\sigma} + mc/\hbar)\gamma^{\alpha\sim}\Big\} = \\
&= (-1)^n\,\mathrm{tr}\Big\{\Big(\gamma^\alpha(\gamma^\sigma k_{1\sigma} + mc/\hbar)\ldots\gamma^\nu(\gamma^\tau k_{n\tau} + mc/\hbar)\Big)^\sim\Big\} = \\
&= (-1)^n \cdot (26.84a) \quad \text{because of } \mathrm{tr}\{A^\sim\} = \mathrm{tr}\{A\} \ . \qquad (26.85)
\end{aligned}$$

This proves theorem (26.83).

26.6 Photon-Photon Scattering

This is the last of the possibly diverging QED diagrams listed in table (25.1). By counting superficially the wavenumber-powers, a logarithmic divergence is to be expected. But actually the computation, which can be found in [58], gives a converging result for the scattering amplitude of photon-photon scattering. That's typical for diagrams with several external

photon lines. Already in the computation of vacuum polarization we found only a logarithmic divergence instead of the suspected quadratic divergence. The explanation is: The superficial estimation of the degree of divergence is based on the assumption, that any field has four mutually independent components. But this is not true in case of the photon field, due to it's gauge invariance. Therefore the actual degree of divergence is often lower than expected for graphs with many external photon lines. As we focus in this chapter on diverging diagrams and their renormalization, we will not compute photon-photon scattering, but refer to the cited literature.

Part 4: Non-Abelian Gauge-Theories

27 Flavor-Isospin

27.1 The Isospin of Nucleons

Both the proton and the neutron can be described as Dirac fields. If we ignore minor details, these fields only differ in their electromagnetic properties and slightly in their masses, while they are behaving almost identically with regard to nuclear forces. Therefore it is assumed in the isospin-model, that both nucleons would be identical particles if the electromagnetic interaction could be switched off. The idea was born in analogy to the Zeeman-effect: Some energy levels of atoms, which differ only by the projections of their spins onto a certain axis of position space, can be split due to a magnetic field applied from outside, but are degenerate without the external magnetic field. In analogy, the isospin model assumes that the doublet of nucleons is split due to electromagnetic forces, but would be degenerate — i. e. the two nucleons would have identical masses — if the electromagnetic forces would not exist.

A Zeeman-doublet is caused by an electron state with spin $S = \hbar/2$, whose projection onto the axis of the external magnetic field (which usually is chosen to be the x^3 axis) can be $S_3 = +\hbar/2$ or $S_3 = -\hbar/2$. In analogy an isospin $I = 1/2$ is assigned to the nucleon. Note that the isospin usually is defined as a dimension-less number, while the dimension of spin is angular momentum. By definition the state with $I^3 = +1/2$ is assigned to the proton, and the state with $I^3 = -1/2$ is assigned to the neutron. The isospin-space is an abstract space, which must not be confused with time-position space. The metric of isospin-space is euclidean (but not Minkowski-metric), and consequently there is no need to discern covariant and contravariant vectors. The state functions of nucleons can be written as products of a two-component iso-spinor, and a rest which is independent of isospin, and identical for proton and neutron:

$$\psi_{\text{proton}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \text{rest}_{\text{nucleon}} \qquad \psi_{\text{neutron}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \text{rest}_{\text{nucleon}} \tag{27.1}$$

This notation of the state functions of nucleons has been introduced by Heisenberg in 1932 [59]. In the sequel we will not explicitly indicate the factors "rest" any more.

The third of the Pauli-matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27.2}$$

is — besides the factor $1/2$ — equal to the I^3-operator of the nucleons. We will use for the Pauli-matrices, if they are acting as operators in isospin-space, the notation $\tau^j \equiv \sigma^j$, to avoid confusion of isospin and spin. The eigenvalue-equations of the operator I^3 are

$$I^3\,\psi_{\text{proton}} \equiv \frac{\tau^3}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}\,\psi_{\text{proton}} \tag{27.3a}$$

$$I^3\,\psi_{\text{neutron}} \equiv \frac{\tau^3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\,\psi_{\text{neutron}}\ . \tag{27.3b}$$

As usual, we are using identical notation for operators and their eigenvalues, e. g. $I^3 = \tau^3/2$ is an operator, and $I^3 = \pm 1/2$ are it's eigenvalues. The ladder-operators $I^1 \pm iI^2 = \frac{1}{2}(\tau^1 \pm i\tau^2)$ convert protons to neutrons, and vice versa:

$$\frac{1}{2}(\tau^1 + i\tau^2)\,\psi_{\text{neutron}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi_{\text{proton}} \tag{27.3c}$$

$$\frac{1}{2}(\tau^1 - i\tau^2)\,\psi_{\text{proton}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \psi_{\text{neutron}} \tag{27.3d}$$

27.2 Global Phase-Transformations

In section 4.4 we applied in equation (4.77) to the Dirac field the global phase transformation

$$\left.\begin{aligned}\psi'(x) &= U\,\psi(x) = e^{+\frac{i}{\hbar}Kq}\,\psi(x)\\ \overline{\psi}'(x) &= \overline{\psi}(x)\,U^\dagger = e^{-\frac{i}{\hbar}Kq}\,\overline{\psi}(x)\end{aligned}\right\} \quad \text{with } K, q \in \mathbb{R}\ . \tag{27.4}$$

This transformation shifts the phase of the field $\psi(x)$ in all points x of space-time by the same angle $Kq/\hbar$. The dimension of the generator q of this transformation is charge, and the dimension of the parameter K is action/charge.

Being a bilinear function of ψ and $\overline{\psi}$, the Lagrangian

$$\mathcal{L} = \overline{\psi}(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\psi = \overline{\psi}U^\dagger(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)U\psi = \mathcal{L}' \tag{27.5}$$

— and consequently also the field-equation — of the free Dirac field is invariant under the global phase transformation. According to Noether's theorem this invariance is correlating with a conserved current density, whose components we computed as

$$j^\rho \overset{(4.16)}{=} C\sum_r \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\phi_r)}\,kq\,\phi_r \quad \text{with} \quad \begin{cases}\phi_r = \psi \text{ or } \overline{\psi}\\ k \equiv \lim\limits_{n\to\infty} K/n\ ,\ n \in \mathbb{N}\ .\end{cases}$$

Defining $C \equiv 1/(i\hbar k)$ we eventually found the components of the conserved current density:

$$j^\rho = \frac{1}{i\hbar k}\left(\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\psi)}kq\psi - \overline{\psi}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\overline{\psi})}kq\right) \overset{(4.87)}{=} q\overline{\psi}c\gamma^\rho\psi \tag{27.6}$$

With the generator q being constant, the set of transformations

$$U \overset{(27.4)}{=} \exp\left\{\frac{i}{\hbar}Kq\right\} \quad , \quad K, q \in \mathbb{R} \tag{27.7}$$

with arbitrary K is a one-dimensional fundamental representation of the

abstract group U(1), see the listing of the systematic nomenclature of matrix groups on page 124 .

Now we are going to evaluate the global phase-transformations of the nucleon-state-functions (27.1). We have proved in section 6.1, that any element of the matrix group SU(2) — which furthermore is the fundamental representation of the abstract group SU(2) — can be written in the form

$$U(\boldsymbol{\Theta}) \overset{(6.17)}{=} \exp\left\{\frac{i}{\hbar}\Theta_k \frac{\hbar\sigma^k}{2}\right\} . \tag{27.8}$$

As explained in section 4.1, the three factors $\hbar\sigma^k/2$ are the generators of this representation, while the three real parameters Θ_k are specifying the respective elements of the group. The comparison of this equation with the global U(1)-phase-transformation (27.7) is suggesting to define the operator

$$g\boldsymbol{I} = g\left(\frac{\tau^1}{2}, \frac{\tau^2}{2}, \frac{\tau^3}{2}\right) \tag{27.9}$$

as three-dimensional isospin-charge-operator, in which the real one-dimensional factor g is a coupling constant with dimension isospin-charge. The isospin-charge-operator is the generator of the global phase transformation

$$\psi' = U\psi = \exp\left\{\frac{i}{\hbar}\tilde{K}g\right\}\psi \equiv \exp\left\{\frac{i}{\hbar}K_j g I^j\right\}\psi \tag{27.10a}$$

$$\text{with} \quad \tilde{K} \equiv K_j I^j = K_j \frac{\tau^j}{2} \quad , \quad K_j \in \mathbb{R} , \tag{27.10b}$$

in which the sum over j is running from 1 to 3. ψ = (27.1) is the state function of the nucleons. It has two isospin components. The dimension of the real parameter-vector $\boldsymbol{K}$ is action/(isospin-charge). The 2×2-matrices in the exponent are interpreted due to the series expansion of the exponential function:

$$U\psi(x) = \exp\left\{\frac{i}{\hbar}K_j g I^j\right\}\psi = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}gK_j I^j\right)^n \psi \tag{27.11}$$

In each term, ψ is multiplied by the 2×2-matrix $(K_j I^j)^n = \tilde{K}^n$. By

$$U = \exp\left\{\frac{i}{\hbar}\, K_j g I^j\right\}\psi = \lim_{n\to\infty}\left(1 + \frac{i}{\hbar}\, g\, \underbrace{\frac{K_j}{n}}_{\equiv k_j}\, I^j\right)^n \quad \text{with } n \in \mathbb{N}$$

the infinitesimal phase transformation can be derived:

$$U_{\text{INF}} = 1 + \frac{i}{\hbar}\, g k_j I^j \tag{27.12}$$

As the phase transformation is unitary, $U^\dagger = U^{-1}$, the Lagrangian

$$\mathcal{L} = \overline{\psi}(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)\psi \tag{27.13}$$

of the nucleons is obviously invariant under the global phase transformation

$$\mathcal{L}' = \overline{\psi}U^\dagger(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)U\psi = \mathcal{L}\ . \tag{27.14}$$

Consequently there exists according to Noether's theorem a conserved current density with the four space-time components

$$j^\rho \overset{(27.6)}{=} C\left(\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\psi)}\, gK_l I^l\psi - \overline{\psi}gK_l I^l\, \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\rho\overline{\psi})}\right)\ . \tag{27.15}$$

As the three components K_l are mutually independent, the constant may be chosen as $CK_l \equiv \delta_{jl}/(i\hbar)$ with $j = 1, 2, 3$, and there are three conserved current densities with space-time components

$$j^{(j)\rho} = g\overline{\psi}c\gamma^\rho I^j\psi \quad \text{with } j = 1, 2, 3\ . \tag{27.16}$$

γ^ρ is a 4×4-matrix in spinor space. ψ is a four-component column spinor, $\overline{\psi}$ is a four-component row spinor. I^j is a 2×2-matrix in isospin-space. ψ is a two-component column-isospinor, $\overline{\psi}$ is a two-component row-isospinor. The factors have been arrange such, that in j^ρ all spinor-indices and all isospin-indices are contracted.

27.3 Local Phase-Transformations

Is the system of nucleons as well invariant under local SU(2)-phase-transformations? This question has been evaluated in 1954 by Yang and Mills [60]. In the local phase transformation

$$\begin{aligned}\psi'(x) = U(x)\,\psi(x) &= \exp\left\{\frac{i}{\hbar}\,\tilde{K}(x)\,g\right\}\psi(x)\\ &= \exp\left\{\frac{i}{\hbar}\,K_j(x)\,g I^j\right\}\psi(x)\ ,\end{aligned} \tag{27.17}$$

K_j may be chosen different at different space-time points x. We merely require the three fields $K_j(x)$ to be everywhere continuous and differentiable. We apply a local phase transformation to the Lagrangian:

$$\begin{aligned}\mathcal{L}' &= \overline{\psi}U^\dagger(i\hbar c\gamma^\rho \mathrm{d}_\rho - mc^2)U\psi\\ &= \overline{\psi}U^\dagger i\hbar c\gamma^\rho U\mathrm{d}_\rho\psi - \overline{\psi}U^\dagger mc^2 U\psi + \overline{\psi}U^\dagger i\hbar c\gamma^\rho(\mathrm{d}_\rho U)\psi\\ &= \mathcal{L} + \overline{\psi}U^\dagger i\hbar c\gamma^\rho(\mathrm{d}_\rho U)\psi = \mathcal{L} - \overline{\psi}c\gamma^\rho g(\mathrm{d}_\rho K_j)I^j\psi\end{aligned} \tag{27.18}$$

The Lagrangian is not invariant under locale phase transformations. This result corresponds to the result (4.96) of the local U(1)-phase-transformation.

In case of U(1) symmetry we could enforce invariance under local phase transformations due to replacing the normal differential operator d_ρ by the covariant differential operator

$$\mathrm{D}_\rho(x) \overset{(4.102)}{\equiv} \mathrm{d}_\rho + \frac{i}{\hbar}qA_\rho(x)\ , \tag{27.19}$$

and upgrading the local phase-transformation to the U(1) gauge-transformation (4.109):

$$\psi(x) \overset{U(x)}{\longrightarrow} \psi'(x) = U(x)\,\psi(x) = \psi(x)\,\exp\left\{\frac{i}{\hbar}\,K(x)\,q\right\} \tag{27.20a}$$

$$A_\rho(x) \overset{U(x)}{\longrightarrow} A'_\rho(x) = A_\rho(x) + \frac{i\hbar}{q}(\mathrm{d}_\rho U)U^\dagger = A_\rho(x) - \mathrm{d}_\rho K(x) \tag{27.20b}$$

We now will transfer this method of securing invariance under local phase transformations to the group SU(2). For that purpose we define the covariant differential operator

$$\mathrm{D}_\rho(x) \equiv \mathrm{d}_\rho + \frac{i}{\hbar} g \tilde{W}_\rho(x) \tag{27.21a}$$

$$\tilde{W}_\rho(x) \equiv W_{j\rho}(x)\, I^j = W_{j\rho}(x)\, \frac{\tau^j}{2} \ . \tag{27.21b}$$

The three charges gI^j are replacing the charge q of (27.19), and the three gauge-fields $W_j(x)$ are replacing in the gauge-group SU(2) the gauge-field $A(x)$ of the gauge-group U(1). $\tilde{W}(x)$ is a vector field with four space-time components. As the nucleon state-function ψ, onto which $\tilde{W}(x)$ is acting, has two isospin-components, we define each single space-time component $\tilde{W}_\rho(x)$ as a 2×2 isospin-matrix. This makes our ansatz as generic as possible. The four space-time components $W_{j\rho}(x)$ of each of the three gauge-fields $W_j(x)$ are one-component numbers with dimension momentum/(isospin-charge). Often $\tilde{W}(x)$ is called gauge-field as well.

By definition, a gauge transformation is acting onto $\tilde{W}(x)$ like this:

$$\tilde{W}'_\rho = \underbrace{U\tilde{W}_\rho U^\dagger}_{W_{j\rho} U \frac{\tau^j}{2} U^\dagger} + \frac{i\hbar}{g} (\mathrm{d}_\rho U) U^\dagger \tag{27.22}$$

Different from the simple form (27.20b) in case of U(1) gauge-symmetry, the phase-transformations U do not commute with their generators $\tau^j/2$. Therefore we can not explicitly compute the derivative of U with respect to x^ρ, because for that purpose we would need to expand U in the series (27.11), and we would not be able to factorize an exponential function from the derivative of that series.

$\tilde{W}'_\rho$ has been defined like this to secure the following transformation of the covariant differential operator under gauge-transformations:

$$\mathrm{D}'_\rho\,\psi' = \mathrm{D}'_\rho\,U\psi = \mathrm{d}_\rho U\psi + \frac{i}{\hbar}\,g\Big(\underbrace{U\tilde{W}_\rho U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\rho U)U^\dagger}_{\tilde{W}'_\rho}\Big)\,U\psi =$$

$$= (\mathrm{d}_\rho U)\psi + U\mathrm{d}_\rho\psi + \frac{i}{\hbar}\,gU\tilde{W}_\rho\psi - (\mathrm{d}_\rho U)\psi =$$

$$= U\,\mathrm{D}_\rho\,\psi \quad\Longrightarrow\quad \mathrm{D}'_\rho = U\,\mathrm{D}_\rho\,U^\dagger \tag{27.23}$$

This is corresponding to the transformation (4.100) of the covariant differential operator of the U(1) gauge-group. Due to this transformation the Lagrangian is invariant under gauge transformations, provided the normal differential operator d_ρ is replaced by the covariant differential operator D_ρ:

$$\mathcal{L}' = \overline{\psi}U^\dagger(i\hbar c\gamma^\rho\,\mathrm{D}'_\rho - mc^2)U\psi = \overline{\psi}(i\hbar c\gamma^\rho U^\dagger\,\mathrm{D}'_\rho\,U - mc^2)\psi =$$

$$= \overline{\psi}(i\hbar c\gamma^\rho\,\mathrm{D}_\rho - mc^2)\psi = \mathcal{L} \tag{27.24}$$

While the derivative $\mathrm{d}_\rho U$ in (27.22) can not be evaluated explicitly for arbitrary phase transformations U, it can be computed for the infinitesimal transformation $U_{\text{INF}} =$ (27.12):

$$\tilde{W}'_\rho \overset{(27.22)}{=} (1 + \frac{i}{\hbar}\,gk_kI^k)\tilde{W}_\rho(1 - \frac{i}{\hbar}\,gk_lI^{l\dagger}) + \frac{i\hbar}{g}(\mathrm{d}_\rho\frac{i}{\hbar}\,gk_kI^k)(1 - \frac{i}{\hbar}\,gk_lI^{l\dagger}) \tag{27.25}$$

The Pauli-matrices (27.2) are self-adjoint, i. e. $I^{l\dagger} = I^l = \tau^l/2$. Neglecting terms $\mathcal{O}(k^2)$, to which also $k_l\mathrm{d}_\rho k_k$ belongs, we find

$$\tilde{W}'_\rho = \tilde{W}_\rho + \frac{i}{\hbar}\,g\left[k_k\frac{\tau^k}{2}, \tilde{W}_\rho\right] - \mathrm{d}_\rho k_j\frac{\tau^j}{2}\ . \tag{27.26}$$

Considering (27.21b), therefore

$$W'_{j\rho}\frac{\tau^j}{2} = W_{j\rho}\frac{\tau^j}{2} - \mathrm{d}_\rho k_j\frac{\tau^j}{2} + \frac{i}{\hbar}\,gk_kW_{l\rho}\left[\frac{\tau^k}{2}, \frac{\tau^l}{2}\right] =$$

$$= W_{j\rho}\frac{\tau^j}{2} - \mathrm{d}_\rho k_j\frac{\tau^j}{2} - \frac{1}{\hbar}\,g\,\epsilon^{klj}k_kW_{l\rho}\,\frac{\tau^j}{2} \tag{27.27}$$

holds for the components $W_{j\rho}$ of the three gauge fields. In the last step the well-known commutator

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] \overset{(6.24)}{=} i\epsilon^{jkl}\frac{\sigma^l}{2} \tag{27.28}$$

has been applied, in which ϵ^{jkl} is the totally antisymmetric Levi-Cività tensor. In the language of group theory, the factors ϵ^{jkl} are the structure-constants of the group SU(2). The common factor $\tau^j/2$ in (27.27) can be canceled:

$$W'_{j\rho} = W_{j\rho} - \mathrm{d}_\rho k_j - \frac{g}{\hbar}\,\epsilon^{klj} k_k W_{l\rho} \tag{27.29a}$$

The combination of this infinitesimal transformation resp. of the finite transformation in the isospin-matrix-form

$$\tilde{W}'_\rho \overset{(27.22)}{=} U\tilde{W}_\rho U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\rho U)U^\dagger \tag{27.29b}$$

with the local phase transformation

$$\psi'(x) = U(x)\,\psi(x) \overset{(27.17)}{=} \exp\left\{\frac{i}{\hbar}\,gK_j(x)\frac{\tau^j}{2}\right\}\psi(x) \tag{27.29c}$$

of the field ψ is the SU(2) gauge-transformation. It is instructive to compare it with the U(1) gauge-transformation (27.20). An additional term is showing up in (27.29a) resp. in (27.29b), because the generators of the phase transformations do not commute, and thus the structure constants of the group SU(2) are different from zero — in contrast to the structure constants of the group U(1). Furthermore in local phase transformations of nucleons we are dealing with three parameter fields $K_j(x)$ and three gauge fields $W_j(x)$, because the group of these transformations is identical to the matrix group SU(2), which is based on three generators (that is charges) $gI^j = g\tau^j/2$. Without proof we state the general rule: The matrix group SU(n) with arbitrary $n \geq 2$ has $n^2 - 1$ generators. Consequently the postulate of local gauge invariance of n-dimensional charged fields is leading to $n^2 - 1$ gauge fields.

When we evaluated the U(1) gauge-transformations in section 4.5 it

turned out that the field-equations of the gauge-field $A(x)$ are almost completely fixed by the postulate of gauge-invariance. We found this combined Lagrangian of the Dirac-field and the U(1) gauge-field:

$$\mathcal{L} \overset{(4.120)}{=} \overline{\psi}\Big(i\hbar c\gamma^\rho(\mathrm{d}_\rho + \frac{i}{\hbar}qA_\rho) - mc^2\Big)\psi - \frac{1}{4\mu_0}F_{\sigma\tau}F^{\sigma\tau}$$
$$F_{\sigma\tau} \equiv \mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma \tag{27.30}$$

Now we are going to extend the Lagrangian (27.24) of the nucleon field such, that we get dynamic field-equations for the three gauge-fields W_j as well. Clearly we must take care that the gauge-invariance will not be damaged by that extension. We assert that the following Lagrangian is the gauge-invariant generalization of (27.30) to the gauge-group SU(2):

$$\mathcal{L} \equiv \overline{\psi}(i\hbar c\gamma^\rho \,\mathrm{D}_\rho - mc^2)\psi - \frac{1}{2}\,\mathrm{tr}\{\tilde{F}_{\sigma\tau}\tilde{F}^{\sigma\tau}\} \tag{27.31a}$$

$$\tilde{F}_{\sigma\tau} \equiv \mathrm{D}_\sigma \tilde{W}_\tau - \mathrm{D}_\tau \tilde{W}_\sigma \equiv F_{j\sigma\tau}\frac{\tau^j}{2} \equiv (\mathrm{D}_\sigma W_{j\tau} - \mathrm{D}_\tau W_{j\sigma})\frac{\tau^j}{2} \tag{27.31b}$$

The field-strength tensor is now defined with covariant derivatives. In case of the gauge-group U(1) it doesn't matter whether the normal or the covariant derivative is applied, because of

$$F_{\sigma\tau} = \mathrm{D}_\sigma A_\tau - \mathrm{D}_\tau A_\sigma \overset{(27.19)}{=} \mathrm{d}_\sigma A_\tau - \mathrm{d}_\tau A_\sigma + \frac{i}{\hbar}q\,\underbrace{[A_\sigma, A_\tau]}_{0}\,. \tag{27.32}$$

In case of the gauge-group SU(2), however, the application of the covariant derivative is compulsory to make the Lagrangian $\mathcal{L}$ gauge-invariant. In appendix A.25

$$\tilde{F}'_{\sigma\tau} \overset{(A.183)}{=} U\tilde{F}_{\sigma\tau}U^\dagger \tag{27.33}$$

is proved. Together with (27.24), this immediately implies the gauge-invariance of $\mathcal{L}$, because (as the trace is invariant under cyclic permutation of the factors)

$$\begin{aligned}\mathrm{tr}\{\tilde{F}'_{\sigma\tau}\tilde{F}'^{\sigma\tau}\} &= \mathrm{tr}\{U\tilde{F}_{\sigma\tau}U^\dagger U\tilde{F}^{\sigma\tau}U^\dagger\} = \\ &= \mathrm{tr}\{U^\dagger U\tilde{F}_{\sigma\tau}U^\dagger U\tilde{F}^{\sigma\tau}\} = \mathrm{tr}\{\tilde{F}_{\sigma\tau}\tilde{F}^{\sigma\tau}\}\ .\end{aligned}$$

The trace may as well be written in the form

$$\begin{aligned}&\mathrm{tr}\{\tilde{F}_{\sigma\tau}\tilde{F}^{\sigma\tau}\} = F_{j\sigma\tau}F_k^{\sigma\tau}\mathrm{tr}\Big\{\frac{\tau^j}{2}\frac{\tau^k}{2}\Big\} = \\ &= \frac{1}{4}F_{j\sigma\tau}F_k^{\sigma\tau}\mathrm{tr}\Big\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\delta_{jk} + i\epsilon^{jkl}\tau^l\Big\} = \frac{1}{4}F_{j\sigma\tau}F_j^{\sigma\tau}\cdot 2\ .\end{aligned} \tag{27.34}$$

The terms with different j and k here summed up to zero. Thereby the Lagrangian (27.31) gets the form

$$\mathcal{L} = \overline{\psi}(i\hbar c\gamma^\rho \mathrm{D}_\rho - mc^2)\psi - \frac{1}{4}F_{j\sigma\tau}F_j^{\sigma\tau}\ , \tag{27.35}$$

in which the similarity to the U(1)-Lagrangian (27.30) becomes very clear. In both cases the product of the field-strength tensors is multiplied by $-1/4$. Only the factor μ_0 is missing here, and the sum is now over $j = 1, 2, 3$.

The three gauge-fields $W_j(x)$ are not observed experimentally. Thus the isospin-charges ascribed to the nucleons are — different from electrical charges — *not* the sources of gauge-fields, which mediate interactions between the particles. Still the efforts, which we (respectively Yang and Mills) have invested into the exploration of the SU(2) gauge-interaction, were not in vain. All our results can be used (with slight modifications) in quantum chromodynamics, which is a SU(3) gauge-interaction. And we will make use of the group SU(2) for the description of weak interactions.

In the "standard model" of elementary particles, isospin-charges are not assigned to nucleons or other hadrons, but to quarks, which are assumed to be their constituents. While the up-quark (about $4\,\mathrm{MeV}/c^2$) and the down-quark (about $7\,\mathrm{MeV}/c^2$) have approximately the same mass, isospin-symmetry (which often is called flavor-symmety) is significantly broken in case of the other quarks (strange about $150\,\mathrm{MeV}/c^2$, charm about $1\,\mathrm{GeV}/c^2$, bottom about $4\,\mathrm{GeV}/c^2$, top about $175\,\mathrm{GeV}/c^2$). Thus flavor-symmetry, which is based on isospin-charges, can define some useful schematic order at least for the hadrons built from light quarks, but is hardly applicable to

characterize hadrons built from heavy quarks.

In contrast, the symmetry of quarks which is based on their color-charges, turned out to be a strictly conserved symmetry. We will evaluate that symmetry in the next chapter.

28 Quantum Chromodynamics

28.1 Quarks and Gluons

Quantum chromodynamics is based on the assumption, that all hadrons are made up by quarks and gluons. Quarks are fermions with an electrical charge, a color charge, and a weak charge. Gluons are boson gauge fields, and their sources are the color charges. Some important properties of quarks and gluons are listed in table 28.1. The color charges separated by commas are to be understood as alternatives, i. e. each quark flavor exists with three different color charges, and there are eight gluons with different color charges. The color charges are named r = red, g = green, b = blue, $\bar{r}$ = anti-red, $\bar{g}$ = anti-green, $\bar{b}$ = anti-blue. It is understood that these names are not related to the colors of everyday language. The anti-color charges are the charges of antiquarks, and e is the electrical charge of the positron.

In (27.1) we described the proton and the neutron as an isospin-doublet. In the same manner we can for example describe the three up-quarks with

flavor	spin	electr. charge	color charge	mass
down	1/2	$-e/3$	$r\,,\ g\,,\ b$	$\approx 7.5\,\mathrm{MeV}/c^2$
up	1/2	$+2e/3$	$r\,,\ g\,,\ b$	$\approx 4.2\,\mathrm{MeV}/c^2$
strange	1/2	$-e/3$	$r\,,\ g\,,\ b$	$\approx 150\,\mathrm{MeV}/c^2$
charme	1/2	$+2e/3$	$r\,,\ g\,,\ b$	$\approx 1.1\,\mathrm{GeV}/c^2$
bottom	1/2	$-e/3$	$r\,,\ g\,,\ b$	$\approx 4.2\,\mathrm{GeV}/c^2$
top	1/2	$+2e/3$	$r\,,\ g\,,\ b$	$\approx 175\,\mathrm{GeV}/c^2$
gluon	1	0	$r\bar{g}\,,\ r\bar{b}\,,\ g\bar{r}\,,$ $g\bar{b}\,,\ b\bar{r}\,,\ b\bar{g}\,,$ $(g\bar{g}-r\bar{r})/\sqrt{2}\,,$ $(g\bar{g}+r\bar{r}-2b\bar{b})/\sqrt{6}$	0

Tab. 28.1 : Properties of quarks and gluons

different color charges as a color-triplet:

$$\psi_{\text{up,red}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \text{rest}_{\text{up}} \qquad \psi_{\text{up,green}} = \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \text{rest}_{\text{up}}$$
$$\psi_{\text{up,blue}} = \begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \text{rest}_{\text{up}} \tag{28.1}$$

Thus each of the six quarks becomes a basis of a three-dimensional fundamental representation of the group SU(3). Mesons (which are made up from one quark and one antiquark) and baryons (which are made up from three quarks), are described in QCD by

$$\psi_{\text{meson}} = \frac{1}{\sqrt{3}}\,\psi_{A,\alpha}\psi_{\bar{B},\bar{\alpha}} \tag{28.2a}$$
$$\psi_{\text{baryon}} = \frac{1}{\sqrt{6}}\,\epsilon^{\alpha\beta\gamma}\,\psi_{A,\alpha}\psi_{B,\beta}\psi_{C,\gamma}\ . \tag{28.2b}$$

Factors with pairs of identical color indices α, β, γ are to be summed-up over red, green, blue. This rule applies as well, if one color index is indicating a color and the other color index is indicating the anticolor:

$$(28.2a) = \frac{1}{\sqrt{3}}(\psi_{A,\text{red}}\psi_{\bar{B},\bar{\text{red}}} + \psi_{A,\text{green}}\psi_{\bar{B},\bar{\text{green}}} + \psi_{A,\text{blue}}\psi_{\bar{B},\bar{\text{blue}}})$$

Due to the totally antisymmetric Levi-Cività tensor $\epsilon^{\alpha\beta\gamma}$, baryon state-functions are antisymmetric (i. e. compliant to the Pauli-principle) even if the three quark-flavors A, B, C are identical and the rest of the function (not explicitly indicated here) is symmetrical. A well-known example for this case is the Δ^{++}-resonance

$$\psi_{\Delta^{++}} = \frac{1}{\sqrt{6}}\,\epsilon^{\alpha\beta\gamma}\,\psi_{\text{up},\alpha}\psi_{\text{up},\beta}\psi_{\text{up},\gamma}\ .$$

The state-functions (28.2) are color-singlets, i. e. they are invariant under arbitrary transformations of the coordinates of the three-dimensional color space. As a most important principle of QCD it turns out, that the state functions of all fields which are observable in isolation must be color-singlets.

All elements of a fundamental representation of the group SU(n) with

$n \geq 2$ can be written in the form

$$U = \exp\left\{\frac{i}{\hbar} K_j \, g \frac{\lambda^j}{2}\right\} = \lim_{r\to\infty} \left(1 + \frac{i}{\hbar} \, g \, \underbrace{\frac{K_j}{r}}_{\equiv k_j} \frac{\lambda^j}{2}\right)^r \tag{28.3a}$$

$$\Longrightarrow \quad U_{\text{INF}} = 1 + \frac{i}{\hbar} \, g k_j \frac{\lambda^j}{2} \; . \tag{28.3b}$$

Here $r \in \mathbb{N}$ is a natural number, and indices j showing up doubly must be summed-up from 1 to $n^2 - 1$. The $n^2 - 1$ factors $g\lambda^j/2$ are the representation's generators. g is a real coupling constant with dimension color-charge. The factors λ^j are dimension-less hermitean $n \times n$-matrices. The dimension of the factors K_j (resp. k_j in the infinitesimal transformation) is action/color-charge.

We constructed the SU(2) generators by means of the 2×2-dimensional Pauli-matrices (27.2). For the construction of the SU(3) generators, Gell-Mann [61] proposed these 3×3-dimensional matrices:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix} \tag{28.4}$$

The commutator of the generators is

$$\left[\frac{\lambda^i}{2}, \frac{\lambda^j}{2}\right] = i f^{ijk} \frac{\lambda^k}{2} \; . \tag{28.5}$$

As in all groups SU(n), the structure constants f^{ijk} are real and totally antisymmetric, i. e. their signs are changing upon any permutation of two

indices. The SU(3) structure constants can be computed from (28.4) and (28.5), or simply be read out the article [61] of Gell-Mann:

ijk	123	147	156	246	257	345	367	458	678
f^{ijk}	1	1/2	−1/2	1/2	1/2	1/2	−1/2	$\sqrt{3}/2$	$\sqrt{3}/2$

All structure constants, which are neither listed here nor can be constructed by permutations of the indices of these f^{ijk}, are zero.

As for the generator-matrices of any SU(n), also for the generator-matrices of SU(3)

$$\mathrm{tr}\{\lambda^j\} = 0 \qquad\qquad \mathrm{tr}\{\lambda^j\lambda^k\} = 2\delta_{jk} \ . \tag{28.6}$$

To match these relations, Gell-Mann chose the curious form for the matrix λ^8, while the seven other matrices are "modified copies" of the Pauli-matrices.

The Lagrangian of the free (not interacting) quark with flavor A is

$$\mathcal{L}_A = \overline{\psi}_A(i\hbar c\gamma^\rho \mathrm{d}_\rho - m_A c^2)\psi_A \ . \tag{28.7}$$

The state-function ψ_A is a color-charge triplet, according to definition (28.1). As the phase transformation (28.3) is unitary, $U^\dagger = U^{-1}$, this Lagrangian obviously is invariant under the global phase transformation

$$\mathcal{L}'_A = \overline{\psi}_{A,\alpha}U^\dagger_{\alpha\beta}(i\hbar c\gamma^\rho \mathrm{d}_\rho - m_A c^2)U_{\beta\gamma}\psi_{A,\gamma} = \mathcal{L}_A \ . \tag{28.8}$$

The color-indices $\alpha\ldots$ have been explicitly indicated here. In most of our formulas, like e. g. (28.7) we drop these indices. Automatic summation is understood over the three colors in case of double color-index α, summation from zero to three in case of double space-time-indices ρ, and summation from one to eight in case of double operator-indices j. According to Noether's theorem, the current density with the four space-time-components

$$j^\rho_j \overset{(27.16)}{=} g\overline{\psi}_{A,\alpha}c\gamma^\rho\frac{\lambda^j_{\alpha\beta}}{2}\psi_{A,\beta} \quad \text{with } j = 1\ldots 8 \tag{28.9}$$

is conserved. Thus each of the eight color-charges $g\lambda^j/2$ is conserved.

Now we will consider the local phase transformation

$$\psi'_A(x) = U(x)\,\psi_A(x) = \exp\left\{\frac{i}{\hbar}\,\tilde{K}(x)\,g\right\}\psi_A(x) = \exp\left\{\frac{i}{\hbar}\,K_j(x)\,g\frac{\lambda^j}{2}\right\}\psi_A(x) \tag{28.10}$$

in which the eight parameters K_j can be chosen different at each space-time point x. We merely require that the eight fields $K_j(x)$ shall be continuous and differentiable at each space-time point. We apply the local phase transformation to the Lagrangian (28.7):

$$\mathcal{L}'_A \overset{(27.18)}{=} \mathcal{L}_A - \overline{\psi}_A c\gamma^\rho g(\mathrm{d}_\rho K_j)\frac{\lambda^j}{2}\,\psi_A \tag{28.11}$$

The Lagrangian is not invariant under local phase transformations. As usual we enforce the invariance by replacing the normal differential operator d_ρ according to (27.21) by the covariant differential operator

$$\mathrm{D}_\rho(x) \equiv \mathrm{d}_\rho + \frac{i}{\hbar}g\tilde{G}_\rho(x) \tag{28.12a}$$

$$\tilde{G}_\rho(x) \equiv G_{j\rho}(x)\,\frac{\lambda^j}{2}\ . \tag{28.12b}$$

In quantum chromodynamics, the eight charges $g\lambda^j/2$ are replacing the charge q in the corresponding equation (27.19) of quantum electrodynamics. And the eight gauge fields $G_j(x)$ are in the gauge-group SU(3) replacing the gauge field $A(x)$ of the gauge-group U(1). The dimension of the gluon fields is

$$[G_j] = \frac{\text{momentum}}{\text{color-charge}}\ . \tag{28.12c}$$

D_ρ, $\tilde{G}_\rho$, and λ^j are matrices with 3×3 color components. A single component of the covariant differential operator is

$$\mathrm{D}_{\rho,\alpha\beta}(x) = \mathbb{1}_{\alpha\beta}\mathrm{d}_\rho + \frac{i}{\hbar}g\tilde{G}_{\rho,\alpha\beta}(x) = \mathbb{1}_{\alpha\beta}\mathrm{d}_\rho + \frac{i}{\hbar}gG_{j\rho}(x)\,\frac{\lambda^j_{\alpha\beta}}{2}\ . \tag{28.12d}$$

We define this change of $\tilde{G}(x)$ under a gauge-transformation:

$$\tilde{G}'_\rho \overset{(27.22)}{=} U\tilde{G}_\rho U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\rho U)U^\dagger = G_{j\rho}U\frac{\lambda^j}{2}U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\rho U)U^\dagger \tag{28.13}$$

Consequently this is the change of the covariant differential operator resp. of one of it's single color components under a gauge-transformation:

$$\mathrm{D}'_\rho \overset{(27.23)}{=} U\,\mathrm{D}_\rho\,U^\dagger \tag{28.14a}$$

$$\mathrm{D}'_{\rho,\alpha\delta} = U_{\alpha\beta}\,\mathrm{D}_{\rho,\beta\gamma}\,U^\dagger_{\gamma\delta} \tag{28.14b}$$

Due to this transformation characteristic, the Lagrangian is invariant under gauge transformations, provided the normal differential operator d_ρ is replaced by the covariant differential operator D_ρ:

$$\begin{aligned}\mathcal{L}'_A &= \overline{\psi}_{A,\alpha}U^\dagger_{\alpha\beta}(i\hbar c\gamma^\rho\,\mathrm{D}'_{\rho,\beta\gamma} - \mathbb{1}_{\beta\gamma}m_A c^2)U_{\gamma\delta}\psi_{A,\delta} = \\ &= \overline{\psi}_{A,\alpha}(i\hbar c\gamma^\rho\,\mathrm{D}_{\rho,\alpha\delta} - \mathbb{1}_{\alpha\delta}m_A c^2)\psi_{A,\delta} = \mathcal{L}_A\end{aligned} \tag{28.15}$$

In case of an infinitesimal local phase transformation (28.3b), the gauge-field's transformation can be indicated explicitly:

$$\boxed{G'_{j\rho} \overset{(27.29a)}{=} G_{j\rho} - \mathrm{d}_\rho k_j - \frac{g}{\hbar}\,f^{klj}k_k G_{l\rho}} \tag{28.16a}$$

The combination of this infinitesimal transformation resp. of the finite transformation in the matrix-form

$$\boxed{\tilde{G}'_\rho \overset{(28.13)}{=} U\tilde{G}_\rho U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\rho U)U^\dagger} \tag{28.16b}$$

with the local phase transformation

$$\boxed{\psi'_A(x) = U(x)\,\psi_A(x) \overset{(28.10)}{=} \exp\left\{\frac{i}{\hbar}\,gK_j(x)\frac{\lambda^j}{2}\right\}\psi_A(x)} \tag{28.16c}$$

is the SU(3) gauge-transformation. The Lagrangian's invariance under this gauge transformation is the basis of quantum chromodynamics.

The combined Lagrangian of the quark field and the gauge fields can

immediately be concluded from (27.31b):

$$\boxed{\mathcal{L}_A \equiv \overline{\psi}_A(i\hbar c\gamma^\rho \mathrm{D}_\rho - m_A c^2)\psi_A - \frac{1}{2}\,\mathrm{Sp}\{\tilde{F}_{\sigma\tau}\tilde{F}^{\sigma\tau}\}} \tag{28.17a}$$

$$\tilde{F}_{\sigma\tau} \equiv \mathrm{D}_\sigma \tilde{G}_\tau - \mathrm{D}_\tau \tilde{G}_\sigma \equiv F_{j\sigma\tau}\frac{\lambda^j}{2} \equiv (\mathrm{D}_\sigma G_{j\tau} - \mathrm{D}_\tau G_{j\sigma})\frac{\lambda^j}{2} \tag{28.17b}$$

Inserting the covariant differential operator (28.12), the field-strength tensor can be written like this:

$$F_{j\sigma\tau}\frac{\lambda^j}{2} = \mathrm{d}_\sigma G_{j\tau}\frac{\lambda^j}{2} - \mathrm{d}_\tau G_{j\sigma}\frac{\lambda^j}{2} + \frac{i}{\hbar}g\Big(G_{k\sigma}G_{j\tau}\frac{\lambda^k}{2}\frac{\lambda^j}{2} - G_{k\tau}G_{j\sigma}\frac{\lambda^k}{2}\frac{\lambda^j}{2}\Big)$$

We rename the contracted indices j in the first term of the second line to l, and in the second term of the second line we rename the contracted indices k to l and j to k:

$$\begin{aligned} F_{j\sigma\tau}\frac{\lambda^j}{2} &= \mathrm{d}_\sigma G_{j\tau}\frac{\lambda^j}{2} - \mathrm{d}_\tau G_{j\sigma}\frac{\lambda^j}{2} + \frac{i}{\hbar}\,gG_{k\sigma}G_{l\tau}\Big(\frac{\lambda^k}{2}\frac{\lambda^l}{2} - \frac{\lambda^l}{2}\frac{\lambda^k}{2}\Big) \\ F_{j\sigma\tau} &\overset{(28.5)}{=} \mathrm{d}_\sigma G_{j\tau} - \mathrm{d}_\tau G_{j\sigma} - \frac{1}{\hbar}\,gf^{jkl}G_{k\sigma}G_{l\tau} \end{aligned} \tag{28.18}$$

The relation

$$\tilde{F}'_{\sigma\tau} \overset{(A.183)}{=} U\tilde{F}_{\sigma\tau}U^\dagger\ , \tag{28.19}$$

which is proved in appendix A.25, holds as well in case of SU(3). As furthermore factors may be permuted cyclically under the trace,

$$\begin{aligned} \mathrm{tr}\{\tilde{F}'_{\sigma\tau}\tilde{F}'^{\sigma\tau}\} &= \mathrm{tr}\{U\tilde{F}_{\sigma\tau}U^\dagger U\tilde{F}^{\sigma\tau}U^\dagger\} \\ &= \mathrm{tr}\{U^\dagger U\tilde{F}_{\sigma\tau}U^\dagger U\tilde{F}^{\sigma\tau}\} = \mathrm{tr}\{\tilde{F}_{\sigma\tau}\tilde{F}^{\sigma\tau}\} \end{aligned}$$

together with (28.15) is securing the gauge-invariance of the QCD Lagrangian $\mathcal{L}_A =$ (28.17).

Alternatively the trace may be written in the form

$$\mathrm{tr}\{\tilde{F}_{\sigma\tau}\tilde{F}^{\sigma\tau}\} = F_{j\sigma\tau}F_k^{\sigma\tau}\mathrm{tr}\Big\{\frac{\lambda^j}{2}\frac{\lambda^k}{2}\Big\} \overset{(28.6)}{=} \frac{1}{4}F_{j\sigma\tau}F_j^{\sigma\tau}\cdot 2\ . \tag{28.20}$$

Thereby the Lagrangian (28.17) becomes

$$\mathcal{L}_A = \overline{\psi}_A(i\hbar c\gamma^\rho \mathrm{D}_\rho - m_A c^2)\psi_A - \frac{1}{4}F_{j\sigma\tau}F_j^{\sigma\tau}\ . \tag{28.21}$$

Now the similarity with the U(1)-Lagrangian (27.30) is quite obvious. In both cases the product of the field-strength tensors is multiplied by $-1/4$. Only the factor μ_0 here has disappeared, and the sum is running over the generator-indices $j = 1 \ldots 8$.

28.2 Confinement and asymptotic Freedom

We have seen, that QCD is constructed quite similar to QED. In particular, gluons are mass-less, exactly like photons. Therefore we would expect an infinite range of the interaction between quarks. But it's well-known that this is not the case. The range of the strong interaction is about 10^{-15}m. This fundamental difference between the electromagnetic and the strong interaction is caused by the fact, that U(1) is an Abelian group, while SU(3) is a non-Abelian group. The consequences of this difference become visible, once the covariant derivative (28.12) and the field-strength tensors (28.18) are explicitly inserted into the QCD Lagrangian:

$$\begin{aligned}\mathcal{L}_A = \overline{\psi}_A\Big(i\hbar c\gamma^\rho(\mathrm{d}_\rho + \frac{i}{\hbar}\,gG_{j\rho}\,\frac{\lambda^j}{2}) - m_A c^2\Big)\psi_A - \\ - \frac{1}{4}\Big(\mathrm{d}_\sigma G_{j\tau} - \mathrm{d}_\tau G_{j\sigma} - \frac{1}{\hbar}\,gf^{jkl}G_{k\sigma}G_{l\tau}\Big)\cdot \\ \cdot\Big(\mathrm{d}^\sigma G_j^\tau - \mathrm{d}^\tau G_j^\sigma - \frac{1}{\hbar}\,gf^{jkl}G_k^\sigma G_l^\tau\Big) =\end{aligned}$$

$$= \underbrace{\overline{\psi}_A(i\hbar c\gamma^\rho \mathrm{d}_\rho - m_A c^2)\psi_A}_{①} \underbrace{- \overline{\psi}_A c\gamma^\rho g G_{j\rho} \frac{\lambda^j}{2} \psi_A}_{②} -$$

$$\underbrace{- \frac{1}{4}(\mathrm{d}_\sigma G_{j\tau} - \mathrm{d}_\tau G_{j\sigma})(\mathrm{d}^\sigma G_j^\tau - \mathrm{d}^\tau G_j^\sigma)}_{③} +$$

$$\underbrace{+ \frac{1}{2\hbar} g f^{jkl} G_{k\sigma} G_{l\tau}(\mathrm{d}^\sigma G_j^\tau - \mathrm{d}^\tau G_j^\sigma)}_{④} \underbrace{- \frac{1}{4\hbar^2} g^2 f^{jkl} f^{jmn} G_{k\sigma} G_{l\tau} G_m^\sigma G_n^\tau}_{⑤} \tag{28.22}$$

In QCD there exists — different from QED — not only a coupling between the Dirac field and it's gauge field, but there also exist self-interactions of the gauge fields:

The term ① is describing the free (not interacting) quark fields, and the term ③ is describing the eight free, not interacting gluon fields. The term ②, which is proportional to the coupling constant g, is describing the coupling between the quark fields and the gluon fields. As the SU(3) structure constants f^{jkl} — different from the U(1) structure constants — are different from zero, the new terms ④ and ⑤, which did not exist in QED, are showing up in QCD. They are describing the self-interactions of the gluon fields. ④ is called $3G$-coupling, because the product of 3 field-amplitudes is contained in each term. Accordingly ⑤ is called $4G$-coupling. Like the coupling between quark fields and gluon fields, $3G$-coupling is proportional to g, while $4G$-coupling is proportional to g^2.

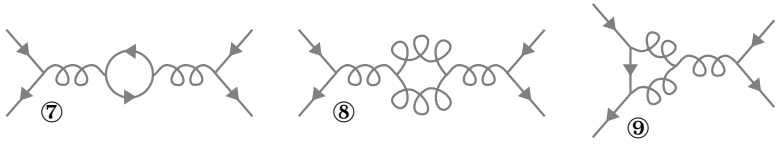
The character of the strong interaction is completely changed due to the self-interactions of gluons. If for example the cross section for the scattering of an electron by a muon is computed in QED, in perturbation computation $\sim e^4$ besides others the graph

is encountered. In section 26.1 we have evaluated the virtual fermion loop, which is called vacuum polarization. It brings about a shielding effect, and thus a running coupling constant, which we have estimated as

$$\alpha(k_\gamma^2) \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \overset{(26.22b)}{\approx} \frac{\alpha(0)}{1 - \dfrac{\alpha(0)}{3\pi}\ln\left\{\dfrac{-k_\gamma^2\hbar^2}{m^2c^2}\right\}} \,. \tag{28.23}$$

k_γ^2 is the virtual photon's wavenumber square. We know from the evaluations in section 24.3.7, that $k_\gamma^2 < 0$ in t-channel scattering. $\alpha(0) \approx 1/137$ is the coupling constant in case of very weak scattering. The approximation (28.23) is only correct, if the modulus of the second term in the denominator is much smaller than 1.

When computing in QCD the scattering of a quark by an other quark in perturbation computation $\sim g^4$, one encounters besides others these graphs:



The graph ⑦ of QCD is clearly equivalent to the graph ⑥ of QED. The virtual fermion loop in ⑦ must however be made from quarks, because leptons don't couple to gluons. We extract from the literature [29, Abschnitt 2.5.2] the result for the graphs ⑦, ⑧, and ⑨, which is corresponding to (28.23):

$$\alpha_s(k^2) \equiv \frac{g^2}{4\pi\hbar c} \approx \frac{\alpha_s(\Lambda^2)}{1 - \dfrac{\alpha_s(\Lambda^2)}{12\pi}(2n_A - 33)\ln\left\{\dfrac{-k^2}{\Lambda^2}\right\}} \tag{28.24}$$

This approximation is only correct, if the modulus of the second term in the denominator is much smaller than 1. α_s is the coupling constant of strong interaction. $-k^2 > 0$ is the wavenumber square transferred in scattering. Λ^2 is a reference value, which must be determined experimentally.

One finds $\Lambda \approx 0.2\,\mathrm{GeV}/(\hbar c)$. And n_A is the number of quark flavors of which the virtual quark-loop in ⑦ can be built. The new term -33 in the denominator is resulting from the graphs ⑧ and ⑨. It's sign is different from the sign of n_A, and it is dominating, because there are only six quark flavors. Therefore $\alpha_s(k^2)$ is decreasing for increasing $|k^2|$, while $\alpha(k_\gamma^2)$ is increasing for increasing $|k_\gamma^2|$. Experimentally determined values of the electromagnetic and the strong running coupling constants have been indicated in (21.20).

The distance r between scattering quarks is (by order of magnitude) $\approx 1/\sqrt{-k^2}$. At small distance r, α_s resp. g are small as well. In that case a perturbation computation, which is considering only one or few orders of g, will lead to reasonable results. In this range the methods of perturbation theory, which are known from QED, can be applied. The relation $g(|k^2|) \to 0$ at $|k^2| \to \infty$ is called asymptotic freedom.

Things are quite different in case $|k^2| \to 0$. If the distance r in-between quarks, which are bound in a hadron, is becoming

$$r \gtrsim 10^{-15}\mathrm{m} \,\widehat{\approx}\, 0.2\,\mathrm{GeV}/(\hbar c) \approx \Lambda \gtrsim \sqrt{|k^2|}\,, \tag{28.25}$$

then the attraction between these quarks becomes so large, that the energy content of the gluon fields is sufficient for pair-creation of additional quarks. Therefore the high-energy scattering of two protons does not produce single quarks, but several color-singlets of the type (28.2). The impenetrable enclosure of quarks into color-singlets is called confinement. As α_s isn't any more small versus 1 in the range (28.25), the usual methods of perturbation computation are not applicable any more. This makes all computations extremely difficult.

29 Electroweak Interaction

QCD and electro-weak interaction are the two pillars of the "standard model of elementary particles". For two reasons, the electro-weak theory is significantly more complicated than QCD: First, it is based on the composite symmetry group $\mathrm{SU}(2) \otimes \mathrm{U}(1)$, while QCD is based on the simple symmetry group SU(3). Second, the gauge bosons of $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ are in addition coupled to a Higgs field, whose symmetry is spontaneously broken. The meaning of these notions will be explained in section 29.3.

The complicated construction of electroweak theory could not be avoided, because it must cope with several requirements which at first sight seem irreconcilable:

(1) It shall be a gauge theory, i. e. it's Lagrangian shall be invariant under local phase transformations, and the forces acting between fermion fields shall be described as gauge-fields.
(2) The relative strengths S of the interactions must be represented correctly: $S_{\mathrm{weak}} : S_{\mathrm{em}} : S_{\mathrm{strong}} \approx 10^{-5} : 10^{-2} : 1$
(3) The ranges R of the interactions must be represented correctly: $R_{\mathrm{weak}} \approx 10^{-18}\mathrm{m}$; $R_{\mathrm{em}} = \infty$; $R_{\mathrm{strong}} \approx 10^{-15}\mathrm{m}$

While it goes without saying, that the theory must comply with the experimental observations mentioned in (2) and (3), the requirement (1) has the character of a guessed guideline, which eventually turned out to be successful. Actually that was a surprise, because (1) seems initially to be an additional, almost unsolvable problem for the following reason:

QED and QCD are gauge theories with mass-less gauge fields (the photons and the gluons). As photons are mass-less, the range $R_{\mathrm{em}} = \infty$ of the electromagnetic interaction is infinite. While gluons as well are mass-less, the range $R_{\mathrm{strong}} \approx 10^{-15}\mathrm{m}$ of the strong interaction is finite. That's caused by the self-interactions of gluon fields. The strength of these self-interactions is proportional to the strong coupling constant g in the 3-gluon-vertex, and

it is proportional to the square g^2 of the strong coupling constant in the 4-gluon-vertex, see the strong Lagrangian $\mathcal{L} = (28.22)$ on page 650.

If the weak interaction shall as well be described by a gauge theory, then it should consequently as well be a non-Abelian gauge theory, because then it's range will be finite. And it's coupling constant should be approximately one-thousand times larger than the strong coupling constant, to adjust the interaction range according to (3). But that's incompatible with (2): The weak interaction shall not be 10^{+3} times stronger, but it shall be weaker than the strong interaction by a factor 10^{-5}.

Obviously the range and the strength of the weak interaction must be somehow decoupled in the theory. We have clarified already for the example of Yukawa-theory in chapter 23, how that can be done: The range of an interaction will be R, if it is mediated by massive bosons with rest-energy

$$Mc^2 \overset{(23.7)}{\approx} \frac{\hbar c}{R} \approx \frac{\hbar c}{10^{-18}\mathrm{m}} \approx 200\,\mathrm{GeV}\ . \tag{29.1}$$

Actually the rest masses of the bosons, which eventually were detected as the medium of weak interactions, are about $80\,\mathrm{GeV}/c^2$ and $91\,\mathrm{GeV}/c^2$. But this solution comes with a new problem: Yukawa-theory isn't a gauge theory, because the gauge fields of a gauge theory must be mass-less. This fact is easily proved:

In the Lagrangian of any field, the mass of the field always is a factor in the product of the field's amplitude and the adjoint field's amplitude. Consider for example the QCD Lagrangian $\mathcal{L} = (28.22)$: In the term $m_A c^2 \overline{\psi}_A \psi_A$ the masses of the various quark flavors A are showing up. Or consider the Lagrangian $\mathcal{L} = (10.10)$ of the Klein-Gordon field with the mass-term $m^2c^4\psi^+\psi$.

Assuming massive gauge-fields, we accordingly should for example insert terms like $m_{W_j} c^2 W_{j\mu} W_j^\mu$ into the QCD Lagrangian $\mathcal{L} = (28.22)$. Under a gauge transformation these terms would be transformed according to (28.13). Thereby terms like $ig(\mathrm{d}_\rho U)U^\dagger/\hbar$ would show up, which could not be compensated by appropriate transformation-products of quark-fields, because these have already been completely compensated by those gaugefield-terms which existed already before the mass-terms were inserted. Thus

a non-zero rest mass of the gauge fields would destroy the theory's gauge invariance.

Eventually a construction (often called GSW-model, because it was mainly detected by Glashow[1], Salam[2], and Weinberg[3]) has been found, in which the requirements (1),(2),(3) became compatible: The model's starting point is a non-Abelian gauge theory with mass-less gauge bosons. Then the model is postulating the existence of a further, scalar field, whose symmetry is spontaneously broken. The gauge bosons couple to that additional scalar field, and thereby receive an effective mass. In the sequel we will explicate this construction step by step.

29.1 The Basis of the Representation

12 fermions (plus their 12 antiparticles) are known, which can interact weakly, namely the 6 leptons and the 6 quarks

$$\nu_e, e^-, \nu_\mu, \mu^-, \nu_\tau, \tau^- \ ; \ d', u, s', c, b', t \ . \tag{29.2}$$

The primes$'$ at three of the quarks are indicating, that they are not identical to the three quarks d, s, b appearing in QCD. Instead the quark d' is consisting mainly of d, but it also contains small admixtures of s and b. Correspondent explanations hold for s' and for b'. The mixture-angles are combined to the Cabibbo-Kobayashi-Maskawa matrix, which must be determined experimentally.

In the GSW-model, left- and right-handed fields are carefully discerned. In the early years of the GSW-model it was assumed that there existed exclusively left-handed neutrinos (spin and direction of propagation are antiparallel) and right-handed antineutrinos (spin and direction of propagation are parallel). Consequently it was assumed that neutrinos are mass-less, for the following reason: If neutrinos had a finite mass, then we could transform ourselves into a reference system which is overtaking the neutrino, thus converting a left-handed neutrino into a right-handed one, and vice

[1] Sheldon Lee Glashow, $*1932$
[2] Abdus Salam, 1926 - 1996
[3] Steven Weinberg, $*1933$

versa. By today it is established experimentally (due to the observation of "neutrino-oscillations"), that neutrinos must have a finite mass. That mass is so small, however, that it could not yet be directly measured. Still we will present the GSW-model with the additional assumption, that neutrinos have a finite mass, and that therefore right-handed neutrinos and left-handed antineutrinos exist.

Just as in (27.1) the state-functions of protons and neutrons have been combined to isospin-doublets, the left-handed components of the 12 fermions are combined to six left-handed "weak isospin-doublets". For example, the left-handed state-functions of the electron-neutrino and the electron are built in the form

$$\psi_{\nu_e,L} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \text{rest}_{\nu_e e^-,L} \tag{29.3a}$$

$$\psi_{e^-,L} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \text{rest}_{\nu_e e^-,L} \tag{29.3b}$$

$$\text{shorthand notation:} \quad \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \equiv \psi_{\nu_e e^-,L} \ . \tag{29.3c}$$

In this manner the six left-handed weak isospin-doublets

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \ ; \ \begin{pmatrix} u \\ d' \end{pmatrix}_L, \begin{pmatrix} c \\ s' \end{pmatrix}_L, \begin{pmatrix} t \\ b' \end{pmatrix}_L \ , \tag{29.4}$$

and the corresponding six right-handed weak isospin-doublets of their antiparticles are built. It is understood, that weak isospin is something different from flavor-isospin, which we considered in chapter 27. Formally, however, the construction is identical in both cases. Each of the doublets (29.4) is the basis of a fundamental representation of the group SU(2), just like the proton-neutron doublet (27.1). We use the notation $\boldsymbol{T}$ for the operator of weak isospin. As usual, it's three components are constructed by means of the Pauli-matrices:

$$T^j \equiv \frac{\tau^j}{2} \ , \ j = 1,2,3 \ , \ \tau^j \equiv \sigma^j = (27.2) \tag{29.5}$$

The weak isospin of all six doublets is $T = 1/2$. The weak isospin-charges of the doublets are

$$g_2 \frac{\tau^j}{2} \ , \ j = 1, 2, 3 \ . \tag{29.6}$$

Just as in case of flavor-isospin, the six functions

$$\begin{pmatrix} \nu_e \\ 0 \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ 0 \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ 0 \end{pmatrix}_L \ ; \ \begin{pmatrix} u \\ 0 \end{pmatrix}_L, \begin{pmatrix} c \\ 0 \end{pmatrix}_L, \begin{pmatrix} t \\ 0 \end{pmatrix}_L$$

are eigenfunctions of the T^3-operator with eigenvalue $+1/2$, while the six functions

$$\begin{pmatrix} 0 \\ e^- \end{pmatrix}_L, \begin{pmatrix} 0 \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} 0 \\ \tau^- \end{pmatrix}_L \ ; \ \begin{pmatrix} 0 \\ d' \end{pmatrix}_L, \begin{pmatrix} 0 \\ s' \end{pmatrix}_L, \begin{pmatrix} 0 \\ b' \end{pmatrix}_L$$

are eigenfunctions of the T^3-operator with eigenvalue $-1/2$. As usual, we are using identical notation for operators and their eigenvalues, e. g. $T^3 = \tau^3/2$ is an operator, and $T^3 = \pm 1/2$ are it's eigenvalues.

The right-handed state-functions of the fermions (29.2) are forming 12 singlets:

$$e^-_R, \nu_{eR}, \mu^-_R, \nu_{\mu R}, \tau^-_R, \nu_{\tau R} \ ; \ d'_R, u_R, s'_R, c_R, b'_R, t_R \tag{29.7}$$

Each of these singlets, and each of their antifields, is considered in the GSW-model a basis of a fundamental representation of the group U(1), whose generator is the weak hypercharge $g_1 Y/2$. The weak isospin of all of these singlets is by definition $T = 0$, and consequently $T_3 = 0$.

The weak hypercharge $g_1 Y/2$ is defined by the relation

$$Y \equiv \frac{2q}{e} - \tau^3 \quad \text{resp.} \quad Y \equiv \frac{2q}{e} - 2T^3 \ . \tag{29.8}$$

The left equation is the operator-form of the relation, the right one the relation's eigenvalue-form. q is the electrical charge, e is the electrical charge of the positron. Therefore the weak hypercharge of the three right-handed

neutrinos is $Y = 0$, and the weak hypercharge of the three right-handed electrically charged leptons is $Y = -2$. The weak hypercharge of u_R, c_R, and t_R is $Y = +4/3$, while the weak hypercharge of d'_R, s'_R, and b'_R is $Y = -2/3$.

The weak doublets as well carry weak hypercharges. The two equations (29.8) shall be interpreted as matrix-equations, where appropriate. For example, application of the operators onto the electron-doublet results into

$$Y \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \equiv \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \equiv \begin{pmatrix} \frac{2q}{e} - 1 & 0 \\ 0 & \frac{2q}{e} + 1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \\ = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = -1 \cdot \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L .$$

The three lepton-doublets in (29.4) have weak hypercharge $Y = -1$, and the three quark-doublets have weak hypercharge $Y = +1/3$. We will continue to write matrix equations in most cases in shorthand notation, like (29.8). The correct interpretation of such equations is confided to the reader's attention.

29.2 Electroweak Gauge-Fields

It's a fundamental assumption of the GSW-model, that the Lagrangian of the 6 weak interacting doublets (29.4) and the 12 weak interacting singlets (29.7) shall be invariant under local phase transformations. That's possible only if the normal differential operator d_μ is replaced by the covariant differential operator

$$\mathrm{D}_\mu(x) \equiv \mathrm{d}_\mu + \frac{i}{\hbar} \frac{g_1}{2} Y B^0_\mu(x) + \frac{i}{\hbar} g_2 W^j_\mu(x) \frac{\tau^j}{2} . \tag{29.9}$$

Here the respective charges $g_1 Y/2$ and $g_2 \tau^j/2$ of that object shall be inserted, onto which this operator is acting. Two examples:

$$\mathrm{D}_\mu \, e^-_R = \mathrm{d}_\mu e^-_R + \frac{i}{\hbar} \frac{g_1}{2} (-2) \, B^0_\mu(x) \, e^-_R \tag{29.10a}$$

$$\mathrm{D}_\mu \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \mathrm{d}_\mu \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L + \frac{i}{\hbar}\,\frac{g_1}{2}\,(-1)\,B^0_\mu(x) \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L + \frac{i}{\hbar}\,g_2 W^j_\mu(x)\,\frac{\tau^j}{2} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \tag{29.10b}$$

The weak hypercharge $g_1 Y/2$ is replacing the electrical charge q, and it's gauge-field $B^0(x)$ is replacing the gauge-field $A(x)$ in the corresponding definition (27.19) of QED. And the weak isospin-charges $g_2\tau^j/2$ and their gauge-fields $W^j(x)$ are replacing the charges $g\tau^j/2$ and gauge-fields $W^j(x)$ in the corresponding definition (27.21) of flavor-isospin gauge theory. The flavor-isospin gauge theory never "became effective", because it's gauge bosons are not observed. Therefore it should not cause confusion, if we — in accord with most of the published literature — are using many notations in GSW-theory, which already have been used in the flavor-isospin gauge theory of Yang and Mills.

The GSW-model not only is postulating, that the bases of electroweak theory are not built with the three quarks d, s, b (which are appearing in strong interaction), but with their mixtures d', s', b'; it also is postulating that the gauge fields acting in electroweak reactions are mixtures built from the fields B^0 and W^j. First we mix the three fields W^j (in case of W^3 the "mixture" actually is merely a re-naming):

$$\begin{pmatrix} W^+_\mu \\ W^0_\mu \\ W^-_\mu \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \end{pmatrix} \tag{29.11}$$

The triplet of the gauge fields (W^+, W^0, W^-) becomes the basis of a three-dimensional representation of the group SU(2) with weak isospin $T = 1$, while the gauge field B^0 becomes the basis of a one-dimensional representation of the group U(1), whose generator is the weak hypercharge $g_1 Y/2$. Different from the gauge fields W, which can mutually interact due to their weak isospin, the gauge field B^0 carries neither weak hypercharge nor weak isospin. To preserve gauge-invariance, all four gauge-fields are mass-less.

field	T	T^3	Y	$q/e = Y/2 + T^3$	mass
W^+	1	+1	0	+1	0
W^0	1	0	0	0	0
W^-	1	−1	0	−1	0
B^0	0	0	0	0	0

(29.12)

To incorporate QED into the GSW-model, now those two gauge fields, which carry no electrical charges, are mixed:

$$\begin{pmatrix} A_\mu \\ Z^0_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B^0_\mu \\ W^0_\mu \end{pmatrix} \tag{29.13}$$

$A(x)$ is the electromagnetic gauge field of QED. The angle θ_W of the orthogonal rotation (29.13) is called Weinberg-angle. According to the GSW-model, not the fields W^1, W^2, W^3, B^0 but the fields W^+, W^-, Z^0, A are the physical, observable gauge fields of the electroweak interaction.

We define the ladder-operators

$$\tau^+ \equiv \frac{1}{2}\left(\tau^1 + i\tau^2\right) \qquad \tau^- \equiv \frac{1}{2}\left(\tau^1 - i\tau^2\right) \tag{29.14}$$

and compute

$$\begin{aligned} &\sqrt{2}(W^+_\mu \tau^- + W^-_\mu \tau^+) = \\ &= \frac{1}{2}\Big(W^1_\mu \tau^1 + i(-W^1_\mu \tau^2 + W^2_\mu \tau^1) + W^2_\mu \tau^2 + \\ &\quad + W^1_\mu \tau^1 + i(+W^1_\mu \tau^2 - W^2_\mu \tau^1) + W^2_\mu \tau^2\Big) = \\ &= W^1_\mu \tau^1 + W^2_\mu \tau^2 \ . \end{aligned} \tag{29.15}$$

Using this result, and using $Y \overset{(29.8)}{=} 2q/e - \tau^3$, the covariant differential operator of the GSW-model becomes

$$\begin{aligned} &\mathrm{D}_\mu \overset{(29.9)}{=} \mathrm{d}_\mu + \frac{i}{\hbar}\,\frac{g_1}{2}\,(2q/e - \tau^3)(A_\mu \cos\theta_W - Z^0_\mu \sin\theta_W) + \\ &+ \frac{i}{\hbar}\Big(\frac{g_2}{\sqrt{2}}(W^+_\mu \tau^- + W^-_\mu \tau^+) + g_2(A_\mu \sin\theta_W + Z^0_\mu \cos\theta_W)\,\frac{\tau^3}{2}\Big) = \end{aligned}$$

$$= \mathrm{d}_\mu + \frac{i}{\hbar\sqrt{g_1^2+g_2^2}}\Big(\frac{g_1 g_2}{2}\,\frac{2q}{e}\,\underbrace{-\frac{g_1 g_2}{2}\,\tau^3 + \frac{g_1 g_2}{2}\,\tau^3}_{0}\Big)A_\mu + \\ + \frac{i}{\hbar\sqrt{g_1^2+g_2^2}}\Big(-\frac{g_1^2}{2}\,\frac{2q}{e} + \frac{g_1^2}{2}\,\tau^3 + \frac{g_2^2}{2}\,\tau^3\Big)Z_\mu^0 + \frac{i}{\hbar}\,\frac{g_2}{\sqrt{2}}(W_\mu^+\tau^- + W_\mu^-\tau^+)\ . \tag{29.16}$$

The gauge field $A(x)$ must show up in the covariant differential operator only as a term $iqA/\hbar$, if the model shall be compatible with QED. Therefore the positrons's electrical charge e and the weak coupling constants g_1 snd g_2 resp. the Weinberg-angle θ_W are constrained by the following condition:

$$\boxed{e = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}} = g_1\cos\theta_W = g_2\sin\theta_W} \tag{29.17}$$

As the values of the two coupling constants are not fixed from the outset, the value of the Weinberg angle as well can not be computed from this relation. Instead it is one of the parameters of the model, which must be determined experimentally. From observations, the value $\sin^2\theta_W \approx 0.22$ has been extracted.

By insertion of condition (29.17), the covariant differential operator becomes

$$\boxed{\begin{aligned}\mathrm{D}_\mu = \mathrm{d}_\mu + \frac{i}{\hbar}\,qA_\mu + \frac{i}{\hbar}\,\frac{g_2}{\sqrt{2}}(W_\mu^+\tau^- + W_\mu^-\tau^+) + \\ + \frac{i}{\hbar\sqrt{g_1^2+g_2^2}}\Big(-\frac{g_1^2 q}{e} + \frac{g_1^2+g_2^2}{2}\,\tau^3\Big)Z_\mu^0\ .\end{aligned}} \tag{29.18}$$

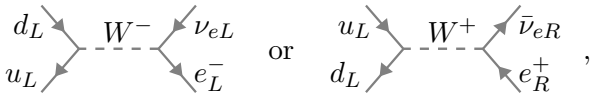
The term with A_μ, and the QED Feynman-diagrams resulting from it, are well-known to us from chapters 24 and 26.

From (29.17) and (29.18) it's obvious, that electrical and weak charges have the same dimension, and that the electrical and the weak gauge fields have the same dimension:

$$[q] = [e] = [g_1] = [g_2] = \text{electroweak charge} \tag{29.19a}$$

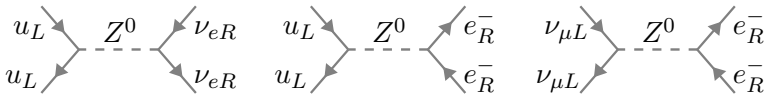
$$[A_\mu] = [W_\mu^+] = [W_\mu^-] = [Z_\mu^0] = \frac{\text{momentum}}{\text{electroweak charge}} \tag{29.19b}$$

The terms with the gauge-fields W^+ and W^- are acting exclusively onto the fermions of the left-handed weak doublets (29.4) (resp. of the right-handed doublets of their antiparticles), because only these fermions carry weak isospin-charge. These terms are resulting into diagrams like



which for example are encountered in the β^-- or β^+-decays of atomic nuclei. Such diagrams are called "charged currents", because the weak gauge-fields W^+ and W^- carry electric charges.

From the last term of the covariant differential operator, uncharged currents like



are resulting. As the gauge bosons W^+, W^-, Z^0 carry weak isospin-charges, they also can interact mutually.

The model, as far as we have described it up to now, is representing all phenomena of the electromagnetic and the weak interactions qualitatively correct. Quantitatively, however, it is a correct description only of the electromagnetic interaction. If we want to describe also the weak interaction quantitatively correct, we must assign masses to the three gauge-fields W^+, W^-, Z^0. How that can be done without destruction of gauge invariance, will be discussed in the next section.

29.3 The Higgs-Mechanism

The vacuum-state $|0\rangle$ is *defined* by the condition

$$|0\rangle \quad \Longleftrightarrow \quad a_{\boldsymbol{k}}\,|0\rangle = \langle 0|\,b_{\boldsymbol{k}}^\dagger = 0 \quad \forall\,\boldsymbol{k}\ . \tag{29.20a}$$

Hence the vacuum state and the state of lowest energy of the field are identical. Usually the expectation value of a field is zero in the vacuum state. Example: The vacuum expectation value of a free, quantized Klein-Gordon field is

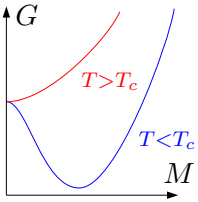
$$\langle 0|\,\phi(x)\,|0\rangle \overset{(15.15a)}{=} \sum_{\boldsymbol{k}} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}}\Big(\langle 0|\,a_{\boldsymbol{k}}\,|0\rangle \exp\{-ikx\} + \\ + \langle 0|\,b^{\dagger}_{\boldsymbol{k}}\,|0\rangle \exp\{+ikx\}\Big) = 0\ . \tag{29.20b}$$

But this is not necessarily the case for all fields. In this section we will encounter a field with

$$|\langle 0|\,\phi(x)\,|0\rangle| = f \neq 0\ . \tag{29.20c}$$

This may happen, if the state $|0\rangle$ of lowest energy does not possess the complete symmetry of the Lagrangian. This case is called "spontaneously broken symmetry".

The classical example for a spontaneously broken symmetry is the ferromagnet: Below the Curie-temperature T_c it's macroscopic magnetization $\boldsymbol{M}$ spontaneously assumes a certain direction in space, even though it's Lagrangian is isotropic. The ferromagnet's free energy G can be described by



$$G = \alpha M^2 + \beta M^4 \quad \begin{cases} \alpha > 0 \text{ if } T > T_c \\ \alpha < 0 \text{ if } T < T_c \\ \beta > 0 \text{ always}\ . \end{cases} \tag{29.21}$$

The Lagrangian of the Klein-Gordon field is

$$\mathcal{L} \overset{(10.10)}{=} \underbrace{c^2\hbar^2(\mathrm{d}_\mu\phi^\dagger)\mathrm{d}^\mu\phi}_{\mathcal{T}} - \underbrace{m^2c^4\phi^\dagger\phi}_{\mathcal{U}}\ . \tag{29.22}$$

In this expression, $\mathcal{T}$ is the density of kinetic energy, and $\mathcal{U}$ is the density of

potential energy, see (3.24). Guided by the example (29.21), we now define the Lagrangian of a scalar field with spontaneously broken symmetry:

$$\mathcal{L} \equiv c^2\hbar^2(\mathrm{D}_\mu\,\phi)^\dagger\,\mathrm{D}^\mu\,\phi - \left(\alpha\phi^\dagger\phi + \beta(\phi^\dagger\phi)^2\right) \tag{29.23}$$

$$\alpha \equiv -\frac{m^2c^4}{2} < 0 \qquad \beta \equiv +\frac{m^2c^4}{4f^2} > 0$$

$$[m] = \text{mass} \qquad [f] = \sqrt{\frac{1}{\text{energy}\cdot\text{volume}}}$$

As this Lagrangian shall be gauge-invariant, instead of d_μ the covariant differential operator $\mathrm{D}_\mu =$ (29.18) has been inserted. $\phi(x)$ shall be a complex scalar field:

$$\phi = \sqrt{\frac{1}{2}}\,(\phi_1 + i\phi_2) = |\phi|\,e^{i\rho}\ ,\ \ \phi_1, \phi_2, \rho \in \mathbb{R}$$

$$\phi^\dagger\phi = |\phi|^2 = \frac{1}{2}\,(\phi_1^2 + \phi_2^2)$$

$$[\phi] \overset{(10.12)}{=} \sqrt{\frac{1}{\text{energy}\cdot\text{volume}}} \tag{29.24}$$

The factor $\sqrt{1/2}$ has been included into this definition, because we later will interpret the real components ϕ_j as elementary fields, and therefore give them from start on the correct normalization. We compute the minimum of the potential energy:

$$\frac{\mathrm{d}}{\mathrm{d}\,|\phi|}\left(\phi^\dagger\phi + \frac{\beta}{\alpha}(\phi^\dagger\phi)^2\right) = 2|\phi| - \frac{4}{2f^2}\,|\phi|^3 = 0$$

$$\Longrightarrow \quad \mathcal{U} \text{ has extrema at } |\phi| = 0 \text{ and } |\phi| = f\ . \tag{29.25}$$

The potential, which in figure 29.1 on the next page is indicated only for $\phi_1 = 0$ resp. for the polar angles $\rho = \pi/2$ and $\rho = 3\pi/2$, has at arbitrary values of ρ a form which resembles the bottom of a wine-bottle. Sometimes the form is also compared to a sombrero. All points $|\phi|\exp\{i\rho\}$ with $|\phi| = f$ and arbitrary ρ, which are indicated by the dashed circle, are equivalent

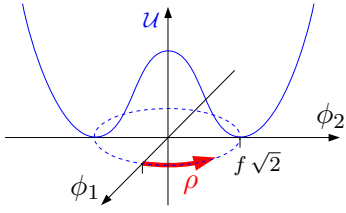


Fig. 29.1 : The potential $\mathcal{U}$ as function of the fields ϕ_1 and ϕ_2 .

minima, which are defining states $|0\rangle$ of lowest energy according to condition (29.20c). Therefore $\langle 0|\,\phi(x)\,|0\rangle = f\exp\{i\rho\} \neq 0$ holds for the field ϕ. The field ϕ will assume in the state of lowest energy spontaneously a certain polar angle ρ, and thus break the symmetry of the Lagrangian (29.23) (which is independent of ρ), just as the magnetization of a ferromagnet assumes at $T < T_c$ spontaneously a certain direction.

$\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ is replaced in the GSW-model by the weak isospin-doublet

$$\phi(x) \equiv \sqrt{\frac{1}{2}}\begin{pmatrix}\phi_3(x) + i\phi_4(x)\\ \phi_1(x) + i\phi_2(x)\end{pmatrix} \qquad \text{with } \phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{R}\ . \tag{29.26}$$

To this doublet the weak isospin $T = 1/2$ is assigned, with

$$T^3\begin{pmatrix}\phi_3(x) + i\phi_4(x)\\ 0\end{pmatrix} = +\frac{1}{2}\begin{pmatrix}\phi_3(x) + i\phi_4(x)\\ 0\end{pmatrix} \tag{29.27a}$$

$$T^3\begin{pmatrix}0\\ \phi_1(x) + i\phi_2(x)\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\ \phi_1(x) + i\phi_2(x)\end{pmatrix}\ . \tag{29.27b}$$

But no electric charge is assigned to ϕ. Therefore it's weak hypercharge is

$$Y\begin{pmatrix}\phi_3(x) + i\phi_4(x)\\ \phi_1(x) + i\phi_2(x)\end{pmatrix} \overset{(29.8)}{=} \begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix}\begin{pmatrix}\phi_3(x) + i\phi_4(x)\\ \phi_1(x) + i\phi_2(x)\end{pmatrix}\ . \tag{29.28}$$

Same as T^3, the hypercharge is well-defined only (i. e. ϕ is an eigenvector of the operator Y only) if the upper or the lower component of the isospinor is zero.

In figure 29.1 we now would have to display $\mathcal{U}$ as a function in a four-di-

mensional space spanned by the four real fields $\phi_1, \phi_2, \phi_3, \phi_4$. The minimum is a circle in with equation

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 2f^2 \ . \tag{29.29}$$

The field $\phi(x)$ will assume in the state $\phi_0(x)$ of lowest energy spontaneously a certain point on that four-dimensional circle. Without loss of generality we may shift this point onto the ϕ_1-axis, such that

$$\phi_0(x) \overset{(29.26)}{=} \sqrt{\frac{1}{2}} \begin{pmatrix} 0 + i0 \\ f\sqrt{2} + i0 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \ . \tag{29.30}$$

This is always possible, because in any case the coordinates of the four-dimensional plane can be rotated such (i. e. linear combinations of the four fields $\phi_1 \dots \phi_4$ can be formed such) that (29.30) results.

We define four fields $\tilde{\phi}_i(x)$ as excitations of $\phi(x)$ from the field's state of lowest energy $\phi_0(x)$:

$$\phi(x) \equiv \phi_0(x) + \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\phi}_3(x) + i\tilde{\phi}_4(x) \\ \tilde{\phi}_1(x) + i\tilde{\phi}_2(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\phi}_3(x) + i\tilde{\phi}_4(x) \\ f\sqrt{2} + \tilde{\phi}_1(x) + i\tilde{\phi}_2(x) \end{pmatrix}$$
$$\text{with } \tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4, f \in \mathbb{R} \tag{29.31}$$

These are inserted into the Lagrangian (29.23):

$$\mathcal{L} \overset{(29.23)}{=} c^2\hbar^2 (\mathrm{D}_\mu \phi^\dagger)\, \mathrm{D}^\mu \phi + \frac{m^2c^4}{2} \phi^\dagger\phi - \frac{m^2c^4}{4f^2} (\phi^\dagger\phi)^2 \tag{29.32a}$$

$$\overset{(29.26)}{=} \sum_{i=1}^{4} \frac{c^2\hbar^2}{2} (\mathrm{D}_\mu \phi_i)^\dagger\, \mathrm{D}^\mu \phi_i + \frac{m^2c^4}{4} \sum_{i=1}^{4} \phi_i\phi_i - \frac{m^2c^4}{16f^2} \Big(\sum_{i=1}^{4} \phi_i\phi_i \Big)^2 \tag{29.32b}$$

$$\overset{(29.31)}{=} \sum_{i=1}^{4} \frac{c^2\hbar^2}{2} (\mathrm{D}_\mu \tilde{\phi}_i)^\dagger\, \mathrm{D}^\mu \tilde{\phi}_i + \frac{m^2c^4}{4} \Big(\sum_{i=1}^{4} \tilde{\phi}_i\tilde{\phi}_i + \sqrt{8} f \tilde{\phi}_1 + 2f^2 \Big) -$$
$$- \frac{m^2c^4}{16f^2} \Big(\sum_{i=1}^{4} \tilde{\phi}_i\tilde{\phi}_i + \sqrt{8} f \tilde{\phi}_1 + 2f^2 \Big)^2$$

$$= \sum_{i=1}^{4} \frac{c^2\hbar^2}{2} \left(\mathrm{D}_\mu \tilde{\phi}_i\right)^\dagger \mathrm{D}^\mu \tilde{\phi}_i - \frac{m^2c^4}{2} (\tilde{\phi}_1)^2 - \frac{m^2c^4}{16f^2} \Big(\sum_{i=1}^{4} \tilde{\phi}_i \tilde{\phi}_i \Big)^2 -$$
$$- \frac{m^2c^4}{f\sqrt{8}} \tilde{\phi}_1 \sum_{i=1}^{4} \tilde{\phi}_i \tilde{\phi}_i + \frac{m^2c^4 f^2}{4} \tag{29.32c}$$

It's plausible to consider the ϕ_i respectively $\tilde{\phi}_i$ in (29.32b) as the true elementary fields, and the weak isospinor $\phi(x)$ just as a combination of elementary fields, like e. g. $\binom{\nu_e}{e^-}_L$ is just a weak isospinor built from the elementary fields $\nu_{e,L}$ and e^-_L.

Bilinear field dependence with a negative coefficient is the hallmark of mass terms in a Lagrangian. The only term of that type is $-m^2c^4(\tilde{\phi}_1)^2/2$. Thus $\tilde{\phi}_1$ is the only massive field in in (29.32c). It's mass is m, as can be concluded from the Lagrangian (10.33) of the real Klein-Gordon field. The two next terms can not be interpreted as mass terms, because they are not quadratic in the field-amplitudes. They are describing self-interactions of the fields. Remember that we encountered similar self-interaction terms when we investigated ψ^3- and ψ^4-theory in chapter 20. Thus the Lagrangian (29.32c) is describing a massive real boson field $\tilde{\phi}_1$ and three mass-less real boson fields $\tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4$. The three massless bosons are called "Goldstone-bosons".

Note that we kept the † (which is marking the adjoint) at the factors $(\mathrm{D}_\mu \tilde{\phi}_i)^\dagger$, because only the fields ϕ_i resp. $\tilde{\phi}_i$ are real, while the gauge fields are complex.

The weak isospinor $\phi(x) =$ (29.31) can be written in two forms:

$$\begin{aligned} \phi(x) &\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\phi}_3(x) + i\tilde{\phi}_4(x) \\ f\sqrt{2} + \tilde{\phi}_1(x) + i\tilde{\phi}_2(x) \end{pmatrix} \equiv \\ &\equiv \frac{1}{\sqrt{2}} \exp\Big\{ i\rho_j(x) \frac{\tau^j}{2} \Big\} \begin{pmatrix} 0 \\ f\sqrt{2} + \chi(x) \end{pmatrix} \\ &\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4, \rho_j, f, \chi \in \mathbb{R} \ , \ j = 1,2,3 \end{aligned} \tag{29.33}$$

This is true, because the weak isospinor with length $f\sqrt{2} + \chi(x)$, which is orientated in negative T^3-direction, can be rotated by means of the rotation-

operator $\exp\{i\rho_j(x)\,\tau^j/2\}$ into any desired direction of weak isospin-space. In both notations, the weak isospinor $\phi(x)$ has four degrees of freedom, which can be parametrized either by $\tilde{\phi}^1, \tilde{\phi}^2, \tilde{\phi}^3, \tilde{\phi}^4$ or by $\rho_1, \rho_2, \rho_3, \chi$.

The cardinal point of the Higgs-mechanism is the postulate, that the Lagrangian (29.32) firstly shall be invariant under local U(1) phase-transformations $\exp\{iK(x)\,g_1 Y/2\}$, whose generator is the weak hypercharge $g_1 Y/2$, and secondly shall be invariant under local rotations $\exp\{iK_j(x)\,g_2\tau^j/2\}$ in weak isospin-space, whose generators are the three weak isospin-charges $g_2\tau^j/2$. For this reason, we have replaced the simple derivatives d_μ by the covariant derivatives $D_\mu =$ (29.9) resp. $D_\mu =$ (29.18).

In electrodynamics, the freedom to choose an arbitrary gauge can for example be used, to eliminate by means of the Coulomb-gauge the unphysical degrees of freedom of longitudinal and time-like polarizations of the photon-field. In the same manner we now make use of the invariance of the field $\phi(x)$ under local rotations in weak isospin-space to choose the gauge

$$\exp\{iK_j(x)\,g_2\tau^j/2\} \equiv \exp\{-i\rho_j(x)\,\tau^j/2\}\ . \tag{29.34}$$

This gauge is called "unitarity gauge", or sometimes "unitary gauge". In this gauge the field assumes the form

$$\phi(x) \overset{(29.33)}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ f\sqrt{2}+\chi(x) \end{pmatrix} \quad \text{with} \quad f,\chi \in \mathbb{R}\ . \tag{29.35}$$

As only the doublet's bottom component is different from zero, the spinor is in the gauge (29.34) an eigenvector of the hypercharge operator with eigenvalue $Y \overset{(29.28)}{=} +1$, and hence because of $q/e = Y/2 + T^3 = 0$ it also is an eigenvector of $\tau^3/2$ with eigenvalue $T^3 = -1/2$.

Comparing (29.33) with (29.35), we see that the three massless Goldstone-bosons $\tilde{\phi}_2, \tilde{\phi}_3, \tilde{\phi}_4$ have disappeared. This means: Like the Coulomb-gauge revealed that longitudinal polarization degrees of freedom are unphysical in case of photons, now the gauge transformation (29.34) has revealed that the three Goldstone-bosons are unphysical degrees of freedom. But degrees of freedom impossibly can simply disappear. In the sequel we will recover the three missing degrees of freedom as additional degrees of freedom of

the gauge fields, i. e. the spontaneous break of symmetry has shifted three degrees of freedom from the Goldstone-bosons to the gauge bosons.

By choice of the gauge (29.34), the Lagrangian (29.32c) assumes the simple form

$$\mathcal{L} = \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{HGB}} \tag{29.36a}$$

$$\mathcal{L}_{\text{Higgs}} = \frac{c^2\hbar^2}{2}\,(\mathrm{d}_\mu\chi)^\dagger \mathrm{d}^\mu\chi - \frac{m^2c^4}{2}\,\chi^2 - \frac{m^2c^4}{f\sqrt{8}}\,\chi^3 - \frac{m^2c^4}{16f^2}\,\chi^4 + \\ + \frac{m^2c^4 f^2}{4} \tag{29.36b}$$

$$\mathcal{L}_{\text{HGB}} \equiv \frac{c^2\hbar^2}{2}\left((\mathrm{D}_\mu\,\phi)^\dagger\,\mathrm{D}^\mu\,\phi - (\mathrm{d}_\mu\phi)^\dagger \mathrm{d}^\mu\phi\right). \tag{29.36c}$$

$\mathcal{L}_{\text{Higgs}}$ is the Lagrangian of the "bare" Higgs field, which is not interacting with other fields. But due to the χ^3- and χ^4-terms it is interacting with itself. The Higgs-field $\chi(x)$ is a real boson field with mass m.

The part $\mathcal{L}_{\text{HGB}}$ of the Lagrangian is describing the interaction of the Higgs field with the weak gauge bosons W^+, W^-, Z^0. The Higgs field can interact with those bosons due to it's weak isospin-charge and it's weak hypercharge. As the Higgs field doesn't carry an electric charge, it does not couple to the gauge boson A.

To make the Lagrangian invariant under local phase transformations, we replaced the normal differential operator d_μ by the covariant differential operator

$$\mathrm{D}_\mu(x) \overset{(29.9)}{=} \mathrm{d}_\mu + \frac{i}{\hbar}\,\frac{g_1}{2}\,Y B^0_\mu(x) + \frac{i}{\hbar}\,g_2 W^j_\mu(x)\,\frac{\tau^j}{2}\,. \tag{29.37}$$

Here we revert to the covariant differential operator in it's original notation instead of the converted form (29.18), because the following computations will be more transparent if the Pauli-matrices τ^j are explicitly visible. Of course the final result is independent of the applied notation. We compute

$$
\begin{aligned}
2(\mathrm{D}_\mu\,\phi)^\dagger\,\mathrm{D}^\mu\,\phi &= (\mathrm{d}_\mu\chi)\mathrm{d}^\mu\chi + \begin{pmatrix}0 & \mathrm{d}_\mu\chi\end{pmatrix}\frac{i}{\hbar}\Big(\frac{g_1}{2}\,Y(B^{0\mu}-B^{0\mu}) + \\
&+ g_2(W^{j\mu}-W^{j\mu})\,\frac{\tau^j}{2}\Big)\begin{pmatrix}0\\ f\sqrt{2}+\chi\end{pmatrix} + \begin{pmatrix}0 & f\sqrt{2}+\chi\end{pmatrix}\frac{1}{\hbar^2}\Big(\\
&\frac{g_1^2}{4}\,Y^2 B^0_\mu B^{0\mu} + \frac{g_1 g_2}{4}\,Y(B^0_\mu W^{j\mu} + B^{0\mu}W^j_\mu)\tau^j + \\
&+ \frac{g_2^2}{4}W^j_\mu W^{k\mu}\tau^j\tau^k\Big)\begin{pmatrix}0\\ f\sqrt{2}+\chi\end{pmatrix}\,. \qquad (29.38)
\end{aligned}
$$

Here $\tau^{j\dagger}=\tau^j$, $W^{j\dagger}_\mu = W^j_\mu$ and $B^{0\dagger}_\mu = B^0_\mu$ has been observed. Thus the factor $\sim(0\ \mathrm{d}_\mu\chi)$ disappears. In the last factor we may set $\left(\begin{smallmatrix}0\\ f\sqrt{2}+\chi\end{smallmatrix}\right)\approx\left(\begin{smallmatrix}0\\ f\sqrt{2}\end{smallmatrix}\right)$ in good approximation, if the field differs only slightly from ϕ_0. Using the anticommutator of the Pauli-matrices $\{\tau^j,\tau^k\}/2=\delta_{kj}$, using $(0\ f)\tau^1\left(\begin{smallmatrix}0\\ f\end{smallmatrix}\right) = (0\ f)\tau^2\left(\begin{smallmatrix}0\\ f\end{smallmatrix}\right)=0$, and using $2(0\ f)\tau^3\left(\begin{smallmatrix}0\\ f\end{smallmatrix}\right)=-2f^2$ and $Y=1$, we find

$$
\begin{aligned}
\mathcal{L}_{\mathrm{HGB}} &= \frac{f^2c^2}{4}\Big(+g_1^2B^0_\mu B^{0\mu} - g_1g_2B^0_\mu W^{3\mu} - g_1g_2W^3_\mu\,B^{0\mu} + g_2^2W^j_\mu W^{j\mu}\Big) \\
&= \frac{f^2c^2}{4}\Big(-g_1B^0_\mu + g_2W^3_\mu\Big)\Big(-g_1B^{0\mu}+g_2W^{3\mu}\Big) + \\
&\quad + \frac{f^2c^2g_2^2}{4}\,(W^1_\mu W^{1\mu}+W^2_\mu W^{2\mu})\,. \qquad (29.39)
\end{aligned}
$$

Now we insert

$$
\begin{aligned}
&2(W^+_\mu)^\dagger W^{+\mu} + 2(W^-_\mu)^\dagger W^{-\mu} \overset{(29.11)}{=} \\
&= (W^1_\mu + iW^2_\mu)^\dagger(W^{1\mu}+iW^{2\mu}) + (W^1_\mu - iW^2_\mu)^\dagger(W^{1\mu}-iW^{2\mu}) \\
&= W^1_\mu W^{1\mu} + iW^1_\mu W^{2\mu} - iW^2_\mu W^{1\mu} + W^2_\mu W^{2\mu} + \\
&\quad + W^1_\mu W^{1\mu} - iW^1_\mu W^{2\mu} + iW^2_\mu W^{1\mu} + W^2_\mu W^{2\mu} \\
&= 2(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu}) \qquad (29.40)
\end{aligned}
$$

and

$$Z_\mu^0 \overset{(29.13)}{=} -B_\mu^0 \sin\theta_W + W_\mu^0 \cos\theta_W \overset{(29.17),(29.11)}{=} \frac{-g_1 B_\mu^0 + g_2 W_\mu^3}{\sqrt{g_1^2 + g_2^2}} \tag{29.41}$$

and factors

$$1 = \zeta_+^2/\zeta_+^2 = \zeta_-^2/\zeta_-^2 = \zeta_0^2/\zeta_0^2 = \zeta_\mathrm{H}^2/\zeta_\mathrm{H}^2$$

into $\mathcal{L}_\mathrm{HGB}$:

$$\begin{aligned}\mathcal{L}_\mathrm{HGB} = {} & \frac{g_2^2 f^2 \zeta_+^2}{4c^2\zeta_\mathrm{H}^2}\, c^4\, \frac{\zeta_\mathrm{H}^2}{\zeta_+^2} W_\mu^+ W^{+\mu} + \frac{g_2^2 f^2 \zeta_-^2}{4c^2\zeta_\mathrm{H}^2}\, c^4\, \frac{\zeta_\mathrm{H}^2}{\zeta_-^2} W_\mu^- W^{-\mu} + \\ & + \frac{(g_1^2 + g_2^2) f^2 \zeta_0^2}{4c^2\zeta_\mathrm{H}^2}\, c^4\, \frac{\zeta_\mathrm{H}^2}{\zeta_0^2} Z_\mu^0 Z^{0\mu}\end{aligned} \tag{29.42}$$

The factors ζ_+, ζ_-, ζ_0, and ζ_H are normalization factors, which are fixed by the integrals over the normalization volume Ω:

$$\begin{aligned}1 &= \frac{1}{\zeta_+^2}\int_\Omega \mathrm{d}^3x\, W_\mu^+ W^{+\mu} = \frac{1}{\zeta_-^2}\int_\Omega \mathrm{d}^3x\, W_\mu^- W^{-\mu} = \\ &= \frac{1}{\zeta_0^2}\int_\Omega \mathrm{d}^3x\, Z_\mu^0 Z^{0\mu} = \frac{1}{\zeta_\mathrm{H}^2}\int_\Omega \mathrm{d}^3x\, \chi^2\end{aligned} \tag{29.43}$$

Therefore the dimensions of the normalization factors must cancel the dimensions of the fields:

$$[\zeta_+] = [\zeta_-] = [\zeta_0] = [W^\pm] = [Z^0] \overset{(29.19)}{=} \frac{\text{momentum}}{\text{charge}} \tag{29.44a}$$

$$[\zeta_\mathrm{H}] = [\phi] \overset{(10.12)}{=} [\chi] = [f] \overset{(29.24),(29.23)}{=} \frac{1}{\sqrt{\text{energy}\cdot\text{volume}}} \tag{29.44b}$$

From the analysis of the dimensions, we see that now mass terms of the gauge bosons have appeared in (29.42):

$$\left[\frac{g_2 f\,\zeta_+}{c\,\zeta_\mathrm{H}}\right] = \left[\frac{g_2 f\,\zeta_-}{c\,\zeta_\mathrm{H}}\right] = \left[\frac{\sqrt{g_1^2 + g_2^2}\, f\,\zeta_0}{c\,\zeta_\mathrm{H}}\right] = \text{mass} \tag{29.44c}$$

Note that the space-like components of the mass-terms have indeed the right, negative sign. For example: $Z^0_\mu Z^{0\mu} = Z^0_0 Z^{00} - Z^0_j Z^{0j}$ with $j = 1, 2, 3$. Consequently these are the properties of the four gauge-bosons in GSW-theory:

field	T	T^3	Y	$q/e = Y/2 + T^3$	mass
W^+	1	+1	0	+1	$g_2 f \zeta_+/(2c\,\zeta_\mathrm{H})$
W^-	1	−1	0	−1	$g_2 f \zeta_-/(2c\,\zeta_\mathrm{H})$
Z^0	n.d.	0	0	0	$\sqrt{g_1^2 + g_2^2}\, f \zeta_0/(2c\,\zeta_\mathrm{H})$
A	n.d.	0	0	0	0

(29.45)

n.d. means "not defined", because Z^0 and A are mixtures of the fields B^0 and $W^3 = W^0$, which don't have identical isospin T, while the component T^3 of both fields is zero, see table (29.12).

The mass-terms of the weak gauge-bosons have been brought about by coupling to the field (29.33). The originally 4 degrees of freedom of that field have been reduced to only one in the form (29.35) due to the spontaneous break of symmetry. The 3 degrees of freedom, which there have disappeared, now turn up again as additional polarisation-degrees of freedom of the weak gauge-bosons. As long as these bosons were mass-less, they had — just like the photon — only two transversal polarization-degrees of freedom. Thanks to the mass-terms in (29.42) they now in addition can be polarized longitudinally.

Although the weak isospin doublet $\phi(x)$ is not identical with the Higgs-field $\chi(x)$, as is obvious from (29.35), the name Higgs-field is often applied synonymously to both $\chi(x)$ and $\phi(x)$ in the literature. The experimentally observable Higgs-boson with mass m according to (29.36b) however is clearly and uniquely the quantum of the field $\chi(x)$.

The GSW-model of electroweak interactions is such an intricate construction, that probably nobody would take it serious, unless it had been impressively verified by experimental observations in the recent decades. In the year 1973 the neutral weak currents, which are mediated by the Z^0-boson, could first time be demonstrated experimentally. The direct experimental verification of the three bosons W^+, W^- (mass approximately $80\,\mathrm{GeV}/c^2$), and Z^0 (mass approximately $91\,\mathrm{GeV}/c^2$) followed in 1983. And

in 2012, after many years of intensive search, the experimentalists eventually announced the observation of the Higgs-boson χ with mass $m \approx 125\,\mathrm{GeV}/c^2$. Obviously not the theorists are to blame for the theory's complicated structure, but Nature herself, who is not considerate of our preference for simple theories.

29.4 ES-tensor of the Higgs field

The Higgs field $\chi(x)$ with Lagrangian $\mathcal{L}_{\text{Higgs}} =$ (29.36b) is a real scalar field, like the real Klein-Gordon field. Hence this is the Higgs field's ES-tensor:

classical Higgs field:

$$\mathcal{T}^{\rho\sigma} \overset{(10.36)}{=} \frac{\partial \mathcal{L}_{\text{Higgs}}}{\partial(\mathrm{d}_\rho \chi)} \mathrm{d}^\sigma \chi - g^{\rho\sigma} \mathcal{L}_{\text{Higgs}} \tag{29.46a}$$

quantized Higgs field:

$$\mathcal{T}^{\rho\sigma} \overset{(15.18)}{=} \frac{\partial \mathcal{L}_{\text{Higgs}}}{\partial(\mathrm{d}_\rho \chi)} \mathrm{d}^\sigma \chi - g^{\rho\sigma} \mathcal{L}_{\text{Higgs}} - Y \tag{29.46b}$$

$Y \equiv$ the sum of all terms in (29.46a) which do not depend on the particle-number operators $a^\dagger_{\boldsymbol{k}}\, a_{\boldsymbol{k}}$

With Y not only the unphysical adders due to the commutators $[a^\dagger_{\boldsymbol{k}}\,,\, a_{\boldsymbol{k}}] = 1$ are removed, but as well the unphysical energy density caused by the last term in (29.36b). Due to measurements of myon decay, the coupling constant g_2 could be determined. Thereby the parameter

$$f = 246\,\mathrm{GeV}(\hbar c)^{-3/2} \tag{29.47}$$

could be computed from the $W^\pm$ mass term (29.45). With this value we get

$$\frac{m^2 c^4 f^2}{4} = \frac{(125\,\mathrm{GeV})^2 (246\,\mathrm{GeV})^2}{4(\hbar c)^3} \approx 2 \cdot 10^{34}\, \frac{\mathrm{J}}{\mathrm{m}^3}\ . \tag{29.48}$$

Comparing this value with the energy density of intergalactic vacuum as concluded from astronomical observations [42]

$$\mathcal{T}_{00}^{\text{vacuum}} \approx 5.4 \cdot 10^{-10}\,\text{J/m}^3 \ , \tag{29.49}$$

we find the ratio

$$\frac{\text{theory}}{\text{observation}} = \frac{(29.48)}{(29.49)} \approx 3.7 \cdot 10^{43} \ . \tag{29.50}$$

This catastrophic mismatch confirms, that the removal of Y in (29.46b) indeed is a sensible measure.

29.5 Fermion Mass

Until now we described all free (i. e. not interacting) elementary fields with spin 1/2, i. e. all leptons and all quarks, by a Lagrangian of the form

$$\mathcal{L}_0 \overset{(8.24)}{=} \overline{\psi}\,(i\hbar c\gamma^\nu \mathrm{d}_\nu - mc^2)\,\psi \ , \tag{29.51}$$

from which the Dirac equation

$$(i\hbar c\gamma^\mu \mathrm{d}_\mu - mc^2)\,\psi \overset{(8.5)}{=} 0 \tag{29.52}$$

can be derived. $\psi(x)$ is a four-component spinor field. Inserting explicitly the γ-matrices (8.15), the Dirac equation is

$$\begin{aligned}(i\hbar c \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \mathrm{d}_0 + i\hbar c \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \mathrm{d}_k - mc^2)\psi = \\ = \begin{pmatrix} -mc^2 & i\hbar c(\mathrm{d}_0 + \sigma^k \mathrm{d}_k) \\ i\hbar c(\mathrm{d}_0 - \sigma^k \mathrm{d}_k) & -mc^2 \end{pmatrix} \psi = 0 \ ,\end{aligned} \tag{29.53}$$

with σ^k being the 2×2-component Pauli matrices. If the rest energy mc^2 is zero or negligible, then the four-component Dirac equation decouples into the Weyl equations (8.115):

$$i\hbar c(\mathrm{d}_0 - \sigma^k \mathrm{d}_k) \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = 0 \qquad , \qquad i\hbar c(\mathrm{d}_0 + \sigma^k \mathrm{d}_k) \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = 0 \tag{29.54}$$

$\binom{L_1}{L_2}$ is a two-component left-handed Weyl-spinor, and $\binom{R_1}{R_2}$ is a two-component right-handed Weyl-spinor.

Now in the GSW-model of electroweak interaction, as outlined in the previous sections, the fermion fields are strictly divided into the left-handed weak isospin doublets (29.4) and the right-handed singlets (29.7). A free-field Lagrangian like (29.51) is not compatible with the GSW-model, because the mass term mixes the left- and right-handed components of the Dirac spinor $\psi(x)$.

If we want to achieve a fermion theory, which is compatible with the GSW-model, then the free fermion fields must be mass-less:

$$\mathcal{L}_0 = \overline{\psi}\, i\hbar c \gamma^\nu \mathrm{d}_\nu\, \psi \tag{29.55}$$

The masses of the fermions then can be introduced by interaction with the Higgs field. As the Higgs field is no gauge field, it is not introduced due to a covariant differential operator. Instead an interaction term can be introduced e. g. for the up- and down-quarks like this:

$$\begin{aligned}
\mathcal{L}_{\text{int}} = -\lambda_d \Bigg[\overline{\begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix}_L} \underbrace{\begin{pmatrix}0\\ f+\frac{\chi}{\sqrt{2}}\end{pmatrix}}_{(29.35)} \psi_{dR} + \overline{\psi_{dR}} \begin{pmatrix}0\\ f+\frac{\chi}{\sqrt{2}}\end{pmatrix}^\dagger \begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix}_L \Bigg] - \\
- \lambda_u \Bigg[\overline{\begin{pmatrix}\psi_d\\ \psi_u\end{pmatrix}_L} \begin{pmatrix}0\\ f+\frac{\chi}{\sqrt{2}}\end{pmatrix} \psi_{uR} + \overline{\psi_{uR}} \begin{pmatrix}0\\ f+\frac{\chi}{\sqrt{2}}\end{pmatrix}^\dagger \begin{pmatrix}\psi_d\\ \psi_u\end{pmatrix}_L \Bigg] \\
= -(f+\frac{\chi}{\sqrt{2}}) \Big(\lambda_d \overline{\psi_{dL}} \psi_{dR} + \lambda_d \overline{\psi_{dR}} \psi_{dL} + \\
+ \lambda_u \overline{\psi_{uL}} \psi_{uR} + \lambda_u \overline{\psi_{uR}} \psi_{uL} \Big) \qquad (29.56a) \\
0 < \lambda_d \in \mathbb{R} \quad , \quad 0 < \lambda_u \in \mathbb{R}
\end{aligned}$$

λ_d and λ_u are new coupling constants. Because of

$$[\mathcal{L}] = \frac{\text{energy}}{\text{volume}} \ , \ [\overline{\psi}\psi] = \frac{1}{\text{volume}} \ , \ [\chi] = [f] = \frac{1}{\sqrt{\text{energy} \cdot \text{volume}}} \ ,$$

these coupling constants must have the dimension

$$[\lambda] = \text{energy}^{3/2} \cdot \text{volume}^{1/2} \ . \tag{29.56b}$$

Due to $\lambda_d f > 0$ and $\lambda_u f > 0$ being positive constants, the terms $\sim f$ in $\mathcal{L}_{\text{int}}$ indeed are mass terms, assigning masses $m_d = \lambda_d f/c^2$ and $m_u = \lambda_u f/c^2$ to the quarks. In contrast, the terms $\sim \chi\overline{\psi}\psi$ in $\mathcal{L}_{\text{int}}$ are no mass terms but interaction terms, as they do not depend bilinearly on the fields.

This is the corresponding expression for the electron and the electron-neutrino:

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= -\lambda_e \left[\overline{\begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}}_L \begin{pmatrix} 0 \\ f + \dfrac{\chi}{\sqrt{2}} \end{pmatrix} \psi_{eR} + \overline{\psi_{eR}} \begin{pmatrix} 0 \\ f + \dfrac{\chi}{\sqrt{2}} \end{pmatrix}^\dagger \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix}_L \right] - \\
&\quad - \lambda_{\nu_e} \left[\overline{\begin{pmatrix} \psi_e \\ \psi_{\nu_e} \end{pmatrix}}_L \begin{pmatrix} 0 \\ f + \dfrac{\chi}{\sqrt{2}} \end{pmatrix} \psi_{\nu_e R} + \overline{\psi_{\nu_e R}} \begin{pmatrix} 0 \\ f + \dfrac{\chi}{\sqrt{2}} \end{pmatrix}^\dagger \begin{pmatrix} \psi_e \\ \psi_{\nu_e} \end{pmatrix}_L \right] \\
&= -(f + \frac{\chi}{\sqrt{2}}) \Big(\lambda_e \, \overline{\psi_{eL}} \psi_{eR} + \lambda_e \, \overline{\psi_{eR}} \psi_{eL} + \\
&\qquad\qquad + \lambda_{\nu_e} \, \overline{\psi_{\nu_e L}} \psi_{\nu_e R} + \lambda_{\nu_e} \, \overline{\psi_{\nu_e R}} \psi_{\nu_e L} \Big)
\end{aligned} \tag{29.57}$$

As the masses of the experimentally observed fermions differ widely, different coupling constants λ are needed for each fermion flavor. All of them must be determined experimentally, as the Standard Model does not give us any clue of their value nor any reason why not all fermions couple to the Higgs field with equal strength.

Appendix

A.1 Derivation of Equation (6.65)

We substitute the indices r, r' by

$$m \equiv r - j - 1 \qquad \text{and} \qquad m' \equiv r' - j - 1\,. \tag{A.1}$$

Furthermore we will write all summations explicitly (i.e. we suspend the summation convention) from now on until (including) equation (A.6).

$$\begin{aligned}\phi^{(j)}_{m'} &\overset{(6.64)}{=} \sum_{m=-j}^{j} D^{(j)}_{m'm}\phi^{(j)}_{m} \overset{(6.56a)}{=} \sum_{m=-j}^{j} D^{(j)}_{m'm}N^{(j)}_{m}u^{j-m}v^{j+m} \\ &\overset{(6.62)}{=} N^{(j)}_{m'}(au+bv)^{j-m'}(-b^*u+a^*v)^{j+m'}\end{aligned} \tag{A.2}$$

The summation is running over m only, but not over m'. Using the binomial theorem, one finds:

$$\begin{aligned}\sum_{m=-j}^{j} D^{(j)}_{m'm}\frac{N^{(j)}_{m}}{N^{(j)}_{m'}}u^{j-m}v^{j+m} &= (au+bv)^{j-m'}(-b^*u+a^*v)^{j+m'} \\ &= \sum_{k=0}^{j-m'}\frac{(j-m')!}{k!(j-m'-k)!}(au)^{j-m'-k}(bv)^{k}\cdot \\ &\quad\cdot\sum_{l=0}^{j+m'}\frac{(j+m')!}{l!(j+m'-l)!}(-b^*u)^{j+m'-l}(a^*v)^{l} \\ &= \sum_{k=0}^{j-m'}\sum_{l=0}^{j+m'}\frac{(j-m')!}{k!(j-m'-k)!}\frac{(j+m')!}{l!(j+m'-l)!}\cdot \\ &\quad\cdot a^{j-m'-k}b^{k}(-b^*)^{j+m'-l}a^{*l}u^{2j-k-l}v^{k+l}\end{aligned} \tag{A.3}$$

The products $u^p v^q$ with different powers of p, q are linearly independent from another. Therefore the left and the right side of the equation must equal for each summand, if $j - m = 2j - k - l$ and $j + m = k + l$. Both conditions are fulfilled, if $m = -j + k + l$. Therefore we introduce a new parameter m on the equation's right side due to the substitution $l \equiv j - k + m$:

$$\begin{aligned}\sum_{m=-j}^{j} D^{(j)}_{m'm} \frac{N^{(j)}_m}{N^{(j)}_{m'}} u^{j-m} v^{j+m} \\ = \sum_{k=0}^{j-m'} \sum_{m=k-j}^{k+m'} \frac{(j-m')!}{k!(j-m'-k)!} \frac{(j+m')!}{(j-k+m)!(m'+k-m)!} \cdot \\ \cdot a^{j-m'-k} b^k (-b^*)^{m'+k-m} a^{*j-k+m} u^{j-m} v^{j+m} \end{aligned} \tag{A.4}$$

The faculty of integers is by definition

$$\left.\begin{array}{ll} z! \equiv 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (z-1) \cdot z & \text{if } z > 0 \\ z! \equiv 1 & \text{if } z = 0 \\ z! \equiv \pm\infty & \text{if } z < 0 \end{array}\right\} z \in \mathbb{Z} \,. \tag{A.5}$$

Using this definition, the limits of the summation over m may be shifted to $\pm j$. Furthermore — because the summation over m now does not depend on k any more — the sequence of the both summations may be changed:

$$\begin{aligned}\sum_{m=-j}^{j} D^{(j)}_{m'm} \frac{N^{(j)}_m}{N^{(j)}_{m'}} u^{j-m} v^{j+m} \\ = \sum_{m=-j}^{j} \sum_{k=0}^{j-m'} \frac{(j-m')!}{k!(j-m'-k)!} \frac{(j+m')!}{(j-k+m)!(m'+k-m)!} \cdot \\ \cdot a^{j-m'-k} b^k (-b^*)^{m'+k-m} a^{*j-k+m} u^{j-m} v^{j+m} \end{aligned} \tag{A.6}$$

This is correct, because all of the additional terms with $m < k - j$ and with $m > k + m'$ are zero, as there are the factors $(j - k + m)! = \pm\infty$ resp. $(m' + k - m)! = \pm\infty$ in their denominators.

As same powers of $u^p v^q$ must equal, one gets:

$$D^{(j)}_{m'm} = \sum_{k=0}^{j-m'} \frac{(j-m')!}{k!(j-m'-k)!} \frac{(j+m')!}{(j-k+m)!(m'+k-m)!} \cdot \cdot \frac{N^{(j)}_{m'}}{N^{(j)}_{m}} \cdot a^{j-m'-k} b^k (-b^*)^{m'+k-m} a^{*j-k+m} \tag{A.7}$$

Inserting the normalization factors (6.56b), the final result becomes:

$$D^{(j)}_{m'm} = \sum_{k=0}^{j-m'} \frac{\sqrt{(j-m')!(j+m')!(j+m)!(j-m)!}}{k!(j-m'-k)!(j-k+m)!(m'+k-m)!} \cdot \cdot a^{j-m'-k} b^k (-b^*)^{m'+k-m} a^{*j-k+m} \tag{A.8}$$

Now we switch back to the previous indices

$$r = j+1+m \qquad , \qquad r' = j+1+m' \ , \tag{A.9}$$

and find equation (6.65):

$$D^{(j)}_{r'r} = \sum_{k=0}^{2j+1-r'} \frac{\sqrt{(2j+1-r')!(r'-1)!(r-1)!(2j+1-r)!}}{k!(2j+1-r'-k)!(-k+r-1)!(r'+k-r)!} \cdot \cdot a^{2j+1-r'-k} b^k (-b^*)^{r'+k-r} a^{*-k+r-1}$$

A.2 Auxiliary Calculation for (6.67)

$$D^{(0)}_{00} = \sum_{k=0}^{0} \frac{\sqrt{0!0!0!0!}}{k!(-k)!(-k)!k!} \cdot a^{-k} b^k (-b^*)^k a^{*-k} = 1 \tag{A.10}$$

$$D^{(\frac{1}{2})}_{-\frac{1}{2}-\frac{1}{2}} = \sum_{k=0}^{1} \frac{\sqrt{1!0!0!1!}}{k!(1-k)!(-k)!k!} \cdot a^{1-k} b^k (-b^*)^k a^{*-k} = a \tag{A.11a}$$

$$D^{(\frac{1}{2})}_{-\frac{1}{2}\frac{1}{2}} = \sum_{k=0}^{1} \frac{\sqrt{1!0!1!0!}}{k!(1-k)!(1-k)!(k-1)!} \cdot a^{1-k} b^k (-b^*)^{k-1} a^{*1-k} = b \tag{A.11b}$$

$$D^{(\frac{1}{2})}_{\frac{1}{2}-\frac{1}{2}} = \sum_{k=0}^{0} \frac{\sqrt{0!1!0!1!}}{k!(-k)!(-k)!(1+k)!} \cdot a^{-k} b^k (-b^*)^{1+k} a^{*-k} = -b^* \quad \text{(A.11c)}$$

$$D^{(\frac{1}{2})}_{\frac{1}{2}\frac{1}{2}} = \sum_{k=0}^{0} \frac{\sqrt{0!1!1!0!}}{k!(-k)!(1-k)!k!} \cdot a^{-k} b^k (-b^*)^{k} a^{*1-k} = a^* \quad \text{(A.11d)}$$

$$D^{(1)}_{-1-1} = \sum_{k=0}^{2} \frac{\sqrt{2!0!0!2!}}{k!(2-k)!(-k)!k!} \cdot \cdot a^{2-k} b^k (-b^*)^{k} a^{*-k} = a^2 \quad \text{(A.12a)}$$

$$D^{(1)}_{-10} = \sum_{k=0}^{2} \frac{\sqrt{2!0!1!1!}}{k!(2-k)!(1-k)!(-1+k)!} \cdot \cdot a^{2-k} b^k (-b^*)^{-1+k} a^{*1-k} = \sqrt{2} ab \quad \text{(A.12b)}$$

$$D^{(1)}_{-11} = \sum_{k=0}^{2} \frac{\sqrt{2!0!2!0!}}{k!(2-k)!(2-k)!(k-2)!} \cdot \cdot a^{2-k} b^k (-b^*)^{k-2} a^{*2-k} = b^2 \quad \text{(A.12c)}$$

$$D^{(1)}_{0-1} = \sum_{k=0}^{1} \frac{\sqrt{1!1!0!2!}}{k!(1-k)!(-k)!(k+1)!} \cdot \cdot a^{1-k} b^k (-b^*)^{k+1} a^{*-k} = -\sqrt{2} ab^* \quad \text{(A.12d)}$$

$$D^{(1)}_{00} = \sum_{k=0}^{1} \frac{\sqrt{1!1!1!1!}}{k!(1-k)!(1-k)!k!} \cdot \cdot a^{1-k} b^k (-b^*)^{k} a^{*1-k} = aa^* - bb^* \quad \text{(A.12e)}$$

$$D^{(1)}_{01} = \sum_{k=0}^{1} \frac{\sqrt{1!1!2!0!}}{k!(1-k)!(2-k)!(k-1)!} \cdot \cdot a^{1-k} b^k (-b^*)^{k-1} a^{*2-k} = \sqrt{2} ba^* \quad \text{(A.12f)}$$

$$D^{(1)}_{1-1} = \sum_{k=0}^{0} \frac{\sqrt{0!2!0!2!}}{k!(-k)!(-k)!(2+k)!} \cdot \cdot a^{-k} b^k (-b^*)^{2+k} a^{*-k} = b^{*2} \quad \text{(A.12g)}$$

$$D^{(1)}_{10} = \sum_{k=0}^{0} \frac{\sqrt{0!2!1!1!}}{k!(-k)!(1-k)!(1+k)!} \cdot \\ \cdot\, a^{-k} b^k (-b^*)^{1+k} a^{*1-k} = -\sqrt{2} b^* a^* \tag{A.12h}$$

$$D^{(1)}_{11} = \sum_{k=0}^{0} \frac{\sqrt{0!2!2!0!}}{k!(-k)!(2-k)!k!} \cdot \\ \cdot\, a^{-k} b^k (-b^*)^{k} a^{*2-k} = a^{*2} \tag{A.12i}$$

$$D^{(\frac{3}{2})}_{-\frac{3}{2}-\frac{3}{2}} = \sum_{k=0}^{3} \frac{\sqrt{3!0!0!3!}}{k!(3-k)!(-k)!k!} \cdot \\ \cdot\, a^{3-k} b^k (-b^*)^{k} a^{*-k} = a^3 \tag{A.13a}$$

$$D^{(\frac{3}{2})}_{-\frac{3}{2}-\frac{1}{2}} = \sum_{k=0}^{3} \frac{\sqrt{3!0!1!2!}}{k!(3-k)!(1-k)!(-1+k)!} \cdot \\ \cdot\, a^{3-k} b^k (-b^*)^{-1+k} a^{*1-k} = \sqrt{3} a^2 b \tag{A.13b}$$

$$D^{(\frac{3}{2})}_{-\frac{3}{2}\frac{1}{2}} = \sum_{k=0}^{3} \frac{\sqrt{3!0!2!1!}}{k!(3-k)!(2-k)!(-2+k)!} \cdot \\ \cdot\, a^{3-k} b^k (-b^*)^{-2+k} a^{*2-k} = \sqrt{3} a b^2 \tag{A.13c}$$

$$D^{(\frac{3}{2})}_{-\frac{3}{2}\frac{3}{2}} = \sum_{k=0}^{3} \frac{\sqrt{3!0!3!0!}}{k!(3-k)!(3-k)!(-3+k)!} \cdot \\ \cdot\, a^{3-k} b^k (-b^*)^{-3+k} a^{*3-k} = b^3 \tag{A.13d}$$

$$D^{(\frac{3}{2})}_{-\frac{1}{2}-\frac{3}{2}} = \sum_{k=0}^{2} \frac{\sqrt{2!1!0!3!}}{k!(2-k)!(-k)!(k+1)!} \cdot \\ \cdot\, a^{2-k} b^k (-b^*)^{k+1} a^{*-k} = -\sqrt{3} a^2 b^* \tag{A.13e}$$

$$D^{(\frac{3}{2})}_{-\frac{1}{2}-\frac{1}{2}} = \sum_{k=0}^{2} \frac{\sqrt{2!1!1!2!}}{k!(2-k)!(1-k)!k!} \cdot \\ \cdot\, a^{2-k} b^k (-b^*)^{k} a^{*1-k} = a^2 a^* - 2abb^* \tag{A.13f}$$

$$D^{(\frac{3}{2})}_{-\frac{1}{2}\frac{1}{2}} = \sum_{k=0}^{2} \frac{\sqrt{2!1!2!1!}}{k!(2-k)!(2-k)!(k-1)!} \cdot$$
$$\cdot\ a^{2-k} b^k (-b^*)^{k-1} a^{*2-k} = 2aba^* - b^2 b^* \tag{A.13g}$$

$$D^{(\frac{3}{2})}_{-\frac{1}{2}\frac{3}{2}} = \sum_{k=0}^{2} \frac{\sqrt{2!1!3!0!}}{k!(2-k)!(3-k)!(k-2)!} \cdot$$
$$\cdot\ a^{2-k} b^k (-b^*)^{k-2} a^{*3-k} = \sqrt{3} b^2 a^* \tag{A.13h}$$

$$D^{(\frac{3}{2})}_{\frac{1}{2}-\frac{3}{2}} = \sum_{k=0}^{1} \frac{\sqrt{1!2!0!3!}}{k!(1-k)!(-k)!(2+k)!} \cdot$$
$$\cdot\ a^{1-k} b^k (-b^*)^{2+k} a^{*-k} = \sqrt{3} a b^{*2} \tag{A.13i}$$

$$D^{(\frac{3}{2})}_{\frac{1}{2}-\frac{1}{2}} = \sum_{k=0}^{1} \frac{\sqrt{1!2!1!2!}}{k!(1-k)!(1-k)!(1+k)!} \cdot$$
$$\cdot\ a^{1-k} b^k (-b^*)^{1+k} a^{*1-k} = -2ab^* a^* + bb^{*2} \tag{A.13j}$$

$$D^{(\frac{3}{2})}_{\frac{1}{2}\frac{1}{2}} = \sum_{k=0}^{1} \frac{\sqrt{1!2!2!1!}}{k!(1-k)!(2-k)!k!} \cdot$$
$$\cdot\ a^{1-k} b^k (-b^*)^{k} a^{*2-k} = aa^{*2} - 2bb^* a^* \tag{A.13k}$$

$$D^{(\frac{3}{2})}_{\frac{1}{2}\frac{3}{2}} = \sum_{k=0}^{1} \frac{\sqrt{1!2!3!0!}}{k!(1-k)!(3-k)!(k-1)!} \cdot$$
$$\cdot\ a^{1-k} b^k (-b^*)^{k-1} a^{*3-k} = \sqrt{3} b a^{*2} \tag{A.13l}$$

$$D^{(\frac{3}{2})}_{\frac{3}{2}-\frac{3}{2}} = \sum_{k=0}^{0} \frac{\sqrt{0!3!0!3!}}{k!(-k)!(-k)!(3+k)!} \cdot$$
$$\cdot\ a^{-k} b^k (-b^*)^{3+k} a^{*-k} = -b^{*3} \tag{A.13m}$$

$$D^{(\frac{3}{2})}_{\frac{3}{2}-\frac{1}{2}} = \sum_{k=0}^{0} \frac{\sqrt{0!3!1!2!}}{k!(-k)!(1-k)!(2+k)!} \cdot$$
$$\cdot\ a^{-k} b^k (-b^*)^{2+k} a^{*1-k} = \sqrt{3} b^{*2} a^* \tag{A.13n}$$

$$D^{(\frac{3}{2})}_{\frac{3}{2}\frac{1}{2}} = \sum_{k=0}^{0} \frac{\sqrt{0!3!2!1!}}{k!(-k)!(2-k)!(1+k)!} \cdot \\ \cdot\ a^{-k} b^k (-b^*)^{1+k} a^{*2-k} = -\sqrt{3} b^* a^{*2} \tag{A.13o}$$

$$D^{(\frac{3}{2})}_{\frac{3}{2}\frac{3}{2}} = \sum_{k=0}^{0} \frac{\sqrt{0!3!3!0!}}{k!(-k)!(3-k)!k!} \cdot a^{-k} b^k (-b^*)^k a^{*3-k} = a^{*3} \tag{A.13p}$$

A.3 The Number of Free Parameters of SL(2,C)

The elements of the group SL(2,$\mathbb{C}$) are 2×2-matrices with complex elements:

$$\begin{pmatrix} a e^{i\alpha} & b e^{i\beta} \\ c e^{i\gamma} & d e^{i\delta} \end{pmatrix} \quad \text{with } a,b,c,d,\alpha,\beta,\gamma,\delta \in \mathbb{R} \tag{A.14}$$

The 8 real parametes of these matrices are restricted by the conditions for the determinant:

$$\det \begin{pmatrix} a e^{i\alpha} & b e^{i\beta} \\ c e^{i\gamma} & d e^{i\delta} \end{pmatrix} = a d e^{i(\alpha+\delta)} - b c e^{i(\beta+\gamma)} = 1$$

$$\Longrightarrow \quad \begin{cases} a d \cos(\alpha+\delta) - b c \cos(\beta+\gamma) = 1 \\ a d \sin(\alpha+\delta) - b c \sin(\beta+\gamma) = 0 \end{cases} \tag{A.15}$$

We want to compute, how many real parameters are needed to determine uniquely a generic element of SL(2,$\mathbb{C}$). For that purpose we consider four cases:

1^{st} *Case:* $a = 0\ ,\ b,c,d \neq 0$
The condition for the determinant is

$$-bce^{i(\beta+\gamma)} \overset{(A.15)}{=} 1$$

$$\Longrightarrow \quad \begin{cases} c = -b^{-1} e^{-i(\beta+\gamma)} \\ \gamma = -\beta + \pi/2 \pm \pi/2\ . \end{cases} \tag{A.16}$$

In this case, the most general form of an element in SL(2,$\mathbb{C}$) is

$$\begin{pmatrix} 0 & be^{i\beta} \\ -b^{-1}e^{i(-\beta+\pi/2\pm\pi/2)} & de^{i\delta} \end{pmatrix} \tag{A.17}$$

There are 4 free parameters plus a double-valued phase. Obviously there is the same number of free parameters, whenever exactly one of the four parameters a, b, c, d is zero, no matter which one.

2^{nd} Case: $a, b, c, d \neq 0$, $\cos(\alpha + \delta) = 0$
The condition is

$$-bc\cos(\beta+\gamma) \overset{(A.15)}{=} 1 \tag{A.18}$$

$$\pm ad - bc\sin(\beta+\gamma) \overset{(A.15)}{=} 0$$

$$\implies \quad \begin{cases} c = \dfrac{1}{-b\cos(\beta+\gamma)} \\ d = \pm\, a^{-1}b\, \dfrac{1}{-b\cos(\beta+\gamma)} \sin(\beta+\gamma) \\ \quad = e^{i(\frac{\pi}{2}\pm\frac{\pi}{2})} a^{-1} \dfrac{\sin(\beta+\gamma)}{\cos(\beta+\gamma)} \,. \end{cases} \tag{A.19}$$

Because of $\delta = -\alpha \pm \pi/2$, the most general form of an element in SL(2,$\mathbb{C}$) in this case is

$$\begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ -b^{-1}\frac{1}{\cos(\beta+\gamma)}e^{i\gamma} & a^{-1}\frac{\sin(\beta+\gamma)}{\cos(\beta+\gamma)}e^{i(-\alpha+\frac{\pi}{2}\pm\pi)} \end{pmatrix} . \tag{A.20}$$

There are 5 free parameters, plus a double-valued phase. In the case $\cos(\beta + \gamma) = 0$, there obviously are the same number of free parameters.

3^{rd} Case: $a, b, c, d \neq 0$, $\sin(\alpha + \delta) = 0$
In this case, there must be $\sin(\beta+\gamma) = 0$ as well. Therefore $\delta = -\alpha + \frac{\pi}{2} \pm \frac{\pi}{2}$ and $\gamma = -\beta + \frac{\pi}{2} \pm \frac{\pi}{2}$ must hold, and consequently

$$ade^{i(\frac{\pi}{2}\pm\frac{\pi}{2})} - bce^{i(\frac{\pi}{2}\pm\frac{\pi}{2})} \overset{(A.15)}{=} 1$$

$$\implies \quad d = (1 + bce^{i(\frac{\pi}{2}\pm\frac{\pi}{2})})a^{-1}e^{-i(\frac{\pi}{2}\pm\frac{\pi}{2})} \tag{A.21}$$

Note, that the two undetermined phases are resulting from two independent sources: The one is resulting from $\gamma = -\beta + \frac{\pi}{2} \pm \frac{\pi}{2}$, the other from $\delta = -\alpha + \frac{\pi}{2} \pm \frac{\pi}{2}$. Only phase-factors, which are resulting from the identical source, may be charged against one another. The most general form of an element in SL(2,$\mathbb{C}$) in this case is

$$\begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ ce^{i(-\beta+\frac{\pi}{2}\pm\frac{\pi}{2})} & (1+bce^{i(\frac{\pi}{2}\pm\frac{\pi}{2})})a^{-1}e^{i(-\alpha)} \end{pmatrix} . \tag{A.22}$$

There are 5 free parameters, and a double-valued phase. While the double-valued phase is showing up in two elements, it is actually in both terms resulting from $\gamma = -\beta + \frac{\pi}{2} \pm \frac{\pi}{2}$, and thus is merely one single indeterminacy.

4^{th} *Case:* $a, b, c, d \neq 0$,
$\cos(\alpha+\delta), \sin(\alpha+\delta), \cos(\beta+\gamma), \sin(\beta+\gamma) \neq 0$
From the condition for the determinant follows

$$d \overset{(A.15)}{=} \frac{1 + bc\cos(\beta+\gamma)}{a\cos(\alpha+\delta)} \tag{A.23}$$

$$c \overset{(A.15)}{=} \frac{a\frac{1+bc\cos(\beta+\gamma)}{a\cos(\alpha+\delta)}\sin(\alpha+\delta)}{b\sin(\beta+\gamma)} \tag{A.24}$$

Thus one finds

$$\begin{aligned} c &= \frac{1 + bc\cos(\beta+\gamma)\sin(\alpha+\delta)}{\cos(\alpha+\delta)b\sin(\beta+\gamma)} \\ &= \Big(\cos(\alpha+\delta)b\sin(\beta+\gamma) - b\cos(\beta+\gamma)\sin(\alpha+\delta)\Big)^{-1} \end{aligned} \tag{A.25}$$

$$d = \frac{1 + b\cdot(A.25)\cdot\cos(\beta+\gamma)}{a\cos(\alpha+\delta)} \tag{A.26}$$

The most general form of an element in SL(2,$\mathbb{C}$) in this case is

$$\begin{pmatrix} ae^{i\alpha} & be^{i\beta} \\ (A.25)e^{i\gamma} & (A.26)e^{i\delta} \end{pmatrix} . \tag{A.27}$$

There are 6 free parameters, i.e. more than in the cases considered before. Therefore a generic element in SL(2,$\mathbb{C}$) is uniquely determined by exactly six real parameters.

A.4 Auxiliary Calculation for (6.15)

We compute the most general form of a matrix $U \in \mathrm{SU}(2)$, considering the boundary conditions (6.13) and (6.14). We consider two cases for a.

1^{st} *Case:* $a = 0$

$$\det U \overset{(6.13)}{=} -bc = +1 \quad \Longrightarrow \quad b \neq 0\ ,\ c = -\frac{1}{b} \tag{A.28}$$

$$bd^* \overset{(6.14b)}{=} 0 \quad \Longrightarrow \quad d = 0 \tag{A.29}$$

$$bb^* \overset{(6.14a)}{=} 1 \quad \Longrightarrow \quad b = \exp\{i\phi\} \text{ with } \phi \in \mathbb{R} \tag{A.30}$$

Thus the most general form of a matrix $U \in \mathrm{SU}(2)$ in case of $a = 0$ is

$$U = \begin{pmatrix} 0 & \exp\{i\phi\} \\ -\exp\{-i\phi\} & 0 \end{pmatrix} \quad \text{with } \phi \in \mathbb{R} \quad . \tag{A.31}$$

2^{nd} *Case:* $a \neq 0$

$$c \overset{(6.14c)}{=} -\frac{db^*}{a^*}$$

$$d \overset{(6.13)}{=} \frac{1+bc}{a} = \frac{1}{a} - \frac{b}{a}\frac{db^*}{a^*}$$

$$d\Big(1 + \frac{bb^*}{aa^*}\Big) \overset{(6.14a)}{=} \frac{d}{aa^*} = \frac{1}{a}$$

$$\Longrightarrow \quad d = a^* \tag{A.32}$$

$$\Longrightarrow \quad c = -b^* \tag{A.33}$$

Thus the most general form of a matrix $U \in \mathrm{SU}(2)$ is

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with} \quad aa^* + bb^* \overset{(6.14a)}{=} 1 \quad , \tag{A.34}$$

as this form does hold both in case $a \neq 0$ and in case $a = 0$, see (A.31).

A.5 Auxiliary Computations for the derivation of (6.38) and (6.40)

We define the unitary transformation

$$T \equiv \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{A.35}$$

T indeed is unitary, because it's adjoint (=transposed complex-conjugate) transformation $T^\dagger$ is identical to it's inverse:

$$TT^\dagger = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$TT^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad T^\dagger = T^{-1} \tag{A.36}$$

Considering $aa^* + bb^* \overset{(6.15)}{=} 1$, the general form of the matrices $\widetilde{W}$ is

$$\widetilde{W} = TWT^{-1} \overset{(6.35)}{=} \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} aa & ab & ba & bb \\ -ab^* & aa^* & -bb^* & ba^* \\ -b^*a & -b^*b & a^*a & a^*b \\ b^*b^* & -b^*a^* & -a^*b^* & a^*a^* \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} ab\sqrt{\frac{1}{2}} - ba\sqrt{\frac{1}{2}} & aa & ab\sqrt{\frac{1}{2}} + ba\sqrt{\frac{1}{2}} & bb \\ aa^*\sqrt{\frac{1}{2}} + bb^*\sqrt{\frac{1}{2}} & -ab^* & aa^*\sqrt{\frac{1}{2}} - bb^*\sqrt{\frac{1}{2}} & ba^* \\ -b^*b\sqrt{\frac{1}{2}} - a^*a\sqrt{\frac{1}{2}} & -b^*a & -b^*b\sqrt{\frac{1}{2}} + a^*a\sqrt{\frac{1}{2}} & a^*b \\ -b^*a^*\sqrt{\frac{1}{2}} + a^*b^*\sqrt{\frac{1}{2}} & b^*b^* & -b^*a^*\sqrt{\frac{1}{2}} - a^*b^*\sqrt{\frac{1}{2}} & a^*a^* \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & aa & ab\sqrt{2} & bb \\ \sqrt{\frac{1}{2}} & -ab^* & (aa^* - bb^*)\sqrt{\frac{1}{2}} & ba^* \\ -\sqrt{\frac{1}{2}} & -b^*a & (aa^* - bb^*)\sqrt{\frac{1}{2}} & a^*b \\ 0 & b^*b^* & -a^*b^*\sqrt{2} & a^*a^* \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & aa & ab\sqrt{2} & bb \\ 0 & -ab^*\sqrt{2} & aa^* - bb^* & a^*b\sqrt{2} \\ 0 & b^*b^* & -a^*b^*\sqrt{2} & a^*a^* \end{pmatrix} \quad . \tag{A.37}$$

The spinors $\widetilde{\chi}$, which constitute the basis of this representation, are of the form

$$\widetilde{\chi} \overset{(6.37)}{=} T\chi \overset{(A.35)}{=} \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi_{11} \\ \chi_{12} \\ \chi_{21} \\ \chi_{22} \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{\frac{1}{2}}(\chi_{12} - \chi_{21}) \\ \chi_{11} \\ \sqrt{\frac{1}{2}}(\chi_{12} + \chi_{21}) \\ \chi_{22} \end{pmatrix} \overset{(6.32)}{=} \begin{pmatrix} \sqrt{\frac{1}{2}}(\phi_1\psi_2 - \phi_2\psi_1) \\ \phi_1\psi_1 \\ \sqrt{\frac{1}{2}}(\phi_1\psi_2 + \phi_2\psi_1) \\ \phi_2\psi_2 \end{pmatrix} . \tag{A.38}$$

A.6 Derivation of (8.41) from (8.40)

$$[\gamma^\nu, S^{\sigma\tau}] \overset{(8.40)}{=} B^{\sigma\tau\nu}{}_\mu \gamma^\mu \tag{A.39}$$

Using the definition

$$B^{\sigma\tau\nu}{}_\mu \overset{(5.33)}{\equiv} i\hbar(g^{\sigma\nu} g^\tau{}_\mu - g^{\tau\nu} g^\sigma{}_\mu) \tag{A.40}$$

of the generators of the Lorentz transformations, this becomes

$$[\gamma^\nu, S^{\sigma\tau}] = i\hbar \tfrac{1}{4}(4g^{\sigma\nu}\gamma^\tau - 4g^{\tau\nu}\gamma^\sigma) . \tag{A.41}$$

We split the both terms on the right-hand side into two parts of same size each, and add zero in the form of four further terms:

$$\begin{aligned}[\gamma^\nu, S^{\sigma\tau}] = i\hbar \tfrac{1}{4}(&2g^{\sigma\nu}\gamma^\tau - \gamma^\sigma\gamma^\nu\gamma^\tau - 2g^{\tau\nu}\gamma^\sigma + \gamma^\tau\gamma^\nu\gamma^\sigma \\ &- 2g^{\tau\nu}\gamma^\sigma + \gamma^\sigma\gamma^\nu\gamma^\tau + 2g^{\sigma\nu}\gamma^\tau - \gamma^\tau\gamma^\nu\gamma^\sigma)\end{aligned}$$

Using

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \overset{(8.9)}{=} 2g^{\mu\nu}\mathbb{1} , \tag{A.42}$$

pairs of neighbored terms on the right-hand side can be combined:

$$
\begin{aligned}
[\gamma^\nu, S^{\sigma\tau}] &= i\hbar\tfrac{1}{4}(\gamma^\nu\gamma^\sigma\gamma^\tau - \gamma^\nu\gamma^\tau\gamma^\sigma - \gamma^\sigma\gamma^\tau\gamma^\nu + \gamma^\tau\gamma^\sigma\gamma^\nu) \\
&= i\hbar\tfrac{1}{4}(\gamma^\nu[\gamma^\sigma,\gamma^\tau] - [\gamma^\sigma,\gamma^\tau]\gamma^\nu) \\
&= \Big[\gamma^\nu, \frac{i\hbar}{4}[\gamma^\sigma,\gamma^\tau]\Big] \qquad \text{(A.43)}
\end{aligned}
$$

A.7 Proof of Theorem (8.43)

We will repeatedly use (8.9) and $g^{\alpha\beta}\overset{(2.5)}{=}g^{\beta\alpha}$:

$$
\begin{aligned}
[S^{\alpha\beta}, S^{\eta\delta}] &= -\frac{\hbar^2}{16}\Big((\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)(\gamma^\eta\gamma^\delta - \gamma^\delta\gamma^\eta) \\
&\quad - (\gamma^\eta\gamma^\delta - \gamma^\delta\gamma^\eta)(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)\Big) \\
&= -\frac{\hbar^2}{16}\Big(\gamma^\alpha(2g^{\beta\eta}\mathbb{1} - \gamma^\eta\gamma^\beta)\gamma^\delta - \gamma^\alpha(2g^{\beta\delta}\mathbb{1} - \gamma^\delta\gamma^\beta)\gamma^\eta \\
&\quad - \gamma^\beta(2g^{\alpha\eta}\mathbb{1} - \gamma^\eta\gamma^\alpha)\gamma^\delta + \gamma^\beta(2g^{\alpha\delta}\mathbb{1} - \gamma^\delta\gamma^\alpha)\gamma^\eta \\
&\quad - \gamma^\eta(2g^{\delta\alpha}\mathbb{1} - \gamma^\alpha\gamma^\delta)\gamma^\beta + \gamma^\eta(2g^{\delta\beta}\mathbb{1} - \gamma^\beta\gamma^\delta)\gamma^\alpha \\
&\quad + \gamma^\delta(2g^{\eta\alpha}\mathbb{1} - \gamma^\alpha\gamma^\eta)\gamma^\beta - \gamma^\delta(2g^{\eta\beta}\mathbb{1} - \gamma^\beta\gamma^\eta)\gamma^\alpha\Big) \\
&= -\frac{\hbar^2}{16}\Big(2g^{\beta\eta}\gamma^\alpha\gamma^\delta - \gamma^\alpha\gamma^\eta\gamma^\beta\gamma^\delta - 2g^{\beta\delta}\gamma^\alpha\gamma^\eta + \gamma^\alpha\gamma^\delta\gamma^\beta\gamma^\eta \\
&\quad - 2g^{\alpha\eta}\gamma^\beta\gamma^\delta + \gamma^\beta\gamma^\eta\gamma^\alpha\gamma^\delta + 2g^{\alpha\delta}\gamma^\beta\gamma^\eta - \gamma^\beta\gamma^\delta\gamma^\alpha\gamma^\eta \\
&\quad - 2g^{\delta\alpha}\gamma^\eta\gamma^\beta + (2g^{\eta\alpha}\mathbb{1} - \gamma^\alpha\gamma^\eta)(2g^{\delta\beta}\mathbb{1} - \gamma^\beta\gamma^\delta) \\
&\quad + 2g^{\delta\beta}\gamma^\eta\gamma^\alpha - (2g^{\eta\beta}\mathbb{1} - \gamma^\beta\gamma^\eta)(2g^{\delta\alpha}\mathbb{1} - \gamma^\alpha\gamma^\delta) \\
&\quad + 2g^{\eta\alpha}\gamma^\delta\gamma^\beta - (2g^{\delta\alpha}\mathbb{1} - \gamma^\alpha\gamma^\delta)(2g^{\eta\beta}\mathbb{1} - \gamma^\beta\gamma^\eta) \\
&\quad - 2g^{\eta\beta}\gamma^\delta\gamma^\alpha + (2g^{\delta\beta}\mathbb{1} - \gamma^\beta\gamma^\delta)(2g^{\eta\alpha}\mathbb{1} - \gamma^\alpha\gamma^\eta)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\hbar^2}{16}\Big(2g^{\beta\eta}\gamma^\alpha\gamma^\delta - 2g^{\beta\delta}\gamma^\alpha\gamma^\eta - 2g^{\alpha\eta}\gamma^\beta\gamma^\delta + 2g^{\alpha\delta}\gamma^\beta\gamma^\eta \\
&\quad - 2g^{\delta\alpha}\gamma^\eta\gamma^\beta + 4g^{\eta\alpha}g^{\delta\beta}\mathbb{1} - 2g^{\eta\alpha}\gamma^\beta\gamma^\delta - 2g^{\delta\beta}(2g^{\alpha\eta}\mathbb{1} - \gamma^\eta\gamma^\alpha) \\
&\quad + 2g^{\delta\beta}\gamma^\eta\gamma^\alpha - 4g^{\eta\beta}g^{\delta\alpha}\mathbb{1} + 2g^{\eta\beta}\gamma^\alpha\gamma^\delta + 2g^{\delta\alpha}(2g^{\beta\eta}\mathbb{1} - \gamma^\eta\gamma^\beta) \\
&\quad + 2g^{\eta\alpha}\gamma^\delta\gamma^\beta - 4g^{\delta\alpha}g^{\eta\beta}\mathbb{1} + 2g^{\delta\alpha}\gamma^\beta\gamma^\eta + 2g^{\eta\beta}(2g^{\alpha\delta}\mathbb{1} - \gamma^\delta\gamma^\alpha) \\
&\quad - 2g^{\eta\beta}\gamma^\delta\gamma^\alpha + 4g^{\delta\beta}g^{\eta\alpha}\mathbb{1} - 2g^{\delta\beta}\gamma^\alpha\gamma^\eta - 2g^{\eta\alpha}(2g^{\beta\delta}\mathbb{1} - \gamma^\delta\gamma^\beta)\Big) \\
&= -\frac{\hbar^2}{16}\Big(4g^{\beta\eta}(\gamma^\alpha\gamma^\delta - \gamma^\delta\gamma^\alpha) - 4g^{\beta\delta}(\gamma^\alpha\gamma^\eta - \gamma^\eta\gamma^\alpha) \\
&\quad - 4g^{\alpha\eta}(\gamma^\beta\gamma^\delta - \gamma^\delta\gamma^\beta) + 4g^{\alpha\delta}(\gamma^\beta\gamma^\eta - \gamma^\eta\gamma^\beta)\Big) \\
&= i\hbar(g^{\beta\eta}S^{\alpha\delta} - g^{\beta\delta}S^{\alpha\eta} - g^{\alpha\eta}S^{\beta\delta} + g^{\alpha\delta}S^{\beta\eta})
\end{aligned}
\tag{A.44}
$$

A.8 Why is the dimension of Dirac-matrices even?

This auxiliary computation is related to the derivation of the Dirac-matrices in section 8.1. We will demonstrate, that the dimension of the γ-matrices must be even. γ^0 and $i\gamma^j$ are hermitean. Any hermitean matrix can be transformed by an appropriate unitary transformation U into diagonal shape. In that shape, the eigenvalues (which are real for a hermitean matrix) then are written in the diagonal, while all off-diagonal elements are zero. We then have

$$
\underbrace{U_0\gamma^0U_0^{-1}}_{\text{diagonal}}\underbrace{U_0\gamma^0U_0^{-1}}_{\text{diagonal}} \overset{(8.9)}{=} U_0\mathbb{1}U_0^{-1} = \mathbb{1} \tag{A.45}
$$

$$
\underbrace{U_j i\gamma^j U_j^{-1}}_{\text{diagonal}}\underbrace{U_j i\gamma^j U_j^{-1}}_{\text{diagonal}} \overset{(8.9)}{=} U_j\mathbb{1}U_j^{-1} = \mathbb{1} \quad , \tag{A.46}
$$

as an exception, no summation over j here!

Thus all eigenvalues of the matrices γ^0 and $i\gamma^j$ are ± 1. The trace of the γ-matrices, i.e. the sum of their diagonal elements, is equal to the sum of their eigenvalues, because the trace is invariant under cyclic permutation of the arguments:

$$\mathrm{tr}(U_0\gamma^0 U_0^{-1}) = \mathrm{tr}(U_0^{-1}U_0\gamma^0) = \mathrm{tr}(\gamma^0) \tag{A.47}$$

$$\mathrm{tr}(U_j i\gamma^j U_j^{-1}) = \mathrm{tr}(U_j^{-1}U_j i\gamma^j) = i\mathrm{tr}(\gamma^j) \tag{A.48}$$

as an exception, no summation over j here!

The trace of the γ^j is zero because of

$$\mathrm{tr}(\gamma^j) = \mathrm{tr}(\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^j) = \mathrm{tr}(\gamma^0\gamma^j\gamma^0) \overset{(8.9)}{=} -\mathrm{tr}(\gamma^0\gamma^0\gamma^j) = -\mathrm{tr}(\gamma^j) = 0 \quad . \tag{A.49a}$$

The trace of γ^0 is zero because of

$$\mathrm{tr}(\gamma^0) = \mathrm{tr}(\underbrace{-\gamma^j\gamma^j}_{\mathbb{1}}\gamma^0) = \mathrm{tr}(-\gamma^j\gamma^0\gamma^j) = \overset{(8.9)}{=} -\mathrm{tr}(-\gamma^j\gamma^j\gamma^0) = -\mathrm{tr}(\gamma^0) = 0\,. \tag{A.49b}$$

as an exception, no summation over j here!

Thus we can conclude: All eigenvalues of γ^0 and $i\gamma^j$ are ± 1, and the sum of the eigenvalues is zero. Therefore the number of eigenvalues $+1$ and the number of eigenvalues -1 must be identical. Hence the γ^ν must have even dimension $2, 4, 6, \ldots$

A.9 Auxiliary Computations for (8.67) and (8.68)

We consider a pure boost, which is parameterized by the multiplet $(\boldsymbol{\Theta}, \boldsymbol{\eta}) = (\mathbf{0}, \boldsymbol{\eta})$. Because of $\sigma^1\sigma^1 = \sigma^2\sigma^2 = \sigma^3\sigma^3 = \mathbb{1}$ and of $\eta \equiv |\boldsymbol{\eta}|$, one gets

$$
\begin{aligned}
D &= \exp\Big\{ -\frac{\eta_j}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \Big\} = \\
&= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \sum_{n=0,2,4,\ldots}^{\infty} \frac{1}{n!} \Big(\frac{\eta}{2}\Big)^n - \frac{\eta_j}{\eta} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n!} \Big(\frac{\eta}{2}\Big)^n \\
&= \begin{pmatrix} \mathbb{1} \cosh\left(\frac{\eta}{2}\right) - \frac{\eta_j}{\eta} \sigma^j \sinh\left(\frac{\eta}{2}\right) & 0 \\ 0 & \mathbb{1} \cosh\left(\frac{\eta}{2}\right) + \frac{\eta_j}{\eta} \sigma^j \sinh\left(\frac{\eta}{2}\right) \end{pmatrix} . \qquad \text{(A.50)}
\end{aligned}
$$

Instead of specifying D as a function of the three components of the rapidity $\boldsymbol{\eta}$, the transformation may as well be specified as a function of the field's three momentum components $\boldsymbol{p}$ in the moving coordinate frame. The velocity of the moving coordinate frame is v (measured in the rest system). The rapidity η is related to this velocity by

$$\tanh \eta = \frac{v}{c} \equiv \beta \qquad \text{(A.51)}$$

$$\cosh \eta = \frac{1}{\sqrt{1 - \tanh^2 \eta}} = \frac{1}{\sqrt{1 - \beta^2}} \equiv \gamma \qquad \text{(A.52)}$$

$$\sinh \eta = \cosh \eta \tanh \eta = \gamma\beta \ . \qquad \text{(A.53)}$$

The relativistic energy E and the relativistic momentum p of the field ψ are related to the rest mass m and the relativistic mass γm by

$$E = \gamma m c^2 \qquad , \qquad p = -\gamma m v = -\gamma\beta m c \ . \qquad \text{(A.54)}$$

Note the minus sign: The boost is a passive one. In the moving coordinates, the field's velocity is $-v$. For the same reason, the unit vector in direction of the coordinate systems's movement has — if measured in the field's rest system — the components η_j/η. The unit vector in direction of the moving field has — if measured in the moving coordinate system — the components $p_j/p = -\eta_j/\eta$, where p is the field's momentum, measured in the moving coordinate system. These relations are inserted into the hyperbolic functions of $\eta/2$:

$$\cosh\frac{\eta}{2} = \sqrt{\frac{1}{2}(\cosh\eta+1)} = \sqrt{\frac{1}{2}(\gamma+1)} = \sqrt{\frac{1}{2}\,\frac{E+mc^2}{mc^2}} = \\ = \frac{E+mc^2}{\sqrt{2mc^2(E+mc^2)}} \tag{A.55}$$

$$\sinh\frac{\eta}{2} = \sqrt{\frac{1}{2}(\cosh\eta-1)} = \sqrt{\frac{1}{2}(\gamma-1)} = \sqrt{\frac{1}{2}\,\frac{E-mc^2}{mc^2}} = \\ = \frac{\sqrt{E^2-m^2c^4}}{\sqrt{2mc^2(E+mc^2)}} = \frac{cp}{\sqrt{2mc^2(E+mc^2)}} \tag{A.56}$$

Insertion into (A.50) gives the result

$$D = \sqrt{\frac{1}{2mc^2(E+mc^2)}}\cdot\begin{pmatrix}(E+mc^2)\mathbb{1}+cp_j\sigma^j & 0\\ 0 & (E+mc^2)\mathbb{1}-cp_j\sigma^j\end{pmatrix}. \tag{A.57}$$

A.10 Derivation of equations (8.76)

Using the definitions

$$N \equiv \sqrt{\frac{1}{2(E+mc^2)}} \tag{A.58a}$$

$$A_+ \equiv E+mc^2+cp_3 \tag{A.58b}$$

$$B \equiv cp_1+icp_2 \tag{A.58c}$$

$$A_- \equiv E+mc^2-cp_3\ , \tag{A.58d}$$

$$E \equiv \hbar\omega_{\boldsymbol{k}} \qquad , \qquad p_j \equiv \hbar k_j$$

the spinors (8.75) are:

$$
{}^1u^{\boldsymbol{k}} = N\begin{pmatrix} A_+ \\ B \\ A_- \\ -B \end{pmatrix},\ {}^2u^{\boldsymbol{k}} = N\begin{pmatrix} B^* \\ A_- \\ -B^* \\ A_+ \end{pmatrix},\ {}^1v^{\boldsymbol{k}} = N\begin{pmatrix} A_+ \\ B \\ -A_- \\ B \end{pmatrix}
$$

$$
{}^2v^{\boldsymbol{k}} = N\begin{pmatrix} B^* \\ A_- \\ B^* \\ -A_+ \end{pmatrix},\quad \begin{aligned} {}^1u^{\boldsymbol{k}\dagger} &= N\begin{pmatrix} A_+ & B^* & A_- & -B^* \end{pmatrix} \\ {}^2u^{\boldsymbol{k}\dagger} &= N\begin{pmatrix} B & A_- & -B & A_+ \end{pmatrix} \\ {}^1v^{\boldsymbol{k}\dagger} &= N\begin{pmatrix} A_+ & B^* & -A_- & B^* \end{pmatrix} \\ {}^2v^{\boldsymbol{k}\dagger} &= N\begin{pmatrix} B & A_- & B & -A_+ \end{pmatrix} \end{aligned}
$$

$$
{}^1u^{-\boldsymbol{k}} = N\begin{pmatrix} A_- \\ -B \\ A_+ \\ B \end{pmatrix},\ {}^2u^{-\boldsymbol{k}} = N\begin{pmatrix} -B^* \\ A_+ \\ B^* \\ A_- \end{pmatrix},\ {}^1v^{-\boldsymbol{k}} = N\begin{pmatrix} A_- \\ -B \\ -A_+ \\ -B \end{pmatrix}
$$

$$
{}^2v^{-\boldsymbol{k}} = N\begin{pmatrix} -B^* \\ A_+ \\ -B^* \\ -A_- \end{pmatrix},\quad \begin{aligned} {}^1u^{-\boldsymbol{k}\dagger} &= N\begin{pmatrix} A_- & -B^* & A_+ & B^* \end{pmatrix} \\ {}^2u^{-\boldsymbol{k}\dagger} &= N\begin{pmatrix} -B & A_+ & B & A_- \end{pmatrix} \\ {}^1v^{-\boldsymbol{k}\dagger} &= N\begin{pmatrix} A_- & -B^* & -A_+ & -B^* \end{pmatrix} \\ {}^2v^{-\boldsymbol{k}\dagger} &= N\begin{pmatrix} -B & A_+ & -B & -A_- \end{pmatrix} \end{aligned} \tag{A.59}
$$

We compute some sums of products of spinor components:

$$
\begin{aligned}
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_+A_+ + BB^* + A_-A_- + BB^*) = 2\hbar\omega_{\boldsymbol{k}} \\
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_+B^* + BA_+ - A_-B^* - BA_-) \neq 0 \\
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_+B + BA_- - A_-B - BA_+) = 0 \\
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_+A_- + BB + A_-A_+ + BB) \neq 0 \\
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_+A_- - BB^* - A_-A_+ + BB^*) = 0
\end{aligned}
$$

$$
\begin{aligned}
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(-A_+B^* + BA_- - A_-B^* + BA_+) = 0 \\
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^1v_1^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^2v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(-A_+B + BA_+ - A_-B + BA_-) = 0 \\
&{}^1u_1^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^2u_1^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^1v_1^{\boldsymbol{k}\dagger}\,{}^2v_4^{\boldsymbol{k}} + {}^2v_1^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_+A_+ - BB - A_+A_+ + BB) = 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*A_+ + A_-B^* - B^*A_- - A_+B^*) = 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*B^* + A_-A_+ + BB^* + A_+A_-) \neq 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*B + A_-A_- + B^*B + A_+A_+) = 2\hbar\omega_{\boldsymbol{k}} \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*A_- + A_-B - B^*A_+ - A_+B) \neq 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*A_- - A_-B^* + B^*A_+ - A_+B^*) = 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(-B^*B^* + A_-A_- + B^*B^* - A_+A_+) \neq 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^2v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(-B^*B + A_-A_+ + B^*B - A_+A_-) = 0 \\
&{}^1u_2^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^2u_2^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^1v_2^{-\boldsymbol{k}\dagger}\,{}^2v_4^{-\boldsymbol{k}} + {}^2v_2^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*A_+ - A_-B + BA_- - A_+B) \neq 0 \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_-A_+ - BB^* - A_+A_- + BB^*) = 0 \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_-B^* - BA_+ + A_+B^* - BA_-) = 0
\end{aligned}
$$

$$
\begin{aligned}
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_-B - BA_- + A_+B - BA_+) = 0 \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_-A_- - BB - A_+A_+ + BB) \neq 0 \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_-A_- + BB^* + A_+A_+ + BB^*) = 2\hbar\omega_{\boldsymbol{k}} \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(-A_-B^* - BA_- + A_+B^* + BA_+) \neq 0 \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^2v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(-A_-B - BA_+ + A_+B + BA_-) = 0 \\
&{}^1u_3^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^2u_3^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^1v_3^{-\boldsymbol{k}\dagger}\,{}^2v_4^{-\boldsymbol{k}} + {}^2v_3^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(A_-A_+ + BB + A_+A_- + BB) \neq 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^1v_1^{-\boldsymbol{k}} + {}^2v_4^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} = \\
&\qquad = N^2(-B^*A_+ + A_+B^* - B^*A_- + A_-B^*) = 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^2u_1^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^1u_1^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^2v_1^{-\boldsymbol{k}} + {}^2v_4^{\boldsymbol{k}\dagger}\,{}^1v_1^{\boldsymbol{k}} = \\
&\qquad = N^2(-B^*B^* + A_+A_+ + B^*B^* - A_+A_+) = 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^1v_2^{-\boldsymbol{k}} + {}^2v_4^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} = \\
&\qquad = N^2(-B^*B + A_+A_- + B^*B - A_-A_+) = 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^2u_2^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^1u_2^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^2v_2^{-\boldsymbol{k}} + {}^2v_4^{\boldsymbol{k}\dagger}\,{}^1v_2^{\boldsymbol{k}} = \\
&\qquad = N^2(-B^*A_- + A_+B - B^*A_-A_+B) \neq 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} + {}^2v_4^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(-B^*A_- - A_+B^* + B^*A_+ + A_-B^*) = 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^2u_3^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^1u_3^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^2v_3^{-\boldsymbol{k}} + {}^2v_4^{-\boldsymbol{k}\dagger}\,{}^1v_3^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*B^* + A_+A_- + B^*B^* + A_-A_+) \neq 0 \\
&{}^1u_4^{\boldsymbol{k}\dagger}\,{}^1u_4^{\boldsymbol{k}} + {}^2u_4^{\boldsymbol{k}\dagger}\,{}^2u_4^{\boldsymbol{k}} + {}^1v_4^{-\boldsymbol{k}\dagger}\,{}^1v_4^{-\boldsymbol{k}} + {}^2v_4^{-\boldsymbol{k}\dagger}\,{}^2v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(B^*B + A_+A_+ + B^*B + A_-A_-) = 2\hbar\omega_{\boldsymbol{k}}
\end{aligned}
$$

$$
\begin{aligned}
&{}^{1}u_4^{\boldsymbol{k}\dagger}\,{}^{2}u_4^{\boldsymbol{k}} + {}^{2}u_4^{\boldsymbol{k}\dagger}\,{}^{1}u_4^{\boldsymbol{k}} + {}^{1}v_4^{-\boldsymbol{k}\dagger}\,{}^{2}v_4^{-\boldsymbol{k}} + {}^{2}v_4^{-\boldsymbol{k}\dagger}\,{}^{1}v_4^{-\boldsymbol{k}} = \\
&\qquad = N^2(-B^*A_+ - A_+B + B^*A_- + A_-B) \neq 0
\end{aligned}
\tag{A.60}
$$

Obviously the general result is

$$
\begin{aligned}
&\sum_{r=1}^{2}\sum_{s=1}^{2}\left({}^{r}u_a^{\boldsymbol{k}\dagger}\,{}^{s}u_b^{\boldsymbol{k}} + {}^{r}v_a^{\boldsymbol{k}\dagger}\,{}^{s}v_b^{\boldsymbol{k}} + {}^{r}u_a^{-\boldsymbol{k}\dagger}\,{}^{s}u_b^{-\boldsymbol{k}} + {}^{r}v_a^{-\boldsymbol{k}\dagger}\,{}^{s}v_b^{-\boldsymbol{k}}\right)\delta_{rs} = \\
&\qquad = 2\cdot\delta_{ab}\,2\hbar\omega_{\boldsymbol{k}}
\end{aligned}
\tag{A.61a}
$$

$$
\begin{aligned}
&\sum_{r=1}^{2}\sum_{s=1}^{2}\left({}^{r}u_a^{\boldsymbol{k}\dagger}\,{}^{s}u_b^{\boldsymbol{k}} + {}^{r}v_a^{\boldsymbol{k}\dagger}\,{}^{s}v_b^{\boldsymbol{k}} + {}^{r}u_a^{-\boldsymbol{k}\dagger}\,{}^{s}u_b^{-\boldsymbol{k}} + {}^{r}v_a^{-\boldsymbol{k}\dagger}\,{}^{s}v_b^{-\boldsymbol{k}}\right) = \\
&\qquad = \text{anything.}
\end{aligned}
\tag{A.61b}
$$

In particular one gets the sum

$$
\begin{aligned}
&\sum_{r=1}^{2}\left({}^{r}u_a^{\boldsymbol{k}}\,{}^{r}u_b^{\boldsymbol{k}\,\dagger} + {}^{r}v_a^{\boldsymbol{-k}}\,{}^{r}v_b^{\boldsymbol{-k}\,\dagger}\right) = \\
&= N^2(A_+A_+ + A_-A_- + B^*B + B^*B)\,\delta_{ab} = \\
&= \frac{2(E+mc^2)^2 + 2c^2p_3^2 + 2c^2p_1^2 + 2c^2p_2^2}{2(E+mc^2)}\,\delta_{ab} = \\
&= \frac{4E(E+mc^2)}{2(E+mc^2)}\,\delta_{ab} = 2E\,\delta_{ab}\ .
\end{aligned}
\tag{A.61c}
$$

Furthermore one gets the spinor products

$$
\begin{aligned}
{}^{1}u^{\boldsymbol{k}\dagger}\,{}^{1}u^{\boldsymbol{k}} &= N^2\begin{pmatrix}A_+ & B^* & A_- & -B^*\end{pmatrix}\begin{pmatrix}A_+\\ B\\ A_-\\ -B\end{pmatrix} \\
&= N^2(A_+^2 + B^*B + A_-^2 + B^*B) = 2E
\end{aligned}
\tag{A.62a}
$$

$$
{}^{2}u^{\boldsymbol{k}\dagger}\,{}^{2}u^{\boldsymbol{k}} = N^2(BB^* + A_-^2 + BB^* + A_+^2) = 2E \tag{A.62b}
$$

$$
{}^{1}v^{\boldsymbol{k}\dagger}\,{}^{1}v^{\boldsymbol{k}} = N^2(A_+^2 + B^*B + A_-^2 + B^*B) = 2E \tag{A.62c}
$$

$$
{}^{2}v^{\boldsymbol{k}\dagger}\,{}^{2}v^{\boldsymbol{k}} = N^2(BB^* + A_-^2 + BB^* + A_+^2) = 2E \tag{A.62d}
$$

$$ {}^{1}u^{\boldsymbol{k}\dagger 2}u^{\boldsymbol{k}} = N^2(A_+B^* + B^*A_- - A_-B^* - B^*A_+) = 0 \tag{A.62e} $$
$$ {}^{2}u^{\boldsymbol{k}\dagger 1}u^{\boldsymbol{k}} = N^2(BA_+ + A_-B - BA_- - A_+B) = 0 \tag{A.62f} $$
$$ {}^{1}v^{\boldsymbol{k}\dagger 2}v^{\boldsymbol{k}} = N^2(A_+B^* + B^*A_- - A_-B^* - B^*A_+) = 0 \tag{A.62g} $$
$$ {}^{2}v^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}} = N^2(BA_+ + A_-B - BA_- - A_+B) = 0 \tag{A.62h} $$

$$ \begin{aligned} {}^{1}u^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}} &= N^2(A_+^2 + B^*B - A_-^2 - B^*B) = \\ &= 2(E+mc^2)cp_3 \end{aligned} \tag{A.63a} $$
$$ {}^{1}v^{\boldsymbol{k}\dagger 1}u^{\boldsymbol{k}} = N^2(A_+^2 + B^*B - A_-^2 - B^*B) = {}^{1}u^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}} \tag{A.63b} $$
$$ {}^{2}u^{\boldsymbol{k}\dagger 2}v^{\boldsymbol{k}} = N^2(BB^* + A_-^2 - BB^* - A_+^2) = -{}^{1}u^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}} \tag{A.63c} $$
$$ {}^{2}v^{\boldsymbol{k}\dagger 2}u^{\boldsymbol{k}} = N^2(BB^* + A_-^2 - BB^* - A_+^2) = -{}^{1}u^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}} \tag{A.63d} $$
$$ {}^{1}u^{\boldsymbol{k}\dagger 2}v^{\boldsymbol{k}} = N^2(A_+B^* + B^*A_- + A_-B^* + BA_+) = 2cp_1 \tag{A.63e} $$
$$ {}^{2}v^{\boldsymbol{k}\dagger 1}u^{\boldsymbol{k}} = N^2(BA_+ + A_-B + BA_- + A_+B) = cp_1 + icp_2 \tag{A.63f} $$
$$ \begin{aligned} {}^{2}u^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}} &= N^2(BA_+ + A_-B - BA_- + A_+B) = \\ &= \frac{(E+mc^2+cp_3)(cp_1+icp_2)}{(E+mc^2)} \end{aligned} \tag{A.63g} $$
$$ \begin{aligned} {}^{1}v^{\boldsymbol{k}\dagger 2}u^{\boldsymbol{k}} &= N^2(A_+B^* + B^*A_- - A_-B^* + B^*A_+) = \\ &= ({}^{2}u^{\boldsymbol{k}\dagger 1}v^{\boldsymbol{k}})^* \end{aligned} \tag{A.63h} $$
$$ {}^{1}u^{\dagger\text{-}\boldsymbol{k}1}v^{\boldsymbol{k}} = N^2(A_-A_+ - B^*B - A_+A_- + B^*B) = 0 \tag{A.63i} $$
$$ {}^{1}u^{\dagger\text{-}\boldsymbol{k}2}v^{\boldsymbol{k}} = N^2(A_-B^* - B^*A_- + A_+B^* - B^*A_+) = 0 \tag{A.63j} $$
$$ {}^{2}u^{\dagger\text{-}\boldsymbol{k}1}v^{\boldsymbol{k}} = N^2(-BA_+ + A_+B - BA_- + A_-B) = 0 \tag{A.63k} $$
$$ {}^{2}u^{\dagger\text{-}\boldsymbol{k}2}v^{\boldsymbol{k}} = N^2(-BB^* + A_+A_- + BB^* - A_-A_+) = 0 \tag{A.63l} $$
$$ {}^{1}v^{\dagger\text{-}\boldsymbol{k}1}u^{\boldsymbol{k}} = N^2(A_-A_+ - B^*B - A_+A_- + B^*B) = 0 \tag{A.63m} $$
$$ {}^{1}v^{\dagger\text{-}\boldsymbol{k}2}u^{\boldsymbol{k}} = N^2(A_-B^* - B^*A_- + A_+B^* - B^*A_+) = 0 \tag{A.63n} $$
$$ {}^{2}v^{\dagger\text{-}\boldsymbol{k}1}u^{\boldsymbol{k}} = N^2(-BA_+ + A_+B - BA_- + A_-B) = 0 \tag{A.63o} $$
$$ {}^{2}v^{\dagger\text{-}\boldsymbol{k}2}u^{\boldsymbol{k}} = N^2(-BB^* + A_+A_- + BB^* - A_-A_+) = 0 \tag{A.63p} $$

$$ {}^{1}u^{\boldsymbol{k}1}u^{\boldsymbol{k}\dagger} = N^2 \begin{pmatrix} A_+A_+ & A_+B^* & A_+A_- & -A_+B^* \\ BA_+ & BB^* & BA_- & -BB^* \\ A_-A_+ & A_-B^* & A_-A_- & -A_-B^* \\ -BA_+ & -BB^* & -BA_- & BB^* \end{pmatrix} \tag{A.63q} $$

$$ {}^{1}u^{\boldsymbol{k}2}u^{\boldsymbol{k}\dagger} = N^2 \begin{pmatrix} A_+B & A_+A_- & -A_+B & A_+A_+ \\ BB & BA_- & -BB & BA_+ \\ A_-B & A_-A_- & -A_-B & A_-A_+ \\ -BB & -BA_- & BB & -BA_+ \end{pmatrix} \tag{A.63r} $$

$$ {}^{2}u^{\boldsymbol{k}1}u^{\boldsymbol{k}\dagger} = N^2 \begin{pmatrix} B^*A_+ & B^*B^* & B^*A_- & -B^*B^* \\ A_-A_+ & A_-B^* & A_-A_- & -A_-B^* \\ -B^*A_+ & -B^*B^* & -B^*A_- & B^*B^* \\ A_+A_+ & A_+B^* & A_+A_- & -A_+B^* \end{pmatrix} \tag{A.63s} $$

$$ {}^{2}u^{\boldsymbol{k}2}u^{\boldsymbol{k}\dagger} = N^2 \begin{pmatrix} B^*B & B^*A_- & -B^*B & B^*A_+ \\ A_-B & A_-A_- & -A_-B & A_-A_+ \\ -B^*B & -B^*A_- & B^*B & -B^*A_+ \\ A_+B & A_+A_- & -A_+B & A_+A_+ \end{pmatrix} \tag{A.63t} $$

A.11 Derivation of equation (8.77)

We have

$$ {}^{r}\bar{u}^{\boldsymbol{k}} = {}^{r}u^{\boldsymbol{k}\dagger}\,\gamma^0 \ . \tag{A.64} $$

The matrix $\gamma^0 =$ (8.15a) permutes the first component of ${}^{r}u^{\boldsymbol{k}\dagger}$ with the third one, and the second with the fourth one. Therefore we find, using the definitions (A.58) and the equations (A.59) of appendix A.10:

$$ {}^{1}u^{\boldsymbol{k}} = N \begin{pmatrix} A_+ \\ B \\ A_- \\ -B \end{pmatrix} \ , \ {}^{2}u^{\boldsymbol{k}} = N \begin{pmatrix} B^* \\ A_- \\ -B^* \\ A_+ \end{pmatrix} \ , \ {}^{1}v^{\boldsymbol{k}} = N \begin{pmatrix} A_+ \\ B \\ -A_- \\ B \end{pmatrix} $$

$$
{}^{2}v^{\boldsymbol{k}} = N \begin{pmatrix} B^* \\ A_- \\ B^* \\ -A_+ \end{pmatrix} \quad , \quad \begin{aligned} {}^{1}\bar{u}^{\boldsymbol{k}} &= N \begin{pmatrix} A_- & -B^* & A_+ & B^* \end{pmatrix} \\ {}^{2}\bar{u}^{\boldsymbol{k}} &= N \begin{pmatrix} -B & A_+ & B & A_- \end{pmatrix} \\ {}^{1}\bar{v}^{\boldsymbol{k}} &= N \begin{pmatrix} -A_- & B^* & A_+ & B^* \end{pmatrix} \\ {}^{2}\bar{v}^{\boldsymbol{k}} &= N \begin{pmatrix} B & -A_+ & B & A_- \end{pmatrix} \end{aligned} \tag{A.65}
$$

Using this result, and using

$$
\begin{aligned}
X_1 &\equiv N^2(A_+A_- - B^*B) = mc^2 \\
X_2 &\equiv N^2(A_+A_+ + B^*B) = \frac{E + cp_3}{mc^2} \\
X_3 &\equiv N^2(A_+B^* + B^*A_-) = \frac{cp_1 - icp_2}{mc^2} \\
X_4 &\equiv N^2(BA_+ + A_-B) = \frac{cp_1 + icp_2}{mc^2} \\
X_5 &\equiv N^2(BB^* + A_-A_-) = \frac{E - cp_3}{mc^2} \ ,
\end{aligned}
$$

one gets the matrix-sum

$$
\begin{aligned}
\sum_{r=1}^{2} {}^{r}u^{\boldsymbol{k}}\, {}^{r}\bar{u}^{\boldsymbol{k}} &= \begin{pmatrix} X_1 & 0 & X_2 & X_3 \\ 0 & X_1 & X_4 & X_5 \\ X_5 & -X_3 & X_1 & 0 \\ -X_4 & X_2 & 0 & X_1 \end{pmatrix} \qquad \text{(A.66a)} \\
&= \begin{pmatrix} mc^2 & 0 & E + cp_3 & cp_1 - icp_2 \\ 0 & mc^2 & cp_1 + icp_2 & E - cp_3 \\ E - cp_3 & -cp_1 + icp_2 & mc^2 & 0 \\ -cp_1 - icp_2 & E + cp_3 & 0 & mc^2 \end{pmatrix} .
\end{aligned}
$$

The matrix-sum

$$
\sum_{r=1}^{2} {}^{r}v^{\boldsymbol{k}}\, {}^{r}\bar{v}^{\boldsymbol{k}} = \begin{pmatrix} -X_1 & 0 & X_2 & X_3 \\ 0 & -X_1 & X_4 & X_5 \\ X_5 & -X_3 & -X_1 & 0 \\ -X_4 & X_2 & 0 & -X_1 \end{pmatrix} \tag{A.66b}
$$

differs from (A.66a) only by the sign of the diagonal elements. Now we compute

$$c\gamma^\mu p_\mu \pm \mathbb{1}mc^2 \overset{(8.15)}{=} \begin{pmatrix} \pm mc^2 & 0 & E+cp_3 & cp_1-icp_2 \\ 0 & \pm mc^2 & cp_1+icp_2 & E-cp_3 \\ E-cp_3 & -cp_1+icp_2 & \pm mc^2 & 0 \\ -cp_1-icp_2 & E+cp_3 & 0 & \pm mc^2 \end{pmatrix},$$

and find

$$\sum_{r=1}^{2} {}^r u^{\boldsymbol{k}}\, {}^r\bar{u}^{\boldsymbol{k}} = c\gamma^\mu p_\mu + mc^2 \tag{A.67a}$$

$$\sum_{r=1}^{2} {}^r v^{\boldsymbol{k}}\, {}^r\bar{v}^{\boldsymbol{k}} = c\gamma^\mu p_\mu - mc^2 \ . \tag{A.67b}$$

A.12 Symmetrical ES-Tensors

If the energydensity-stress-tensor is symmetrical, i. e. if $\mathcal{T}^{\rho\sigma} = \mathcal{T}^{\sigma\rho}$, then the equation of continuity

$$\mathrm{d}_\nu \mathcal{M}^{\nu\rho\sigma} \overset{(4.73)}{=} 0 \quad \text{with } \rho\sigma = 10, 20, 30, 23, 31, 12 \ , \tag{A.68}$$

which is describing the conservation of the six angular-momentum densities, assumes (after multiplication by c) the simple form

$$\begin{aligned} c\mathrm{d}_\nu \mathcal{M}^{\nu\rho\sigma} &= \mathrm{d}_\nu(x^\rho \mathcal{T}^{\nu\sigma} - x^\sigma \mathcal{T}^{\nu\rho}) \\ &= g_\nu{}^\rho \mathcal{T}^{\nu\sigma} + x^\rho \mathrm{d}_\nu \mathcal{T}^{\nu\sigma} - g_\nu{}^\sigma \mathcal{T}^{\nu\rho} - x^\sigma \mathrm{d}_\nu \mathcal{T}^{\nu\rho} \\ &= \underbrace{\mathcal{T}^{\rho\sigma} - \mathcal{T}^{\sigma\rho}}_{0} + x^\rho \underbrace{\mathrm{d}_\nu \mathcal{T}^{\nu\sigma}}_{0} - x^\sigma \underbrace{\mathrm{d}_\nu \mathcal{T}^{\nu\rho}}_{0} = 0 \ . \end{aligned} \tag{A.69}$$

(A.68) does hold as well for non-interacting vector- or spinor-fields. But in their momentum density

$$\mathcal{M}^{\nu\rho\sigma} \overset{(5.100),(6.101)}{=} x^\rho \mathcal{T}^{\nu\sigma}/c - x^\sigma \mathcal{T}^{\nu\rho}/c + \mathcal{S}^{\nu\rho\sigma} \tag{A.70}$$

there must show up additional spin-densities $\mathcal{S}$, which are compensating due to

$$c\mathrm{d}_\nu \mathcal{M}^{\nu\rho\sigma} = \underbrace{\mathcal{T}^{\rho\sigma} - \mathcal{T}^{\sigma\rho} + c\mathrm{d}_\nu \mathcal{S}^{\nu\rho\sigma}}_{0} + x^\rho \underbrace{\mathrm{d}_\nu \mathcal{T}^{\nu\sigma}}_{0} - x^\sigma \underbrace{\mathrm{d}_\nu \mathcal{T}^{\nu\rho}}_{0} = 0 \tag{A.71}$$

the missing symmetry of the energydensity-stress-tensors $\mathcal{T}$.

We now want to construct as well for vector- and spinor-fields energy-stress-tensors

$$\widetilde{\mathcal{T}}^{\rho\sigma} \equiv \mathcal{T}^{\rho\sigma} + \mathrm{d}_\nu X^{\nu\rho\sigma} \quad \text{with } \widetilde{\mathcal{T}}^{\sigma\rho} = \widetilde{\mathcal{T}}^{\rho\sigma} \ , \tag{A.72}$$

which shall — in contrast to $\mathcal{T}$ — be symmetrical, but at the same time shall result into the identical conserved quantities as $\mathcal{T}$. From this requirement, a condition for the tensor X can be derived:

$$\begin{aligned} & \mathrm{d}_\rho \mathcal{T}^{\rho\sigma} = \mathrm{d}_\rho \widetilde{\mathcal{T}}^{\rho\sigma} = \mathrm{d}_\rho \mathcal{T}^{\rho\sigma} + \mathrm{d}_\rho \mathrm{d}_\nu X^{\nu\rho\sigma} = 0 \\ \Longrightarrow \quad & \mathrm{d}_\rho \mathrm{d}_\nu X^{\nu\rho\sigma} = 0 = \mathrm{d}_\nu \mathrm{d}_\rho X^{\rho\nu\sigma} = \mathrm{d}_\rho \mathrm{d}_\nu X^{\rho\nu\sigma} \\ \Longrightarrow \quad & \mathrm{d}_\rho \mathrm{d}_\nu \left(X^{\nu\rho\sigma} - X^{\rho\nu\sigma} \right) = 0 \\ \Longrightarrow \quad & X^{\rho\nu\sigma} = -X^{\nu\rho\sigma} \end{aligned} \tag{A.73a}$$

$\mathcal{T}$ and $\widetilde{\mathcal{T}}$ result into the same densities of energy and momentum, if the tensor X is skew-symmetric in it's both first indices ρ and ν, and if X is analytical, i. e. if $\mathrm{d}_\nu \mathrm{d}_\rho X^{\nu\rho\sigma} = \mathrm{d}_\rho \mathrm{d}_\nu X^{\nu\rho\sigma}$.

A second equation for X can be found by inserting (A.72) into the continuity-equation (A.69) of fields with symmetrical energy-stress-tensor, and comparing the result with (A.71):

$$\mathrm{d}_\nu X^{\nu\rho\sigma} - \mathrm{d}_\nu X^{\nu\sigma\rho} = c\mathrm{d}_\nu \mathcal{S}^{\nu\rho\sigma} \tag{A.73b}$$

This expression is changing it's sign under the permutation of ρ and σ. Thus S is anti-symmetric in it's both last indices. But it can not be concluded from (A.73b), that X is skew-symmetric in it's both last indices. One merely

can exclude, that X is symmetric in it's both last indices, because then $\mathrm{d}_\nu \mathcal{S}^{\nu\rho\sigma} = 0$ would hold.

As there are 6 permutations possible for 3 indices, the most general ansatz is

$$\begin{aligned} \frac{X^{\nu\rho\sigma}}{c} &= a\mathcal{S}^{\nu\rho\sigma} + b\mathcal{S}^{\sigma\nu\rho} + c\mathcal{S}^{\rho\sigma\nu} + d\mathcal{S}^{\rho\nu\sigma} + e\mathcal{S}^{\sigma\rho\nu} + f\mathcal{S}^{\nu\sigma\rho} \\ &= g\mathcal{S}^{\nu\rho\sigma} + h\mathcal{S}^{\sigma\nu\rho} + i\mathcal{S}^{\rho\sigma\nu} \\ &\text{with} \quad g \equiv a - f \quad h \equiv b - e \quad i \equiv c - d \ . \end{aligned} \tag{A.74}$$

Here the skew-symmetry of $\mathcal{S}$ in it's both last indices has been used. Because of (A.73a),

$$\begin{aligned} (\text{A.74}) &= -g\mathcal{S}^{\rho\nu\sigma} - h\mathcal{S}^{\sigma\rho\nu} - i\mathcal{S}^{\nu\sigma\rho} \\ &= +g\mathcal{S}^{\rho\sigma\nu} + h\mathcal{S}^{\sigma\nu\rho} + i\mathcal{S}^{\nu\rho\sigma} \qquad \Longrightarrow \quad g = i \ . \end{aligned}$$

Again the skew-symmetry of $\mathcal{S}$ in it's both last indices has been used in the second line. Now from condition (A.73b) follows

$$\begin{aligned} &\mathrm{d}_\nu \Big(g\mathcal{S}^{\nu\rho\sigma} + h\mathcal{S}^{\sigma\nu\rho} + g\mathcal{S}^{\rho\sigma\nu} - g\mathcal{S}^{\nu\sigma\rho} - h\mathcal{S}^{\rho\nu\sigma} - g\mathcal{S}^{\sigma\rho\nu} \Big) \ = \\ &= \mathrm{d}_\nu \Big(2g\mathcal{S}^{\nu\rho\sigma} + (h+g)\mathcal{S}^{\sigma\nu\rho} + (g+h)\mathcal{S}^{\rho\sigma\nu} \Big) = \mathrm{d}_\nu \mathcal{S}^{\nu\rho\sigma} \\ &\Longrightarrow \quad g = \frac{1}{2} \qquad h = -g \ . \end{aligned}$$

Therefore

$$X^{\nu\rho\sigma} = \frac{c}{2} \Big(\mathcal{S}^{\rho\sigma\nu} - \mathcal{S}^{\sigma\nu\rho} + \mathcal{S}^{\nu\rho\sigma} \Big) \tag{A.75}$$

does meet both conditions (A.73), and

$$\widetilde{\mathcal{T}}^{\rho\sigma} \equiv \mathcal{T}^{\rho\sigma} + \frac{c}{2} \mathrm{d}_\nu \Big(\mathcal{S}^{\rho\sigma\nu} - \mathcal{S}^{\sigma\nu\rho} + \mathcal{S}^{\nu\rho\sigma} \Big) \tag{A.76}$$

is a symmetrical energy-momentum-tensor for arbitrary fields, whether they have spin or not. 10 equations of continuity (1 for energy density, 3 for momentum density, and 6 for angular-momentum density) follow from $\widetilde{\mathcal{T}}^{\rho\sigma}$:

$$\mathrm{d}_\rho \widetilde{\mathcal{T}}^{\rho\sigma} = \mathrm{d}_\rho \mathcal{T}^{\rho\sigma} = 0 \tag{A.77a}$$

$$\begin{aligned}\mathrm{d}_\nu \mathcal{M}^{\nu\rho\sigma} &= \mathrm{d}_\nu \Big(x^\rho \widetilde{\mathcal{T}}^{\nu\sigma}/c - x^\sigma \widetilde{\mathcal{T}}^{\nu\rho}/c \Big) = \\ &= \mathrm{d}_\nu \Big(x^\rho \mathcal{T}^{\nu\sigma}/c - x^\sigma \mathcal{T}^{\nu\rho}/c + \mathcal{S}^{\nu\rho\sigma} \Big) = 0\end{aligned} \tag{A.77b}$$

We now will compute the symmetrized tensor $\widetilde{\mathcal{T}}^{\rho\sigma}$ of the electromagnetic field. The electromagnetic field's spin density is

$$\mathcal{S}^{\nu\rho\sigma} \overset{(5.98)}{=} \frac{1}{i\hbar c} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\nu A^\alpha)} B^{\rho\sigma\alpha}{}_\beta A^\beta \ . \tag{A.78a}$$

$$B^{\rho\sigma\alpha}{}_\beta \overset{(5.33)}{=} i\hbar (g^{\rho\alpha} g^\sigma{}_\beta - g^{\sigma\alpha} g^\rho{}_\beta) \tag{A.78b}$$

are the components of the generator of the Lorentz-transformations (5.35). Inserting

$$\frac{\partial \mathcal{L}}{\partial(\mathrm{d}_\rho A_\alpha)} \overset{(4.123)}{=} -\frac{1}{\mu_0} F^{\rho\alpha} \ , \tag{A.79}$$

one gets

$$\mathcal{S}^{\nu\rho\sigma} = \frac{-1}{c\mu_0} \Big(F^{\nu\rho} A^\sigma - F^{\nu\sigma} A^\rho \Big) \ . \tag{A.80}$$

Thus one finds the tensor

$$\begin{aligned}X^{\nu\rho\sigma} \overset{(A.75)}{=} \frac{-1}{2\mu_0} \Big(&(F^{\rho\sigma} A^\nu - F^{\rho\nu} A^\sigma) - (F^{\sigma\nu} A^\rho - F^{\sigma\rho} A^\nu) + \\ &+ (F^{\nu\rho} A^\sigma - F^{\nu\sigma} A^\rho) \Big) = -\frac{1}{\mu_0} F^{\nu\rho} A^\sigma \ .\end{aligned} \tag{A.81}$$

As required, this tensor is skew-symmetric in the indices ν and ρ. Using the field-equation

$$\mathrm{d}_\nu F^{\nu\rho} \overset{(4.125)}{=} 0 \tag{A.82}$$

of the homogeneous field, one finds the tensor

$$\widetilde{\mathcal{T}}^{\rho\sigma} = \mathcal{T}^{\rho\sigma} + \mathrm{d}_\nu X^{\nu\rho\sigma} = \frac{1}{\mu_0} F^{\nu\rho}\mathrm{d}^\sigma A_\nu - g^{\rho\sigma}\mathcal{L} - \frac{1}{\mu_0} F^{\nu\rho}\mathrm{d}_\nu A^\sigma$$
$$= -\frac{1}{\mu_0} F^{\rho\nu}F^\sigma{}_\nu - g^{\rho\sigma}\mathcal{L}\ , \tag{A.83}$$

which is symmetrical under permutation of ρ and σ.

Now we compute the symmetrized ES-tensor of the Dirac-field. The Dirac-fields spin density is

$$\mathcal{S}^{\nu\rho\sigma} \overset{(6.99)}{=} \frac{1}{i\hbar c}\Big(\overline{\psi}\, S^{\rho\sigma}\frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\nu\overline{\psi})} + \frac{\partial\mathcal{L}}{\partial(\mathrm{d}_\nu\psi)}\, S^{\rho\sigma}\psi\Big) \overset{(8.24)}{=} \overline{\psi}\gamma^\nu S^{\rho\sigma}\psi\ . \tag{A.84a}$$

$$S^{\rho\sigma} \overset{(8.43)}{=} \frac{i\hbar}{4}[\gamma^\rho,\gamma^\sigma] \tag{A.84b}$$

are the components of the generator of the spinor transformations $D =$ (8.45). Thus the tensor

$$X^{\nu\rho\sigma} \overset{(A.75)}{=} \frac{i\hbar c}{8}\,\overline{\psi}K^{\nu\rho\sigma}\psi$$
$$K^{\nu\rho\sigma} \equiv \gamma^\rho\gamma^\sigma\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu\gamma^\rho + \\ + \gamma^\sigma\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma^\rho$$

is found. As required, K — and consequently X — is skew-symmetric in the both first indices ν and ρ. Thus the divergence of $X^{\nu\rho\sigma}$ becomes

$$\mathrm{d}_\nu X^{\nu\rho\sigma} = \frac{i\hbar c}{8}\Big((\mathrm{d}_\nu\overline{\psi})K^{\nu\rho\sigma}\psi + \overline{\psi}K^{\nu\rho\sigma}\mathrm{d}_\nu\psi\Big)\ .$$

This expression can be simplified by means of the Dirac-equations

$$i\hbar c\mathrm{d}_\nu\overline{\psi}\gamma^\nu \overset{(8.26b)}{=} -mc^2\,\overline{\psi}$$
$$i\hbar c\gamma^\nu\mathrm{d}_\nu\psi \overset{(8.26a)}{=} +mc^2\,\psi\ .$$

For that purpose we compute two versions of $K^{\nu\rho\sigma}$, in which by use of the commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \overset{(8.9)}{=} 2g^{\mu\nu}$$

the γ^ν are shifted in all products of three γ-matrices either to the very left or to the very right.

$$\begin{aligned} K^{\nu\rho\sigma} &= 2g^{\sigma\nu}\gamma^\rho - 2g^{\rho\nu}\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\sigma - 2g^{\rho\nu}\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\sigma - \\ &\quad - 2g^{\sigma\nu}\gamma^\rho + \gamma^\nu\gamma^\sigma\gamma^\rho + 2g^{\rho\nu}\gamma^\sigma - 2g^{\sigma\nu}\gamma^\rho + \gamma^\nu\gamma^\sigma\gamma^\rho + \\ &\quad + \gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\sigma\gamma^\rho \\ &= +2\gamma^\nu\gamma^\rho\gamma^\sigma - 2g^{\rho\nu}\gamma^\sigma - 2g^{\sigma\nu}\gamma^\rho + 2g^{\sigma\rho}\gamma^\nu \qquad \text{(A.85a)} \\ K^{\nu\rho\sigma} &= \gamma^\rho\gamma^\sigma\gamma^\nu - 2g^{\nu\sigma}\gamma^\rho + \gamma^\rho\gamma^\sigma\gamma^\nu - 2g^{\nu\rho}\gamma^\sigma + \gamma^\sigma\gamma^\rho\gamma^\nu + \\ &\quad + \quad \gamma^\sigma\gamma^\rho\gamma^\nu + 2g^{\nu\rho}\gamma^\sigma - 2g^{\nu\sigma}\gamma^\rho + \gamma^\rho\gamma^\sigma\gamma^\nu - 2g^{\nu\sigma}\gamma^\rho + \\ &\quad + 2g^{\nu\rho}\gamma^\sigma - \gamma^\sigma\gamma^\rho\gamma^\nu \\ &= 2\gamma^\rho\gamma^\sigma\gamma^\nu - 6g^{\nu\sigma}\gamma^\rho + 2g^{\nu\rho}\gamma^\sigma + 2g^{\sigma\rho}\gamma^\nu \qquad \text{(A.85b)} \end{aligned}$$

The skew-symmetry in the indices ν and ρ is no more obvious. Still is must exist, unless a computational error creped in. Using the Dirac-equations, the green marked terms in the divergence of X mutually compensate:

$$\begin{aligned} \mathrm{d}_\nu X^{\nu\rho\sigma} = \frac{i\hbar c}{4}\Big(&(\mathrm{d}_\nu\overline{\psi})(\gamma^\nu\gamma^\rho\gamma^\sigma - g^{\rho\nu}\gamma^\sigma - g^{\sigma\nu}\gamma^\rho + g^{\sigma\rho}\gamma^\nu)\psi + \\ &+ \overline{\psi}(\gamma^\rho\gamma^\sigma\gamma^\nu - 3g^{\nu\sigma}\gamma^\rho + g^{\nu\rho}\gamma^\sigma + g^{\sigma\rho}\gamma^\nu)\mathrm{d}_\nu\psi\Big) \end{aligned}$$

This results into the energydensity-stress-tensor

$$\begin{aligned} \widetilde{\mathcal{T}}^{\rho\sigma} &= i\hbar c\,\overline{\psi}\gamma^\rho\mathrm{d}^\sigma\psi - g^{\rho\sigma}\,\overline{\psi}(i\hbar c\gamma^\nu\mathrm{d}_\nu - mc^2)\psi \\ &\quad + \frac{i\hbar c}{4}\Big(- (\mathrm{d}^\rho\overline{\psi})\gamma^\sigma\psi - (\mathrm{d}^\sigma\overline{\psi})\gamma^\rho\psi - 3\overline{\psi}\gamma^\rho\mathrm{d}^\sigma\psi + \overline{\psi}\gamma^\sigma\mathrm{d}^\rho\psi\Big) \\ &= \frac{i\hbar c}{4}\Big(- (\mathrm{d}^\rho\overline{\psi})\gamma^\sigma\psi - (\mathrm{d}^\sigma\overline{\psi})\gamma^\rho\psi + \overline{\psi}\gamma^\rho\mathrm{d}^\sigma\psi + \overline{\psi}\gamma^\sigma\mathrm{d}^\rho\psi\Big) - \\ &\quad - g^{\rho\sigma}\underbrace{\overline{\psi}(i\hbar c\gamma^\nu\mathrm{d}_\nu - mc^2)\psi}_{\mathcal{L}}\,, \qquad \text{(A.86)} \end{aligned}$$

which obviously is symmetrical under permutation of ρ and σ.

A.13 The charge of the Klein-Gordon field

The conserved charge of the classical Klein-Gordon field is

$$Q \overset{(10.32)}{=} \int_\Omega \mathrm{d}^3x \, \frac{iq}{\hbar}\Big(\pi\phi - \phi^*\pi^*\Big) \,. \tag{A.87}$$

Replacing the classical field functions by the quantized field operators, we get the charge operator

$$Q \equiv \int_\Omega \mathrm{d}^3x \, \underbrace{\frac{iq}{\hbar}(\pi\phi - \phi^\dagger\pi^\dagger)}_{\mathcal{Q}} \,. \tag{A.88}$$

With the field operators (15.2) this becomes

$$\begin{aligned}\mathcal{Q} = \frac{iq}{\hbar}\sum_{\boldsymbol{k},\boldsymbol{f}} \frac{i\hbar}{2\Omega}\sqrt{\frac{\omega_{\boldsymbol{f}}}{\omega_{\boldsymbol{k}}}}\Big((a^\dagger_{\boldsymbol{f}} - b_{\text{-}\boldsymbol{f}})(a_{\boldsymbol{k}} + b^\dagger_{\text{-}\boldsymbol{k}})\exp\{+i(\boldsymbol{k}-\boldsymbol{f})\boldsymbol{x}\} + \\ + (a^\dagger_{\boldsymbol{k}} + b_{\text{-}\boldsymbol{k}})(a_{\boldsymbol{f}} - b^\dagger_{\text{-}\boldsymbol{f}})\exp\{+i(\boldsymbol{f}-\boldsymbol{k})\boldsymbol{x}\}\Big) \,.\end{aligned} \tag{A.89}$$

Integration over the normalization volume Ω results with (7.12) into

$$\begin{aligned}Q = \int_\Omega \mathrm{d}^3x \, \mathcal{Q} = \frac{iq}{\hbar}\sum_{\boldsymbol{k}} \frac{i\hbar}{2}\Big(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} + a^\dagger_{\boldsymbol{k}}b^\dagger_{\text{-}\boldsymbol{k}} - b_{\text{-}\boldsymbol{k}}a_{\boldsymbol{k}} - b_{\text{-}\boldsymbol{k}}b^\dagger_{\text{-}\boldsymbol{k}} + \\ + a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} - a^\dagger_{\boldsymbol{k}}b^\dagger_{\text{-}\boldsymbol{k}} + b_{\text{-}\boldsymbol{k}}a_{\boldsymbol{k}} - b_{\text{-}\boldsymbol{k}}b^\dagger_{\text{-}\boldsymbol{k}}\Big) \,.\end{aligned} \tag{A.90}$$

As the sum is symmetric over all positive and negative wave numbers $\boldsymbol{k}$, in two terms $-\boldsymbol{k}$ may be replaced by $\boldsymbol{k}$:

$$\begin{aligned}Q &= -q\sum_{\boldsymbol{k}}(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} - b_{\boldsymbol{k}}b^\dagger_{\boldsymbol{k}}) = \\ &= -q\sum_{\boldsymbol{k}}(a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}} - b^\dagger_{\boldsymbol{k}}b_{\boldsymbol{k}} - \underbrace{[b_{\boldsymbol{k}}, b^\dagger_{\boldsymbol{k}}]}_{1})\end{aligned} \tag{A.91}$$

A.14 The charge of the Dirac field

Using $\gamma^0\gamma^0 = \mathbb{1}$ and $\psi(x) =$ (16.1a) and $\psi^\dagger(x) =$ (16.7), the operator of the Dirac field's conserved charge is

$$
\begin{aligned}
Q &\overset{(4.89)}{=} \int_\Omega \mathrm{d}^3x\, q\, \psi^\dagger\psi = \int_\Omega \mathrm{d}^3x \sum_{\boldsymbol{f},s,\boldsymbol{k},r} \frac{q}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \\
&\quad \Big({}^{s}a^\dagger_{\boldsymbol{f}}\, {}^{s}u^{\dagger\boldsymbol{f}} \exp\{+ifx\} + {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\dagger\boldsymbol{f}} \exp\{-ifx\}\Big) \cdot \\
&\qquad \cdot \Big({}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + {}^{r}b^\dagger_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\}\Big) \\
&= \int_\Omega \mathrm{d}^3x \sum_{\boldsymbol{f},s,\boldsymbol{k},r} \frac{q}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \Big({}^{s}a^\dagger_{\boldsymbol{f}}\, {}^{s}u^{\dagger\boldsymbol{f}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{+i(f-k)x\} \\
&\quad + {}^{s}a^\dagger_{\boldsymbol{f}}\, {}^{s}u^{\dagger\boldsymbol{f}}\, {}^{r}b^\dagger_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+i(f+k)x\} \\
&\quad + {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\dagger\boldsymbol{f}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i(f+k)x\} \\
&\quad + {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\dagger\boldsymbol{f}}\, {}^{r}b^\dagger_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{-i(f-k)x\}\Big) \\
&\overset{(16.3)}{=} \sum_{s,\boldsymbol{k},r} \frac{q}{2\hbar\omega_{\boldsymbol{k}}} \Big({}^{s}a^\dagger_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}u^{\dagger\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} + {}^{s}a^\dagger_{\text{-}\boldsymbol{k}}\, {}^{r}b^\dagger_{\boldsymbol{k}}\, {}^{s}u^{\dagger\text{-}\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{i2k^0x^0\} \\
&\quad + {}^{s}b_{\text{-}\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}v^{\dagger\text{-}\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i2k^0x^0\} + {}^{s}b_{\boldsymbol{k}}\, {}^{r}b^\dagger_{\boldsymbol{k}}\, {}^{s}v^{\dagger\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}}\Big) \\
&\overset{(8.76)}{=} \sum_{\boldsymbol{k},r} q\Big({}^{r}a^\dagger_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}} + {}^{r}b_{\boldsymbol{k}}\, {}^{r}b^\dagger_{\boldsymbol{k}}\Big) \\
&\overset{(16.5)}{=} \sum_{\boldsymbol{k},r} q\Big({}^{r}a^\dagger_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}} - {}^{r}b^\dagger_{\boldsymbol{k}}\, {}^{r}b_{\boldsymbol{k}} + \underbrace{\{ {}^{r}b^\dagger_{\boldsymbol{k}}, {}^{r}b_{\boldsymbol{k}}\}}_{1}\Big) \,. \qquad \text{(A.92)}
\end{aligned}
$$

A.15 Proof of the commutation relations (17.67)

We have

$$
\begin{aligned}
[\tilde{A}_\mu(t,\boldsymbol{x}),\tilde{\pi}^\tau(t,\boldsymbol{y})] = \sum_{\boldsymbol{k},\boldsymbol{f}} \sum_{\alpha,\beta=0}^{3} \sum_{\kappa=0}^{3} (e^{(\mu)}\cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\kappa)}\cdot e^{(\beta)}_{\boldsymbol{f}})\cdot \\
\cdot \frac{c\,i\hbar}{2\Omega}\sqrt{\frac{1}{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}}\Big(-f^0 g^{\tau\kappa}+f^\tau g^{0\kappa}-g^{0\tau}f^\kappa\Big)\Big(\\
-[c^{(\alpha)}_{\boldsymbol{k}},c^{(\beta)}_{\boldsymbol{f}}]\exp\{-i(kx+fy)\} \\
+[c^{(\alpha)}_{\boldsymbol{k}},c^{(\beta)\dagger}_{\boldsymbol{f}}]\exp\{-i(kx-fy)\} \\
-[c^{(\alpha)\dagger}_{\boldsymbol{k}},c^{(\beta)}_{\boldsymbol{f}}]\exp\{+i(kx-fy)\} \\
+[c^{(\alpha)\dagger}_{\boldsymbol{k}},c^{(\beta)\dagger}_{\boldsymbol{f}}]\exp\{+i(kx+fy)\}\Big)\,.
\end{aligned}
\tag{A.93}
$$

To check, whether the commutation relations (17.67) really follow from (17.66), we insert them into (A.93):

$$
\begin{aligned}
[\tilde{A}_\mu(t,\boldsymbol{x}),\tilde{\pi}^\tau(t,\boldsymbol{y})] = \sum_{\boldsymbol{k}} \sum_{\alpha,\beta=0}^{3} \sum_{\kappa=0}^{3} (e^{(\mu)}\cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\kappa)}\cdot e^{(\beta)}_{\boldsymbol{k}})\cdot \\
\cdot \frac{c\,i\hbar}{2\Omega\omega_{\boldsymbol{k}}}\Big(-k^0 g^{\tau\kappa}+k^\tau g^{0\kappa}-g^{0\tau}k^\kappa\Big)\cdot \\
\cdot(-g_{\alpha\beta})\Big(\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}+\exp\{-i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}\Big)
\end{aligned}
\tag{A.94}
$$

As the commutator must be computed with same time $x^0 = y^0$, the time component has disappeared from the exponents. The sequence of summations over $\boldsymbol{k}$ in the second exponential function may be changed, using

$$
\begin{aligned}
e^{(\alpha)}_{-\boldsymbol{k}} &= +e^{(\alpha)}_{\boldsymbol{k}} \quad \text{for } \alpha = 0 \\
e^{(\alpha)}_{-\boldsymbol{k}} &= -e^{(\alpha)}_{\boldsymbol{k}} \quad \text{for } \alpha = 1,2,3 \\
\omega_{-\boldsymbol{k}} &= \omega_{\boldsymbol{k}}\,.
\end{aligned}
\tag{A.95}
$$

As the sign change of the factors $e^{(\alpha)}_{\boldsymbol{k}}$ and $e^{(\beta)}_{\boldsymbol{k}}$ happens at the same time due to $(-g_{\alpha\beta})$, and thus compensates, one gets

$$[\tilde{A}_\mu(t,\boldsymbol{x}),\tilde{\pi}^\tau(t,\boldsymbol{y})] = \sum_{\boldsymbol{k}}\sum_{\alpha,\beta=0}^{3}\sum_{\kappa=0}^{3}(e^{(\mu)}\cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\kappa)}\cdot e^{(\beta)}_{\boldsymbol{k}})\cdot$$
$$\cdot\frac{c\,i\hbar}{2\Omega\omega_{\boldsymbol{k}}}\Big(\underbrace{-k^0 g^{\tau\kappa}+k^\tau g^{0\kappa}-g^{0\tau}k^\kappa}_{Y_a}$$
$$\underbrace{-k^0 g^{\tau\kappa}+(2g^\tau{}_0k^0-k^\tau)g^{0\kappa}-g^{0\tau}(2g^\kappa{}_0k^0-k^\kappa)}_{Y_b}\Big)\cdot$$
$$\cdot(-g_{\alpha\beta})\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}. \tag{A.96}$$

We have

$$\begin{aligned}Y_a+Y_b &= 2k^0(-g^{\tau\kappa}+g^\tau{}_0g^\kappa{}_0-g^{0\tau}g^{0\kappa})\\ &= 2k^0\cdot\begin{cases}-1 & \text{for } \tau=0 \quad , \quad \kappa=\tau=0\\ 0 & \text{for } \tau=0 \quad , \quad \kappa\neq\tau=0\\ +1 & \text{for } \tau\neq 0 \quad , \quad \kappa=\tau\neq 0\\ 0 & \text{for } \tau\neq 0 \quad , \quad \kappa\neq\tau\neq 0\end{cases}\\ &= -2k^0 g_{\tau\kappa}\end{aligned} \tag{A.97}$$

Using this result, and using $k^0=\omega_{\boldsymbol{k}}/c$,

$$[\tilde{A}_\mu(t,\boldsymbol{x}),\tilde{\pi}^\tau(t,\boldsymbol{y})] = i\hbar\underbrace{\frac{1}{\Omega}\sum_{\boldsymbol{k}}\exp\{+i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})\}}_{\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}\cdot$$
$$\cdot\underbrace{\sum_{\kappa=0}^{3}(-g_{\tau\kappa})\sum_{\alpha,\beta=0}^{3}(-g_{\alpha\beta})(e^{(\mu)}\cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\tau)}\cdot e^{(\beta)}_{\boldsymbol{k}})}_{Y_c} \tag{A.98}$$

follows. For the evaluation of Y_c, we expand the vectors $e^{(\mu)}$ and $e^{(\tau)}$ with respect to the unit vectors $e^{(\alpha)}_{\boldsymbol{k}}$:

$$e^{(\mu)} \overset{(K.10d)}{=} \sum_{\gamma=0}^{3}\sum_{\alpha=0}^{3} g^{\gamma\alpha}(e_{\boldsymbol{k}}^{(\alpha)}\cdot e^{(\mu)})e_{\boldsymbol{k}}^{(\gamma)} \tag{A.99a}$$

$$e^{(\tau)} \overset{(K.10d)}{=} \sum_{\delta=0}^{3}\sum_{\beta=0}^{3} g^{\delta\beta}(e_{\boldsymbol{k}}^{(\beta)}\cdot e^{(\tau)})e_{\boldsymbol{k}}^{(\delta)} \tag{A.99b}$$

Multiplying the respective sides of these two equations, one gets

$$\underbrace{e^{(\mu)}\cdot e^{(\tau)}}_{\overset{(K.10e)}{=} g_{\mu\tau}} = \sum_{\gamma,\alpha,\delta,\beta=0}^{3} g^{\gamma\alpha}g^{\delta\beta}(e_{\boldsymbol{k}}^{(\alpha)}\cdot e^{(\mu)})(e_{\boldsymbol{k}}^{(\beta)}\cdot e^{(\tau)})\underbrace{(e_{\boldsymbol{k}}^{(\gamma)}\cdot e_{\boldsymbol{k}}^{(\delta)})}_{\overset{(K.10e)}{=} g_{\gamma\delta}}$$

$$g_{\mu\tau} = \sum_{\alpha\beta=0}^{3} g^{\alpha\beta}(e_{\boldsymbol{k}}^{(\alpha)}\cdot e^{(\mu)})(e_{\boldsymbol{k}}^{(\beta)}\cdot e^{(\tau)})\ . \tag{A.100}$$

Note, that the indices in $g_{\mu\tau}$ are resulting from the names of $e^{(\mu)}$ and $e^{(\tau)}$. Therefore they are not automatically summed-up according to the summation convention. For that reason, there is also no summation over τ in

$$Y_c \overset{(A.98),(A.100)}{=} + \sum_{\kappa=0}^{3} g_{\tau\kappa}\, g_{\mu\tau} = g_{\mu}{}^{\tau}\ . \tag{A.101}$$

Therefore (A.98) is identical to (17.66). This proves (17.67).

A.16 Computation of the operators (16.16)

Our starting point is the Hamilton-density

$$\mathcal{H}' \overset{(8.104)}{=} -\frac{i\hbar c}{2}(\mathrm{d}_0\overline{\psi})\gamma^0\psi + \frac{i\hbar c}{2}\overline{\psi}\gamma^0\mathrm{d}_0\psi \tag{A.102a}$$

and the physical momentum density

$$\mathcal{P}'^{j} \overset{(8.105)}{=} \frac{i\hbar}{2}\,\overline{\psi}\gamma^0\mathrm{d}^j\psi - (\mathrm{d}^j\overline{\psi})\frac{i\hbar}{2}\gamma^0\psi \tag{A.102b}$$

of the classical (not quantized) Dirac field. These quantities have been derived in section 8.6 from the alternative Lagrangian $\mathcal{L}' = (8.99)$, which is more symmetrical in the fields ψ and $\overline{\psi}$ than the Lagrangian $\mathcal{L} = (8.24)$ used in other places.

Due to the field's quantization, the classical amplitudes ψ and $\overline{\psi}$ are replaced by the field operators

$$\psi(x) = \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\} \Big) \tag{A.103a}$$

$$\overline{\psi}(x) = \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big({}^{r}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}\overline{u}^{\boldsymbol{k}} \exp\{+ikx\} + {}^{r}b_{\boldsymbol{k}}\, {}^{r}\overline{v}^{\boldsymbol{k}} \exp\{-ikx\} \Big) \,. \tag{A.103b}$$

Consequently one finds the operator of energy density

$$\begin{aligned} \mathcal{H} \overset{(8.104)}{=} -\frac{i\hbar c}{2} \sum_{\boldsymbol{k},\boldsymbol{f},r,s} \frac{1}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}} \Big(& \Big(if^0\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}\overline{u}^{\boldsymbol{f}} \exp\{+ifx\} - if^0\, {}^{s}b_{\boldsymbol{f}}\, {}^{s}\overline{v}^{\boldsymbol{f}} \exp\{-ifx\} \Big)\gamma^0 \\ & \Big({}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\} \Big) \\ & - \Big({}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}\overline{u}^{\boldsymbol{f}} \exp\{+ifx\} + {}^{s}b_{\boldsymbol{f}}\, {}^{s}\overline{v}^{\boldsymbol{f}} \exp\{-ifx\} \Big)\gamma^0 \\ & \Big(- ik^0\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-ikx\} + ik^0\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+ikx\} \Big) \Big) \end{aligned}$$

$$
\begin{aligned}
\mathcal{H} = +\frac{\hbar}{2} \sum_{\boldsymbol{k},\boldsymbol{f},r,s} \frac{1}{\Omega 2\hbar\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}} \Big(& \\
& \omega_{\boldsymbol{f}}\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{f}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \exp\{+i(f-k)x\} \\
& + \omega_{\boldsymbol{f}}\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{f}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \exp\{+i(f+k)x\} \\
& - \omega_{\boldsymbol{f}}\, {}^{s}b_{\boldsymbol{f}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{f}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i(f+k)x\} \\
& - \omega_{\boldsymbol{f}}\, {}^{s}b_{\boldsymbol{f}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{f}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \exp\{-i(f-k)x\} \\
& + \omega_{\boldsymbol{k}}\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{f}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \exp\{+i(f-k)x\} \\
& - \omega_{\boldsymbol{k}}\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{f}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \exp\{+i(f+k)x\} \\
& + \omega_{\boldsymbol{k}}\, {}^{s}b_{\boldsymbol{f}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{f}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i(f+k)x\} \\
& - \omega_{\boldsymbol{k}}\, {}^{s}b_{\boldsymbol{f}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{f}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \exp\{-i(f-k)x\} \Big) \,. \qquad \text{(A.104)}
\end{aligned}
$$

Integrating over the total normalization volume Ω, using the form (16.3) of the Kronecker symbol, and using $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, one finds the Hamilton operator

$$
\begin{aligned}
H = \int_{\Omega} \mathrm{d}^3x\, \mathcal{H}(x) = \frac{1}{4} \sum_{\boldsymbol{k},r,s} \Big(& {}^{s}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{k}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \\
& + {}^{s}a^{\dagger}_{-\boldsymbol{k}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}u^{-\boldsymbol{k}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \exp\{i2k^0x^0\} \\
& - {}^{s}b_{-\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}v^{-\boldsymbol{k}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i2k^0x^0\} \\
& - {}^{s}b_{\boldsymbol{k}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{k}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} + {}^{s}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{k}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \\
& - {}^{s}a^{\dagger}_{-\boldsymbol{k}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}u^{-\boldsymbol{k}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \exp\{i2k^0x^0\} \\
& + {}^{s}b_{-\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}v^{-\boldsymbol{k}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i2k^0x^0\} - {}^{s}b_{\boldsymbol{k}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{k}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \Big) \\
= \frac{1}{2} \sum_{\boldsymbol{k},r,s} \Big(& {}^{s}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{k}\,\dagger}\, {}^{r}u^{\boldsymbol{k}} - {}^{s}b_{\boldsymbol{k}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{k}\,\dagger}\, {}^{r}v^{\boldsymbol{k}} \Big) \,. \qquad \text{(A.105)}
\end{aligned}
$$

Using eventually

$$
{}^{r}u^{\boldsymbol{k}\dagger}\, {}^{s}u^{\boldsymbol{k}} \overset{(A.62)}{=} {}^{r}v^{\boldsymbol{k}\dagger}\, {}^{s}v^{\boldsymbol{k}} \overset{(A.62)}{=} 2E\,\delta_{rs} = 2\hbar\omega_{\boldsymbol{k}}\,\delta_{rs} \,, \qquad \text{(A.106)}
$$

one gets the Hamilton operator

$$
\begin{aligned}
H &= \sum_{\boldsymbol{k},r} \hbar\omega_{\boldsymbol{k}} \Big({}^{r}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}} - {}^{r}b_{\boldsymbol{k}}\, {}^{r}b^{\dagger}_{\boldsymbol{k}} \Big) \\
&= \sum_{\boldsymbol{k},r} \hbar\omega_{\boldsymbol{k}} \Big({}^{r}a^{\dagger}_{\boldsymbol{k}}\, {}^{r}a_{\boldsymbol{k}} + {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}b_{\boldsymbol{k}} - \{ {}^{r}b_{\boldsymbol{k}} \,,\, {}^{r}b^{\dagger}_{\boldsymbol{k}} \} \Big) \,, \qquad \text{(A.107)}
\end{aligned}
$$

which is identical to (16.13) in the case of discrete fields.

As operator of the physical momentum density one gets

$$
\begin{aligned}
\mathcal{P}'^{j} = -\frac{\hbar}{2} \sum_{\boldsymbol{k},\boldsymbol{f},r,s} \frac{1}{2\hbar\Omega\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{f}}}} \Big(\\
- {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}u^{\boldsymbol{f}\dagger} k^{j}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{+i(f-k)x\} \\
+ {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}u^{\boldsymbol{f}\dagger} k^{j}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+i(f+k)x\} \\
- {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\boldsymbol{f}\dagger} k^{j}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i(f+k)x\} \\
+ {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\boldsymbol{f}\dagger} k^{j}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{-i(f-k)x\} \\
- f^{j}\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}u^{\boldsymbol{f}\dagger}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{+i(f-k)x\} \\
- f^{j}\, {}^{s}a^{\dagger}_{\boldsymbol{f}}\, {}^{s}u^{\boldsymbol{f}\dagger}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{+i(f+k)x\} \\
+ f^{j}\, {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\boldsymbol{f}\dagger}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} \exp\{-i(f+k)x\} \\
+ f^{j}\, {}^{s}b_{\boldsymbol{f}}\, {}^{s}v^{\boldsymbol{f}\dagger}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \exp\{-i(f-k)x\} \Big) \,. \qquad \text{(A.108)}
\end{aligned}
$$

Integrating over the total normalization volume Ω, using the form (16.3) of the Kronecker symbol, and using the relation $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, one gets the momentum operator

$$
\begin{aligned}
P^{j} = \int_{\Omega} \mathrm{d}^{3}x\, \mathcal{P}^{j}(x) = -\sum_{\boldsymbol{k},r,s} \frac{\hbar k^{j}}{2\hbar\omega_{\boldsymbol{k}}} \Big(\\
- {}^{s}a^{\dagger}_{\boldsymbol{k}}\, {}^{s}u^{\boldsymbol{k}+}\, {}^{r}a_{\boldsymbol{k}}\, {}^{r}u^{\boldsymbol{k}} + {}^{s}b_{\boldsymbol{k}}\, {}^{s}v^{\boldsymbol{k}+}\, {}^{r}b^{\dagger}_{\boldsymbol{k}}\, {}^{r}v^{\boldsymbol{k}} \Big) \,,
\end{aligned}
$$

which eventually by means of (A.106) gets the shape

$$P^j = \sum_{\boldsymbol{k},r} \hbar k^j \left({}^r a^\dagger_{\boldsymbol{k}} \, {}^r a_{\boldsymbol{k}} - {}^r b_{\boldsymbol{k}} \, {}^r b^\dagger_{\boldsymbol{k}} \right) , \tag{A.109}$$

which is identical to (16.14) in case of discrete fields.

A.17 Computation of the Hamilton operator (17.69)

We want to demonstrate, that the Hamilton operator (17.69) results from the Hamilton density (17.68) of the quantized field $\tilde{A}(x)$.

The Lagrangian (17.59) can be written in a compacter form due to renaming of some contracted indices:

$$\begin{aligned}
\tilde{\mathcal{L}} &\overset{(17.59)}{=} -\frac{1}{2\mu_0}\Big((\mathrm{d}_\tau \tilde{A}_\sigma)\mathrm{d}^\tau \tilde{A}^\sigma - (\mathrm{d}_\sigma \tilde{A}_\tau)\mathrm{d}^\tau \tilde{A}^\sigma + (\mathrm{d}_\tau \tilde{A}^\tau)\mathrm{d}_\sigma \tilde{A}^\sigma \Big) \\
&= +\frac{1}{2\mu_0}(\mathrm{d}_\tau \tilde{A}_\sigma)\Big(- \mathrm{d}^\tau \tilde{A}^\sigma + \mathrm{d}^\sigma \tilde{A}^\tau - g^{\tau\sigma}\mathrm{d}_\lambda \tilde{A}^\lambda \Big) \\
&= \frac{1}{2\mu_0}(\mathrm{d}_\tau \tilde{A}_\sigma)\mathrm{d}_\mu \tilde{A}_\nu \Big(- g^{\tau\mu}g^{\sigma\nu} + g^{\sigma\mu}g^{\tau\nu} - g^{\tau\sigma}g^{\mu\nu} \Big)
\end{aligned} \tag{A.110}$$

Using $\tilde{A} = (17.61)$, $\tilde{A}_\sigma \overset{(K.10c)}{=} e^{(\sigma)} \cdot \tilde{A}$ and $\tilde{\pi}^\sigma = (17.64)$, one finds the Hamilton-density-operator

$$
\begin{aligned}
\tilde{\mathcal{H}} = & \sum_{\sigma=0}^{3} \sum_{\boldsymbol{f}} \sum_{\beta=0}^{3} \sum_{\kappa=0}^{3} \sqrt{\frac{\hbar}{\mu_0 2\omega_{\boldsymbol{f}} \Omega}} (e^{(\kappa)} \cdot e_{\boldsymbol{f}}^{(\beta)}) \cdot \\
& \cdot i \Big(- f^0 g^{\sigma\kappa} + f^\sigma g^{0\kappa} - g^{0\sigma} f^\kappa \Big) \cdot \\
& \cdot \Big(- c_{\boldsymbol{f}}^{(\beta)} \exp\{-ifx\} + c_{\boldsymbol{f}}^{(\beta)\dagger} \exp\{+ifx\} \Big) \cdot \\
& \quad \cdot \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} i \sqrt{\frac{\mu_0 c^2 \hbar \omega_{\boldsymbol{k}}}{2\Omega}} (e^{(\sigma)} \cdot e_{\boldsymbol{k}}^{(\alpha)}) \cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{k}}^{(\alpha)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{+ikx\} \Big) - \\
& - \frac{1}{2\mu_0} \sum_{\sigma=0}^{3} \sum_{\boldsymbol{f}} \sum_{\beta=0}^{3} i f^\tau \sqrt{\frac{\mu_0 c^2 \hbar}{2\omega_{\boldsymbol{f}} \Omega}} (e^{(\sigma)} \cdot e_{\boldsymbol{f}}^{(\beta)}) \cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{f}}^{(\beta)} \exp\{-ifx\} + c_{\boldsymbol{f}}^{(\beta)\dagger} \exp\{+ifx\} \Big) \cdot \\
& \quad \cdot \sum_{\kappa=0}^{3} \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} ik^\mu \sqrt{\frac{\mu_0 c^2 \hbar}{2\omega_{\boldsymbol{k}} \Omega}} (e^{(\kappa)} \cdot e_{\boldsymbol{k}}^{(\alpha)}) \cdot \\
& \quad \cdot \Big(- c_{\boldsymbol{k}}^{(\alpha)} \exp\{-ikx\} + c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{+ikx\} \Big) \\
& \quad \cdot \Big(- g^{\tau\mu} g^{\sigma\kappa} + g^{\sigma\mu} g^{\tau\kappa} - g^{\tau\sigma} g^{\mu\kappa} \Big) \, . \qquad \text{(A.111)}
\end{aligned}
$$

More summation symbols than usual have been used here, because according to the summation convention, only over space-time-indices and over spinor-indices is automatically summed-up, but not over the names (σ) of the unit vectors, which are marked by brackets.

$$\tilde{\mathcal{H}} = \sum_{\boldsymbol{f},\boldsymbol{k}} \sum_{\sigma,\beta,\kappa,\alpha=0}^{3} \frac{c\hbar}{2\Omega} \Big(- \sqrt{\frac{\omega_{\boldsymbol{k}}}{\omega_{\boldsymbol{f}}}} (e^{(\kappa)} \cdot e_{\boldsymbol{f}}^{(\beta)})(e^{(\sigma)} \cdot e_{\boldsymbol{k}}^{(\alpha)})$$
$$(-f^0 g^{\sigma\kappa} + f^\sigma g^{0\kappa} - g^{0\sigma} f^\kappa) +$$
$$+ \frac{c}{2} \sqrt{\frac{1}{\omega_{\boldsymbol{f}} \omega_{\boldsymbol{k}}}} (e^{(\sigma)} \cdot e_{\boldsymbol{f}}^{(\beta)})(e^{(\kappa)} \cdot e_{\boldsymbol{k}}^{(\alpha)}) \cdot$$
$$(-f^\tau k_\tau g^{\sigma\kappa} + f_\kappa k_\sigma - f_\sigma k_\kappa) \Big) \cdot$$
$$\cdot \Big(c_{\boldsymbol{f}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)} \exp\{-i(f+k)x\} - c_{\boldsymbol{f}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{-i(f-k)x\}$$
$$- c_{\boldsymbol{f}}^{(\beta)\dagger} c_{\boldsymbol{k}}^{(\alpha)} \exp\{+i(f-k)x\} + c_{\boldsymbol{f}}^{(\beta)\dagger} c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{+i(f+k)x\} \Big)$$

Integrating over the total normalization volume Ω, one finds the Hamilton-operator

$$\tilde{H} = \int_{\Omega} \mathrm{d}^3 x \, \tilde{\mathcal{H}} \, . \tag{A.112}$$

To get the Kronecker-symbol according to (16.3) in this integration, we switch the sequence of summation over $\boldsymbol{k}$ in two terms. Using $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, one gets

$$
\begin{aligned}
\tilde{H} = \sum_{\boldsymbol{k}} \sum_{\sigma,\beta,\kappa,\alpha=0}^{3} \frac{c\hbar}{2} \Big(& -(e^{(\kappa)} \cdot e_{\boldsymbol{k}}^{(\beta)})(e^{(\sigma)} \cdot e_{\boldsymbol{k}}^{(\alpha)})(-k^0 g^{\sigma\kappa} + k^\sigma g^{0\kappa} - g^{0\sigma} k^\kappa) + \\
& + \frac{c}{2} \frac{1}{\omega_{\boldsymbol{k}}} (e^{(\sigma)} \cdot e_{\boldsymbol{k}}^{(\beta)})(e^{(\kappa)} \cdot e_{\boldsymbol{k}}^{(\alpha)}) (-\underbrace{k^\tau k_\tau}_{0} g^{\sigma\kappa} + \underbrace{k_\kappa k_\sigma - k_\sigma k_\kappa}_{0}) \Big) \\
& (-c_{\boldsymbol{k}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)\dagger} - c_{\boldsymbol{k}}^{(\beta)\dagger} c_{\boldsymbol{k}}^{(\alpha)}) \\
+ \sum_{\boldsymbol{k}} \sum_{\sigma,\beta,\kappa,\alpha=0}^{3} \frac{c\hbar}{2} \Big(& -(e^{(\kappa)} \cdot e_{\boldsymbol{k}}^{(\beta)})(e^{(\sigma)} \cdot e_{-\boldsymbol{k}}^{(\alpha)}) \\
& (-k^0 g^{\sigma\kappa} + k^\sigma g^{0\kappa} - g^{0\sigma} k^\kappa) + \\
& + \frac{c}{2} \frac{1}{\omega_{\boldsymbol{k}}} (e^{(\sigma)} \cdot e_{\boldsymbol{k}}^{(\beta)})(e^{(\kappa)} \cdot e_{-\boldsymbol{k}}^{(\alpha)}) \Big(\underbrace{-k^\tau (2 g_\tau{}^0 k_0 - k_\tau)}_{-2k^0 k_0} g^{\sigma\kappa} + \\
& + \underbrace{k_\kappa (2 g_\sigma{}^0 k_0 - k_\sigma) - k_\sigma (2 g_\kappa{}^0 k_0 - k_\kappa)}_{2k_0 (k_\kappa g_\sigma{}^0 - k_\sigma g_\kappa{}^0)} \Big) \Big) \\
& (c_{\boldsymbol{k}}^{(\beta)} c_{-\boldsymbol{k}}^{(\alpha)} \exp\{-i 2k^0 x^0\} + c_{\boldsymbol{k}}^{(\beta)\dagger} c_{-\boldsymbol{k}}^{(\alpha)\dagger} \exp\{+i 2k^0 x^0\}) \,. \qquad \text{(A.113)}
\end{aligned}
$$

Now we use

$$
\begin{aligned}
e_{-\boldsymbol{k}}^{(\alpha)} &= +e_{\boldsymbol{k}}^{(\alpha)} \quad \text{if } \alpha = 0 \\
e_{-\boldsymbol{k}}^{(\alpha)} &= -e_{\boldsymbol{k}}^{(\alpha)} \quad \text{if } \alpha \neq 0 \qquad \text{(A.114)}
\end{aligned}
$$

and $k^0 = \omega_{\boldsymbol{k}}/c$. Furthermore we rotate the coordinate system such, that it is congruent with the coordinate system which is aligned to $\boldsymbol{k}$, i. e. we choose

$$
e^{(\alpha)} \equiv e_{\boldsymbol{k}}^{(\alpha)} \qquad \Longrightarrow \qquad (e^{(\sigma)} \cdot e_{\boldsymbol{k}}^{(\alpha)}) \overset{\text{(K.10e)}}{=} g^{\sigma\alpha} \,. \qquad \text{(A.115)}
$$

$$
\begin{aligned}
\tilde{H} = & \sum_{\boldsymbol{k}} \sum_{\sigma,\beta,\kappa,\alpha=0}^{3} \frac{c\hbar}{2}\Big(- g^{\kappa\beta} g^{\sigma\alpha}(-k^0 g^{\sigma\kappa} + k^\sigma g^{0\kappa} - g^{0\sigma} k^\kappa)\Big) \\
& (-c_{\boldsymbol{k}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)+} - c_{\boldsymbol{k}}^{(\beta)+} c_{\boldsymbol{k}}^{(\alpha)}) \\
& + \sum_{\boldsymbol{k}} \sum_{\sigma,\beta,\kappa,\alpha=0}^{3} \frac{c\hbar}{2}\Big(- g^{\kappa\beta} g^{\sigma\alpha}(2g_\alpha{}^0 - 1)(-k^0 g^{\sigma\kappa} + k^\sigma g^{0\kappa} - g^{0\sigma} k^\kappa) + \\
& \quad + g^{\sigma\beta} g^{\kappa\alpha}(2g_\alpha{}^0 - 1)(-k^0 g^{\sigma\kappa} + k_\kappa g_\sigma{}^0 - k_\sigma g_\kappa{}^0)\Big) \\
& (c_{\boldsymbol{k}}^{(\beta)} c_{\boldsymbol{-k}}^{(\alpha)} \exp\{-i2k^0 x^0\} + c_{\boldsymbol{k}}^{(\beta)\dagger} c_{\boldsymbol{-k}}^{(\alpha)\dagger} \exp\{+i2k^0 x^0\}) \\
= & \sum_{\boldsymbol{k}} \sum_{\beta,\alpha=0}^{3} \frac{c\hbar}{2}(-k^0 g^{\beta\alpha} + k_\alpha g^0{}_\beta - k_\beta g^0{}_\alpha)(+c_{\boldsymbol{k}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)\dagger} + c_{\boldsymbol{k}}^{(\beta)\dagger} c_{\boldsymbol{k}}^{(\alpha)}) \\
& + \sum_{\boldsymbol{k}} \sum_{\beta,\alpha=0}^{3} \frac{c\hbar}{2}(2g_\alpha{}^0 - 1) \\
& (k^0 g^{\beta\alpha} - k_\alpha g^0{}_\beta + k_\beta g^0{}_\alpha - k^0 g^{\alpha\beta} + k^\alpha g^{0\beta} - k^\beta g^{0\alpha}) \\
& (c_{\boldsymbol{k}}^{(\beta)} c_{\boldsymbol{-k}}^{(\alpha)} \exp\{-i2k^0 x^0\} + c_{\boldsymbol{k}}^{(\beta)\dagger} c_{\boldsymbol{-k}}^{(\alpha)\dagger} \exp\{+i2k^0 x^0\})
\end{aligned}
\tag{A.116}
$$

We split the summation over α and β:

$$
\tilde{H} = \tilde{H}_{\alpha=\beta} + \tilde{H}_{\alpha=0\neq\beta} + \tilde{H}_{\alpha\neq 0=\beta} + \tilde{H}_{0\neq\alpha\neq\beta\neq 0} \tag{A.117}
$$

One finds

$$\begin{aligned}
\tilde{H}_{\alpha=\beta} &\overset{(A.116)}{=} \sum_{\boldsymbol{k}}\sum_{\alpha=0}^{3}\frac{c\hbar}{2}\Big(-k^0(2g_0{}^{\alpha}-1)+k_\alpha g^0{}_\alpha - k_\alpha g^0{}_\alpha\Big)\\
&\quad (c^{(\alpha)}_{\boldsymbol{k}}c^{(\alpha)\dagger}_{\boldsymbol{k}}+c^{(\alpha)\dagger}_{\boldsymbol{k}}c^{(\alpha)}_{\boldsymbol{k}})\\
&\quad +\sum_{\boldsymbol{k}}\sum_{\alpha=0}^{3}\frac{c\hbar}{2}(2g_\alpha{}^0-1)\Big(k^0(2g_\alpha{}^0-1)-k_\alpha g^0{}_\alpha+k_\alpha g^0{}_\alpha\\
&\qquad -k^0(2g_\alpha{}^0-1)+k^\alpha g^{0\alpha}-k^\alpha g^{0\alpha}\Big)\\
&\quad (c^{(\alpha)}_{\boldsymbol{k}}c^{(\alpha)}_{\text{-}\boldsymbol{k}}\exp\{-i2k^0x^0\}+c^{(\alpha)\dagger}_{\boldsymbol{k}}c^{(\alpha)\dagger}_{\text{-}\boldsymbol{k}}\exp\{+i2k^0x^0\})\\
&= \sum_{\boldsymbol{k}}\sum_{\alpha=0}^{3}\frac{\hbar\omega_{\boldsymbol{k}}}{2}(1-2g_0{}^{\alpha})(c^{(\alpha)}_{\boldsymbol{k}}c^{(\alpha)\dagger}_{\boldsymbol{k}}+c^{(\alpha)\dagger}_{\boldsymbol{k}}c^{(\alpha)}_{\boldsymbol{k}})\ . \qquad (A.118)
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{\alpha=0\neq\beta} &= \sum_{\boldsymbol{k}}\sum_{j=1}^{3}\frac{c\hbar}{2}(-k^0g^{j0}+k_0g^0{}_j-k_jg^0{}_0)\\
&\quad (+c^{(j)}_{\boldsymbol{k}}c^{(0)\dagger}_{\boldsymbol{k}}+c^{(j)\dagger}_{\boldsymbol{k}}c^{(0)}_{\boldsymbol{k}})\\
&\quad +\sum_{\boldsymbol{k}}\sum_{j=1}^{3}\frac{c\hbar}{2}(2g_0{}^0-1)\\
&\quad (k^0g^{j0}-k_0g^0{}_j+k_jg^0{}_0-k^0g^{0j}+k^0g^{0j}-k^jg^{00})\\
&\quad (c^{(j)}_{\boldsymbol{k}}c^{(0)}_{\text{-}\boldsymbol{k}}\exp\{-i2k^0x^0\}+c^{(j)\dagger}_{\boldsymbol{k}}c^{(0)\dagger}_{\text{-}\boldsymbol{k}}\exp\{+i2k^0x^0\})\\
&= -\sum_{\boldsymbol{k}}\sum_{j=1}^{3}\frac{c\hbar k_j}{2}(+c^{(j)}_{\boldsymbol{k}}c^{(0)\dagger}_{\boldsymbol{k}}+c^{(j)\dagger}_{\boldsymbol{k}}c^{(0)}_{\boldsymbol{k}})\\
&\quad +\sum_{\boldsymbol{k}}\sum_{j=1}^{3}\frac{c\hbar}{2}(k_j-k^j)\Big(c^{(j)}_{\boldsymbol{k}}c^{(0)}_{\text{-}\boldsymbol{k}}\exp\{-i2k^0x^0\}+\\
&\quad +c^{(j)\dagger}_{\boldsymbol{k}}c^{(0)\dagger}_{\text{-}\boldsymbol{k}}\exp\{+i2k^0x^0\}) \qquad (A.119)
\end{aligned}$$

$$
\begin{aligned}
\tilde{H}_{\alpha\neq 0=\beta} = & \sum_{\boldsymbol{k}} \sum_{j=1}^{3} \frac{c\hbar}{2} (-k^0 g^{0j} + k_j g^0{}_0 - k_0 g^0{}_j) \\
& (+c_{\boldsymbol{k}}^{(0)} c_{\boldsymbol{k}}^{(j)\dagger} + c_{\boldsymbol{k}}^{(0)\dagger} c_{\boldsymbol{k}}^{(j)}) \\
& + \sum_{\boldsymbol{k}} \sum_{j=1}^{3} \frac{c\hbar}{2} (2 g_j{}^0 - 1) \\
& (k^0 g^{0j} - k_j g^0{}_0 + k_0 g^0{}_j - k^0 g^{j0} + k^j g^{00} - k^0 g^{0j}) \\
& (c_{\boldsymbol{k}}^{(0)} c_{\text{-}\boldsymbol{k}}^{(j)} \exp\{-i2k^0 x^0\} + c_{\boldsymbol{k}}^{(0)\dagger} c_{\text{-}\boldsymbol{k}}^{(j)\dagger} \exp\{+i2k^0 x^0\}) \\
= & \sum_{\boldsymbol{k}} \sum_{j=1}^{3} \frac{c\hbar k_j}{2} (c_{\boldsymbol{k}}^{(0)} c_{\boldsymbol{k}}^{(j)\dagger} + c_{\boldsymbol{k}}^{(0)\dagger} c_{\boldsymbol{k}}^{(j)}) \\
& - \sum_{\boldsymbol{k}} \sum_{j=1}^{3} \frac{c\hbar}{2} (+k_j - k^j) \Big(c_{\text{-}\boldsymbol{k}}^{(0)} c_{\boldsymbol{k}}^{(j)} \exp\{-i2k^0 x^0\} + \\
& + c_{\text{-}\boldsymbol{k}}^{(0)\dagger} c_{\boldsymbol{k}}^{(j)\dagger} \exp\{+i2k^0 x^0\} \Big)
\end{aligned}
\tag{A.120}
$$

As the summation is symmetrical over all positive and negative $\boldsymbol{k}$, in the last line $-\boldsymbol{k}$ and $\boldsymbol{k}$ could be permuted. As furthermore the Fourier-operators commute according to (17.67) because of $(j) \neq (0)$, we get

$$
\tilde{H}_{\alpha=0\neq\beta} + \tilde{H}_{\alpha\neq 0=\beta} = 0 \; . \tag{A.121}
$$

Furthermore

$$
\begin{aligned}
\tilde{H}_{0\neq\alpha\neq\beta\neq 0} = & \sum_{\boldsymbol{k}} \sum_{j,l=1}^{3} (1 + g^{jl}) \frac{c\hbar}{2} (-k^0 g^{jl} + k_l g^0{}_j - k_j g^0{}_l) \\
& (+c_{\boldsymbol{k}}^{(j)} c_{\boldsymbol{k}}^{(l)\dagger} + c_{\boldsymbol{k}}^{(j)\dagger} c_{\boldsymbol{k}}^{(l)}) \\
& + \sum_{\boldsymbol{k}} \sum_{j,l=1}^{3} (1 + g^{jl}) \frac{c\hbar}{2} (2 g_l{}^0 - 1) \\
& (k^0 g^{jl} - k_l g^0{}_j + k_j g^0{}_l - k^0 g^{lj} + k^l g^{0j} - k^j g^{0l}) \\
& (c_{\boldsymbol{k}}^{(j)} c_{\text{-}\boldsymbol{k}}^{(l)} \exp\{-i2k^0 x^0\} + c_{\boldsymbol{k}}^{(j)\dagger} c_{\text{-}\boldsymbol{k}}^{(l)\dagger} \exp\{+i2k^0 x^0\}) = 0 \; .
\end{aligned}
$$

Thus one finds the Hamilton operator

$$\begin{aligned}\tilde{H} &= \underbrace{\tilde{H}_{\alpha=\beta}}_{(\text{A.118})} + \underbrace{\tilde{H}_{\alpha=0\neq\beta} + \tilde{H}_{\alpha\neq 0=\beta}}_{0} + \underbrace{\tilde{H}_{0\neq\alpha\neq\beta\neq 0}}_{0} = \\ &= \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \frac{\hbar\omega_{\boldsymbol{k}}}{2}(1-2{g_0}^{\alpha})(c_{\boldsymbol{k}}^{(\alpha)} c_{\boldsymbol{k}}^{(\alpha)\dagger} + c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)}) \\ &= -\sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g^{\alpha\beta} \frac{\hbar\omega_{\boldsymbol{k}}}{2} (\underbrace{c_{\boldsymbol{k}}^{(\alpha)} c_{\boldsymbol{k}}^{(\alpha)\dagger} - c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)}}_{1} + c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)} + c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)})\end{aligned}$$

As in case of all elementary (i. e. continuous) quantum fields, we postulate as law of nature, that the unphysical term which does not depend on the particle-number operator $c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)}$ must be removed, to achieve the correct result:

$$\tilde{H} = -\sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g^{\alpha\beta} \hbar\omega_{\boldsymbol{k}} c_{\boldsymbol{k}}^{(\alpha)\dagger} c_{\boldsymbol{k}}^{(\alpha)} \tag{A.122}$$

A.18 Computation of the momentum operator (17.71)

We want to demonstrate, that the momentum operator (17.71) follows from the momentum density (17.68) of the quantized field $\tilde{A}(x)$.

Using $\tilde{\pi}^{\tau}(x) =$ (17.64), $\tilde{A}(x) =$ (17.61), and $\tilde{A}_{\tau}(x) =$ (17.65), one gets

$$\begin{aligned}\tilde{\mathcal{P}}^{j} &= \tilde{\pi}^{\tau}(x)\, \mathrm{d}^{j} \tilde{A}_{\tau}(x) \\ &= -\sum_{\boldsymbol{f},\boldsymbol{k}} \sum_{\beta,\alpha=0}^{3} \sum_{\kappa,\tau=0}^{3} \frac{c\hbar(-f^0 g^{\tau\kappa} + f^{\tau} g^{0\kappa} - g^{0\tau} f^{\kappa}) k^{j}}{2\Omega\sqrt{\omega_{\boldsymbol{f}}\omega_{\boldsymbol{k}}}} \\ &\quad (e^{(\kappa)} \cdot e_{\boldsymbol{f}}^{(\beta)})(e^{(\tau)} \cdot e_{\boldsymbol{k}}^{(\alpha)}) \\ &\quad \Big(c_{\boldsymbol{f}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)} \exp\{-i(f+k)x\} - c_{\boldsymbol{f}}^{(\beta)} c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{-i(f-k)x\} \\ &\quad - c_{\boldsymbol{f}}^{(\beta)\dagger} c_{\boldsymbol{k}}^{(\alpha)} \exp\{i(f-k)x\} + c_{\boldsymbol{f}}^{(\beta)\dagger} c_{\boldsymbol{k}}^{(\alpha)\dagger} \exp\{i(f+k)x\} \Big) \,. \end{aligned} \tag{A.123}$$

Because there is not automatic summation over the names (κ) of the unit vectors, the summation symbols have been written explicitly. The momentum operator

$$\tilde{P}^j = \int\limits_{\Omega} \mathrm{d}^3x\, \tilde{\mathcal{P}}^j \tag{A.124}$$

is the integral of $\tilde{\mathcal{P}}^j$ over the total normalization volume Ω. To get in this integration the Kronecker-symbol according to (16.3), we switch before the integration in two terms the sequence of summation over $\boldsymbol{k}$. Using $\omega_{-\boldsymbol{k}} = \omega_{\boldsymbol{k}}$ and $e^{(\alpha)}_{-\boldsymbol{k}} = e^{(\alpha)}_{\boldsymbol{k}}(2g^{\alpha}{}_0 - 1)$ one gets

$$\begin{aligned}
\tilde{P}^j = &-\sum_{\boldsymbol{k}} \sum_{\beta,\alpha=0}^{3} \sum_{\kappa,\tau=0}^{3} \frac{c\hbar(-k^0 g^{\tau\kappa} + k^\tau g^{0\kappa} - g^{0\tau}k^\kappa)k^j}{2\omega_{\boldsymbol{k}}} \\
&(e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}})(e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}})\Big(- c^{(\beta)}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} - c^{(\beta)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}}\Big) \\
&+\sum_{\boldsymbol{k}} \sum_{\beta,\alpha=0}^{3} \sum_{\kappa,\tau=0}^{3} \frac{c\hbar(-k^0 g^{\tau\kappa} + k^\tau g^{0\kappa} - g^{0\tau}k^\kappa)k^j}{2\omega_{\boldsymbol{k}}} \\
&(e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}})(e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}})(2g^{\alpha}{}_0 - 1) \\
&\Big(c^{(\beta)}_{\boldsymbol{k}} c^{(\alpha)}_{-\boldsymbol{k}} \exp\{-i2k^0x^0\} + c^{(\beta)\dagger}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{-\boldsymbol{k}} \exp\{i2k^0x^0\}\Big)\,.
\end{aligned} \tag{A.125}$$

We split $\tilde{P}^j$ into the five terms

$$\tilde{P}^j = \tilde{P}^j_{\alpha=\beta=0} + \tilde{P}^j_{\alpha=\beta\neq 0} + \tilde{P}^j_{\alpha=0\neq\beta} + \tilde{P}^j_{\alpha\neq 0=\beta} + \tilde{P}^j_{0\neq\alpha\neq\beta\neq 0}\,.$$

In the sum $\tilde{P}^j_{\alpha=\beta=0}$, also $\tau = \kappa = 0$ must hold because of the factors $(e^{(\kappa)} \cdot e^{(0)}_{\boldsymbol{k}}) = g^{\kappa 0}$:

$$\tilde{P}^j_{\alpha=\beta=0} = \sum_{\boldsymbol{k}} \frac{c\hbar(-k^0+k^0-k^0)k^j}{2\omega_{\boldsymbol{k}}} \left(c^{(0)}_{\boldsymbol{k}} c^{(0)\dagger}_{\boldsymbol{k}} + c^{(0)\dagger}_{\boldsymbol{k}} c^{(0)}_{\boldsymbol{k}}\right) + \sum_{\boldsymbol{k}} \frac{c\hbar(-k^0+k^0-k^0)k^j}{2\omega_{\boldsymbol{k}}} \left(c^{(0)}_{\boldsymbol{k}} c^{(0)}_{\text{-}\boldsymbol{k}} \exp\{-i2k^0x^0\} + c^{(0)\dagger}_{\boldsymbol{k}} c^{(0)\dagger}_{\text{-}\boldsymbol{k}} \exp\{i2k^0x^0\}\right)$$

The second sum over $\boldsymbol{k}$ is zero, because for each term with $+\boldsymbol{k}$ there is another term with $-\boldsymbol{k}$, and these two terms differ by nothing than the sign of k^j. Using $k^0 = \omega_{\boldsymbol{k}}/c$, therefore

$$\tilde{P}^j_{\alpha=\beta=0} = -\sum_{\boldsymbol{k}} \frac{\hbar k^j}{2} \left(c^{(0)}_{\boldsymbol{k}} c^{(0)\dagger}_{\boldsymbol{k}} + c^{(0)\dagger}_{\boldsymbol{k}} c^{(0)}_{\boldsymbol{k}}\right) . \tag{A.126}$$

In the sum $\tilde{P}^j_{\alpha=\beta\neq 0}$, also $\tau \neq 0$ and $\kappa \neq 0$ must hold because of the factors $(e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}})$:

$$\tilde{P}^j_{\alpha=\beta\neq 0} = +\sum_{\boldsymbol{k}} \sum_{\alpha=1}^{3} \sum_{\kappa,\tau=1}^{3} \frac{c\hbar k^0 g^{\tau\kappa} k^j}{2\omega_{\boldsymbol{k}}} (e^{(\kappa)} \cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}}) \left(- c^{(\alpha)}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} - c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}}\right) + \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\kappa,\tau=0}^{3} \frac{c\hbar k^0 g^{\tau\kappa} k^j}{2\omega_{\boldsymbol{k}}} (e^{(\kappa)} \cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}}) \left(c^{(\alpha)}_{\boldsymbol{k}} c^{(\alpha)}_{\text{-}\boldsymbol{k}} \exp\{-i2k^0x^0\} + c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\text{-}\boldsymbol{k}} \exp\{i2k^0x^0\}\right)$$

In the second sum over $\boldsymbol{k}$, the term with $-\boldsymbol{k}$ differs by nothing than the sign of k^j from the term with $+\boldsymbol{k}$. Therefore in total the sum is zero. We expand $e^{(\alpha)}_{\boldsymbol{k}}$ with respect to $e^{(\kappa)}$ and with respect to $e^{(\tau)}$, and multiply the both expansions. Because of $\alpha \neq 0$, the result is

$$e^{(\alpha)}_{\boldsymbol{k}} \cdot e^{(\alpha)}_{\boldsymbol{k}} \overset{\text{(K.10e)}}{=} -1 = \sum_{\kappa,\tau=1}^{3} (e^{(\kappa)} \cdot e^{(\alpha)}_{\boldsymbol{k}})(e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}}) \underbrace{e^{(\kappa)} \cdot e^{(\tau)}}_{g^{\kappa\tau}} .$$

Using $k^0 = \omega_{\boldsymbol{k}}/c$, we thus find

$$\tilde{P}^j_{\alpha=\beta\neq 0} = +\sum_{\boldsymbol{k}} \sum_{\alpha=1}^{3} \frac{\hbar k^j}{2} \left(c^{(\alpha)}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} + c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} \right) . \tag{A.127}$$

In the sum $\tilde{P}^j_{\alpha=0\neq\beta}$, also $\tau = 0$ and $\kappa \neq 0$ must hold because of the factors $(e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}})$:

$$\begin{aligned}
\tilde{P}^j_{\alpha=0\neq\beta} = &+\sum_{\boldsymbol{k}} \sum_{\beta=1}^{3} \sum_{\kappa=1}^{3} \frac{c\hbar k^\kappa k^j}{2\omega_{\boldsymbol{k}}} \\
&(e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}}) \Big(- c^{(\beta)}_{\boldsymbol{k}} c^{(0)\dagger}_{\boldsymbol{k}} - c^{(\beta)\dagger}_{\boldsymbol{k}} c^{(0)}_{\boldsymbol{k}} \Big) \\
&- \sum_{\boldsymbol{k}} \sum_{\beta=1}^{3} \sum_{\kappa=1}^{3} \frac{c\hbar k^\kappa k^j}{2\omega_{\boldsymbol{k}}} (e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}}) \\
&\Big(c^{(\beta)}_{\boldsymbol{k}} c^{(0)}_{-\boldsymbol{k}} \exp\{-i2k^0x^0\} + c^{(\beta)\dagger}_{\boldsymbol{k}} c^{(0)\dagger}_{-\boldsymbol{k}} \exp\{i2k^0x^0\} \Big)
\end{aligned}$$

In the sum $\tilde{P}^j_{\alpha\neq 0=\beta}$, also $\tau \neq 0$ and $\kappa = 0$ must hold because of the factors $(e^{(\kappa)} \cdot e^{(\beta)}_{\boldsymbol{k}})$:

$$\begin{aligned}
\tilde{P}^j_{\alpha\neq 0=\beta} = &-\sum_{\boldsymbol{k}} \sum_{\alpha=1}^{3} \sum_{\tau=1}^{3} \frac{c\hbar k^\tau k^j}{2\omega_{\boldsymbol{k}}} \\
&(e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}}) \Big(- c^{(0)}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} - c^{(0)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} \Big) \\
&- \sum_{\boldsymbol{k}} \sum_{\alpha=1}^{3} \sum_{\tau=1}^{3} \frac{c\hbar k^\tau k^j}{2\omega_{\boldsymbol{k}}} (e^{(\tau)} \cdot e^{(\alpha)}_{\boldsymbol{k}}) \\
&\Big(c^{(0)}_{\boldsymbol{k}} c^{(\alpha)}_{-\boldsymbol{k}} \exp\{-i2k^0x^0\} + c^{(0)\dagger}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{-\boldsymbol{k}} \exp\{i2k^0x^0\} \Big)
\end{aligned}$$

The Fourier-operators in this equation commute because of (17.67). As the summation is running symmetrically over all positive and negative $\boldsymbol{k}$, one may furthermore permute $\boldsymbol{k}$ and $-\boldsymbol{k}$ in the second summand. Doing that, there is a sign change due to $e^{(\alpha)}_{-\boldsymbol{k}} = -e^{(\alpha)}_{\boldsymbol{k}}$:

$$\tilde{P}^j_{\alpha\neq 0=\beta} = -\sum_{\boldsymbol{k}}\sum_{\alpha=1}^{3}\sum_{\tau=1}^{3}\frac{c\hbar k^\tau k^j}{2\omega_{\boldsymbol{k}}} (e^{(\tau)}\cdot e^{(\alpha)}_{\boldsymbol{k}})\Big(-c^{(\alpha)}_{\boldsymbol{k}}c^{(0)\dagger}_{\boldsymbol{k}} - c^{(\alpha)\dagger}_{\boldsymbol{k}}c^{(0)}_{\boldsymbol{k}}\Big) + \sum_{\boldsymbol{k}}\sum_{\alpha=1}^{3}\sum_{\tau=1}^{3}\frac{c\hbar k^\tau k^j}{2\omega_{\boldsymbol{k}}}(e^{(\tau)}\cdot e^{(\alpha)}_{\boldsymbol{k}}) \Big(c^{(\alpha)}_{\boldsymbol{k}}c^{(0)}_{\text{-}\boldsymbol{k}}\exp\{-i2k^0x^0\} + c^{(\alpha)\dagger}_{\boldsymbol{k}}c^{(0)\dagger}_{\text{-}\boldsymbol{k}}\exp\{i2k^0x^0\}\Big)$$

Now it becomes obvious, that

$$\tilde{P}^j_{\alpha=0\neq\beta} + \tilde{P}^j_{\alpha\neq 0=\beta} = 0\ . \tag{A.128}$$

Eventually we compute $\tilde{P}^j_{0\neq\alpha\neq\beta\neq 0}$. In this term, also $\tau\neq 0$ and $\kappa\neq 0$ must hold because of the factors $(e^{(\kappa)}\cdot e^{(\beta)}_{\boldsymbol{k}})$:

$$\begin{aligned}\tilde{P}^j_{0\neq\alpha\neq\beta\neq 0} &= -\sum_{\boldsymbol{k}}\sum_{\beta,\alpha=1}^{3}\sum_{\kappa,\tau=1}^{3}(1-g^\alpha{}_\beta)\frac{c\hbar(-k^0g^{\tau\kappa})k^j}{2\omega_{\boldsymbol{k}}} (e^{(\kappa)}\cdot e^{(\beta)}_{\boldsymbol{k}})(e^{(\tau)}\cdot e^{(\alpha)}_{\boldsymbol{k}})\Big(-c^{(\beta)}_{\boldsymbol{k}}c^{(\alpha)\dagger}_{\boldsymbol{k}} - c^{(\beta)\dagger}_{\boldsymbol{k}}c^{(\alpha)}_{\boldsymbol{k}}\Big)\\ &\quad -\sum_{\boldsymbol{k}}\sum_{\beta,\alpha=1}^{3}\sum_{\kappa,\tau=1}^{3}(1-g^\alpha{}_\beta)\frac{c\hbar(-k^0g^{\tau\kappa})k^j}{2\omega_{\boldsymbol{k}}} (e^{(\kappa)}\cdot e^{(\beta)}_{\boldsymbol{k}})(e^{(\tau)}\cdot e^{(\alpha)}_{\boldsymbol{k}}) \Big(c^{(\beta)}_{\boldsymbol{k}}c^{(\alpha)}_{\text{-}\boldsymbol{k}}\exp\{-i2k^0x^0\} + c^{(\beta)\dagger}_{\boldsymbol{k}}c^{(\alpha)\dagger}_{\text{-}\boldsymbol{k}}\exp\{i2k^0x^0\}\Big)\\ &= 0\end{aligned} \tag{A.129}$$

This term is zero, as becomes visible if the coordinate system spanned by the unit vectors $e^{(\kappa)}$ is rotated such, that $e^{(\kappa)} = e^{(\kappa)}_{\boldsymbol{k}}$ holds for the unit vectors. There are only terms different from zero in case $\kappa=\beta\,,\ \tau=\alpha$. There is a factor $g^{\tau\kappa}$ in each of the summands. Therefore a summand can be different from zero only if $\beta=\alpha$. But the summation is running over $\beta\neq\alpha$ only due to the factor $(1-g^\alpha{}_\beta)$. Therefore in $\tilde{P}^j_{0\neq\alpha\neq\beta\neq 0}$ there is not

any summand different from zero.

In total, we have found:

$$\begin{aligned}\tilde{P}^j &= \underbrace{\tilde{P}^j_{\alpha=\beta=0}}_{\text{(A.126)}} + \underbrace{\tilde{P}^j_{\alpha=\beta\neq 0}}_{\text{(A.127)}} + \underbrace{\tilde{P}^j_{\alpha=0\neq\beta} + \tilde{P}^j_{\alpha\neq 0=\beta}}_{0} + \underbrace{\tilde{P}^j_{0\neq\alpha\neq\beta\neq 0}}_{0} \\ &= \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \frac{\hbar k^j}{2} \left(c^{(\alpha)}_{\boldsymbol{k}} c^{(\alpha)\dagger}_{\boldsymbol{k}} + c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} \right) (1 - 2g^{\alpha}{}_0) \\ &= \sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \hbar k^j \left(c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} + \tfrac{1}{2}[c^{(\alpha)}_{\boldsymbol{k}}, c^{(\alpha)\dagger}_{\boldsymbol{k}}] \right) (1 - 2g^{\alpha}{}_0) \\ &= -\sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g^{\alpha\beta} \hbar k^j \left(c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} + \tfrac{1}{2}[c^{(\alpha)}_{\boldsymbol{k}}, c^{(\alpha)\dagger}_{\boldsymbol{k}}] \right) \end{aligned} \tag{A.130}$$

As in case of all elementary (i. e. continuous) quantum fields, we postulate as law of nature that the commutator, as it does not depend on the particle-number operator $c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}}$, must be removed to achieve the correct result:

$$\tilde{P}^j = -\sum_{\boldsymbol{k}} \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} g^{\alpha\beta} \hbar k^j c^{(\alpha)\dagger}_{\boldsymbol{k}} c^{(\alpha)}_{\boldsymbol{k}} \tag{A.131}$$

A.19 Auxiliary computation for (14.22)

We want to prove, that

$$\langle \phi | S | \psi \rangle \overset{(14.22)}{=} \int_{\Omega} \mathrm{d}^3 x \, \phi^*(\boldsymbol{x}) S \psi(\boldsymbol{x}) \tag{A.132}$$

does hold for arbitrary operators S. In the most general case, S can be a polynomial of numbers and differential operators. If S contains only numbers, but no differential operators, then we have

$$\langle\phi|S|\psi\rangle = \int\limits_{\Omega} \mathrm{d}^3y \int\limits_{\Omega} \mathrm{d}^3x \underbrace{\langle\phi|\boldsymbol{y}\rangle}_{\phi^*(\boldsymbol{y})} \underbrace{\langle\boldsymbol{y}|S|\boldsymbol{x}\rangle}_{S\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x})} \underbrace{\langle\boldsymbol{x}|\psi\rangle}_{\psi(\boldsymbol{x})} = \int\limits_{\Omega} \mathrm{d}^3x\, \phi^*(\boldsymbol{x})\, S\, \psi(\boldsymbol{x}) \ . \tag{A.133}$$

To find the matrix element of differential operators in the position representation, we make use of the fact that we already know the commutator of one differential operator, i. e. the k-component of the momentum operator

$$p_k = \frac{\hbar}{i} \frac{\mathrm{d}}{\mathrm{d}x_k} \tag{A.134}$$

with the j-component of the position operator x_j:

$$x_j p_k - p_k x_j \overset{(14.9)}{=} i\hbar\delta_{jk} \tag{A.135}$$

It's easy to guess the corresponding matrix-element in the position representation, if we firstly restrict to a one-dimensional evaluation:

$$\langle y|xp - px|z\rangle = i\hbar\langle y|z\rangle = i\hbar\delta(y-z) \tag{A.136}$$

x, y, z and p are all within the same space dimension. We just are using different letters, to make the position operator x and it's eigenvalues y and z discernible.

The matrix element (A.136) can be written in the form

$$\langle y|xp - px)|z\rangle = \frac{\hbar}{i}\,(y-z)\,\frac{\mathrm{d}\,\delta(y-z)}{\mathrm{d}y} \ . \tag{A.137}$$

This is true, because $\delta(y-z)$ is different from zero only at $y = z$, but not at the boundaries of the normalization volume Ω, and thus partial integration results into

$$\int\limits_{\Omega^{1/3}} \mathrm{d}y\,(y-z)\,\frac{\mathrm{d}\,\delta(y-z)}{\mathrm{d}\,y} = \tag{A.138}$$

$$= \underbrace{\Big[(y-z)\delta(y-z)\Big]_{\Omega^{1/3}}}_{0} - \int\limits_{\Omega^{1/3}} \mathrm{d}y\,\underbrace{\frac{\mathrm{d}\,(y-z)}{\mathrm{d}\,y}}_{1}\,\delta(y-z)\ .$$

Alternatively, the matrix element (A.136) can be computed as follows:

$$\langle y|xp-px)|z\rangle = \int\limits_{\Omega^{1/3}} \mathrm{d}w\Big(\langle y|x|w\rangle\langle w|p|z\rangle - \langle y|p|w\rangle\langle w|x|z\rangle\Big) =$$

$$= \int\limits_{\Omega^{1/3}} \mathrm{d}w\Big(w\delta(y-w)\langle w|p|z\rangle - \langle y|p|w\rangle z\delta(w-z)\Big) = (y-z)\langle y|p|z\rangle \tag{A.139}$$

Comparing (A.137), (A.139) and (A.134), we get

$$\langle y|\frac{\mathrm{d}}{\mathrm{d}\,x}|z\rangle = \frac{\mathrm{d}\,\delta(y-z)}{\mathrm{d}\,y}\ . \tag{A.140}$$

For the $(m+1)$th power of the differential operator, we find due to partial integration

$$\langle y|\frac{\mathrm{d}^{m+1}}{\mathrm{d}\,x^{m+1}}|z\rangle = \int\limits_{\Omega^{1/3}} \mathrm{d}w\,\langle y|\frac{\mathrm{d}^m}{\mathrm{d}\,x^m}|w\rangle\langle w|\frac{\mathrm{d}}{\mathrm{d}\,x}|z\rangle =$$

$$= \int\limits_{\Omega^{1/3}} \mathrm{d}w\,\langle y|\frac{\mathrm{d}^m}{\mathrm{d}\,x^m}|w\rangle\,\frac{\mathrm{d}\,\delta(w-z)}{\mathrm{d}\,w} =$$

$$= \underbrace{\Big[\langle y|\frac{\mathrm{d}^m}{\mathrm{d}\,x^m}|w\rangle\,\delta(w-z)\Big]_{\Omega^{1/3}}}_{0} - \int\limits_{\Omega^{1/3}} \mathrm{d}w\,\delta(w-z)\,\frac{\mathrm{d}}{\mathrm{d}\,w}\Big(\langle y|\frac{\mathrm{d}^m}{\mathrm{d}\,x^m}|w\rangle\Big) =$$

$$= -\frac{\mathrm{d}}{\mathrm{d}\,z}\Big(\langle y|\frac{\mathrm{d}^m}{\mathrm{d}\,x^m}|z\rangle\Big)\ . \tag{A.141}$$

Because the matrix element with $m=1$ is already known from (A.140), the

matrix elements of all powers of the differential operator can be computed by means of this recursion formula:

$$\begin{aligned}
\langle y|\frac{\mathrm{d}^2}{\mathrm{d}x^2}|z\rangle &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\,\delta(y-z)}{\mathrm{d}y}\right) = +\frac{\mathrm{d}^2\,\delta(y-z)}{\mathrm{d}y^2} \\
\langle y|\frac{\mathrm{d}^3}{\mathrm{d}x^3}|z\rangle &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}^2\,\delta(y-z)}{\mathrm{d}y^2}\right) = +\frac{\mathrm{d}^3\,\delta(y-z)}{\mathrm{d}y^3} \\
\langle y|\frac{\mathrm{d}^m}{\mathrm{d}x^m}|z\rangle &= +\frac{\mathrm{d}^m\,\delta(y-z)}{\mathrm{d}y^m}
\end{aligned} \tag{A.142}$$

This formula holds for the one-dimensional case. It's plausible, that the extension to the three-dimensional case is given by

$$\langle \boldsymbol{y}|\frac{\mathrm{d}^m}{\mathrm{d}\boldsymbol{x}^m}|\boldsymbol{z}\rangle = +\frac{\mathrm{d}^m\,\delta(\boldsymbol{y}-\boldsymbol{z})}{\mathrm{d}\boldsymbol{y}^m}\ . \tag{A.143}$$

Thus for $S \equiv \frac{\mathrm{d}^m}{\mathrm{d}\boldsymbol{x}^m}$ the matrix element in the position representation is

$$\begin{aligned}
\langle\phi|S|\psi\rangle &= \int\limits_\Omega \mathrm{d}^3y \int\limits_\Omega \mathrm{d}^3x\ \underbrace{\langle\phi|\boldsymbol{y}\rangle}_{\phi^*(\boldsymbol{y})}\langle\boldsymbol{y}|\frac{\mathrm{d}^m}{\mathrm{d}\boldsymbol{x}^m}|\boldsymbol{x}\rangle\underbrace{\langle\boldsymbol{x}|\psi\rangle}_{\psi(\boldsymbol{x})} \\
&= \int\limits_\Omega \mathrm{d}^3y\,\phi^*(\boldsymbol{y}) \int\limits_\Omega \mathrm{d}^3x\,\frac{\mathrm{d}^m\delta(\boldsymbol{y}-\boldsymbol{x})}{\mathrm{d}\boldsymbol{y}^m}\,\psi(\boldsymbol{x}) \\
&= \int\limits_\Omega \mathrm{d}^3y\,\phi^*(\boldsymbol{y}) \int\limits_\Omega \mathrm{d}^3x\,(-1)^m\,\frac{\mathrm{d}^m\delta(\boldsymbol{y}-\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}^m}\,\psi(\boldsymbol{x})\ .
\end{aligned} \tag{A.144}$$

Now we integrate m-times partially over $\boldsymbol{x}$. Because the delta function or the nth derivative of the delta function are contained in the primitive, it's value is zero at the boundaries of the integration volume Ω. Thus we get

$$\begin{aligned}
&(-1)^m \int\limits_\Omega \mathrm{d}^3x\ \frac{\mathrm{d}^m\delta(\boldsymbol{y}-\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}^m}\,\psi(\boldsymbol{x}) \ = \\
&= (-1)^m(-1)^m \int\limits_\Omega \mathrm{d}^3x\ \delta(\boldsymbol{y}-\boldsymbol{x})\,\frac{\mathrm{d}^m\,\psi(\boldsymbol{x})}{\mathrm{d}\boldsymbol{x}^m} = +\frac{\mathrm{d}^m\,\psi(\boldsymbol{y})}{\mathrm{d}\boldsymbol{y}^m}\ .
\end{aligned}$$

This is inserted into (A.144):

$$\langle\phi|S|\psi\rangle = \langle\phi|\frac{\mathrm{d}^m}{\mathrm{d}\boldsymbol{x}^m}|\psi\rangle = \int\limits_\Omega \mathrm{d}^3y\,\phi^*(\boldsymbol{y})\frac{\mathrm{d}^m\psi(\boldsymbol{y})}{\mathrm{d}\boldsymbol{y}^m} = \\ = \int\limits_\Omega \mathrm{d}^3x\,\phi^*(\boldsymbol{x})S\psi(\boldsymbol{x}) \tag{A.145}$$

This formula for the matrix element in the position representation holds for arbitrary operators S, no matter which powers of numbers and/or differential operators they contain.

A.20 The Laplace Operator

The Laplace operator is
a) in Cartesian coordinates

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}\ , \tag{A.146a}$$

b) in cylindrical coordinates

$$\Delta = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\Big(\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\Big) + \frac{1}{\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}\ , \tag{A.146b}$$

c) in spherical coordinates

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r^2\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Big(\sin\vartheta\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Big) + \frac{1}{r^2\sin^2\vartheta}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\ . \tag{A.146c}$$

A.21 Derivation of Equation (23.28)

Using the definitions

$$\begin{pmatrix} B^{\boldsymbol{k}} \\ A^{\boldsymbol{k}}_{-} \\ A^{\boldsymbol{k}}_{+} \end{pmatrix} \equiv \sqrt{\frac{1}{2(E+mc^2)}} \begin{pmatrix} c\hbar k_1 + ic\hbar k_2 \\ E + mc^2 - c\hbar k_3 \\ E + mc^2 + c\hbar k_3 \end{pmatrix} \tag{A.147}$$
$$\text{with } E = \hbar\omega_{\boldsymbol{k}} > 0 \ , \ k_j = -k^j \ ,$$

we compute the spinors

$${}^{1}u^{\boldsymbol{f}} \overset{(8.75a)}{=} \begin{pmatrix} A^{\boldsymbol{f}}_{+} \\ B^{\boldsymbol{f}} \\ A^{\boldsymbol{f}}_{-} \\ -B^{\boldsymbol{f}} \end{pmatrix} \qquad {}^{2}u^{\boldsymbol{f}} \overset{(8.75b)}{=} \begin{pmatrix} B^{\boldsymbol{f}*} \\ A^{\boldsymbol{f}}_{-} \\ -B^{\boldsymbol{f}*} \\ A^{\boldsymbol{f}}_{+} \end{pmatrix} \tag{A.148a}$$

$${}^{1}v^{\boldsymbol{f}} \overset{(8.75c)}{=} \begin{pmatrix} A^{\boldsymbol{f}}_{+} \\ B^{\boldsymbol{f}} \\ -A^{\boldsymbol{f}}_{-} \\ B^{\boldsymbol{f}} \end{pmatrix} \qquad {}^{2}v^{\boldsymbol{f}} \overset{(8.75d)}{=} \begin{pmatrix} B^{\boldsymbol{f}*} \\ A^{\boldsymbol{f}}_{-} \\ B^{\boldsymbol{f}*} \\ -A^{\boldsymbol{f}}_{+} \end{pmatrix} \ . \tag{A.148b}$$

Furthermore we compute the spinor product

$$\begin{aligned} {}^{r_\alpha}\bar{u}^{\boldsymbol{f}} \, {}^{r_\beta}u^{\boldsymbol{k}} &= {}^{r_\alpha}u^{\boldsymbol{f}+} \, \gamma^0 \, {}^{r_\beta}u^{\boldsymbol{k}} = \\ &\overset{(8.15a)}{=} \begin{pmatrix} {}^{r_\alpha}u^{\boldsymbol{f}*}_1 & {}^{r_\alpha}u^{\boldsymbol{f}*}_2 & {}^{r_\alpha}u^{\boldsymbol{f}*}_3 & {}^{r_\alpha}u^{\boldsymbol{f}*}_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} {}^{r_\beta}u^{\boldsymbol{k}}_1 \\ {}^{r_\beta}u^{\boldsymbol{k}}_2 \\ {}^{r_\beta}u^{\boldsymbol{k}}_3 \\ {}^{r_\beta}u^{\boldsymbol{k}}_4 \end{pmatrix} \\ &= {}^{r_\alpha}u^{\boldsymbol{f}*}_3 \, {}^{r_\beta}u^{\boldsymbol{k}}_1 + {}^{r_\alpha}u^{\boldsymbol{f}*}_4 \, {}^{r_\beta}u^{\boldsymbol{k}}_2 + {}^{r_\alpha}u^{\boldsymbol{f}*}_1 \, {}^{r_\beta}u^{\boldsymbol{k}}_3 + {}^{r_\alpha}u^{\boldsymbol{f}*}_2 \, {}^{r_\beta}u^{\boldsymbol{k}}_4 \ . \end{aligned} \tag{A.149}$$

In particular, one finds

$${}^{1}\bar{u}^{\boldsymbol{f}} \, {}^{1}u^{\boldsymbol{k}} = A^{\boldsymbol{f}}_{-} \, A^{\boldsymbol{k}}_{+} - B^{\boldsymbol{f}*} \, B^{\boldsymbol{k}} + A^{\boldsymbol{f}}_{+} \, A^{\boldsymbol{k}}_{-} - B^{\boldsymbol{f}*} \, B^{\boldsymbol{k}} \tag{A.150a}$$

$${}^{1}\bar{u}^{\boldsymbol{f}} \, {}^{2}u^{\boldsymbol{k}} = A^{\boldsymbol{f}}_{-} \, B^{\boldsymbol{k}*} - B^{\boldsymbol{f}*} \, A^{\boldsymbol{k}}_{-} - A^{\boldsymbol{f}}_{+} \, B^{\boldsymbol{k}*} + B^{\boldsymbol{f}*} \, A^{\boldsymbol{k}}_{+} \tag{A.150b}$$

$${}^{2}\bar{u}^{\boldsymbol{f}} \, {}^{1}u^{\boldsymbol{k}} = -B^{\boldsymbol{f}} \, A^{\boldsymbol{k}}_{+} + A^{\boldsymbol{f}}_{+} \, B^{\boldsymbol{k}} + B^{\boldsymbol{f}} \, A^{\boldsymbol{k}}_{-} - A^{\boldsymbol{f}}_{-} \, B^{\boldsymbol{k}} \tag{A.150c}$$

$${}^{2}\bar{u}^{\boldsymbol{f}} \, {}^{2}u^{\boldsymbol{k}} = -B^{\boldsymbol{f}} \, B^{\boldsymbol{k}*} + A^{\boldsymbol{f}}_{+} \, A^{\boldsymbol{k}}_{-} - B^{\boldsymbol{f}} \, B^{\boldsymbol{k}*} + A^{\boldsymbol{f}}_{-} \, A^{\boldsymbol{k}}_{+} \tag{A.150d}$$

$${}^{1}\bar{u}^{\boldsymbol{f}} \, {}^{1}v^{\boldsymbol{k}} = A^{\boldsymbol{f}}_{-} \, A^{\boldsymbol{k}}_{+} - B^{\boldsymbol{f}*} \, B^{\boldsymbol{k}} - A^{\boldsymbol{f}}_{+} \, A^{\boldsymbol{k}}_{-} + B^{\boldsymbol{f}*} \, B^{\boldsymbol{k}} \tag{A.150e}$$

$$
\begin{aligned}
{}^{1}\bar{u}^{\boldsymbol{f}}\ {}^{2}v^{\boldsymbol{k}} &= A_{-}^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - B^{\boldsymbol{f}*}\ A_{-}^{\boldsymbol{k}} + A_{+}^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - B^{\boldsymbol{f}*}\ A_{+}^{\boldsymbol{k}} && \text{(A.150f)}\\
{}^{2}\bar{u}^{\boldsymbol{f}}\ {}^{1}v^{\boldsymbol{k}} &= -B^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} + A_{+}^{\boldsymbol{f}}\ B^{\boldsymbol{k}} - B^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} + A_{-}^{\boldsymbol{f}}\ B^{\boldsymbol{k}} && \text{(A.150g)}\\
{}^{2}\bar{u}^{\boldsymbol{f}}\ {}^{2}v^{\boldsymbol{k}} &= -B^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} + A_{+}^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} + B^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - A_{-}^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} && \text{(A.150h)}\\
{}^{1}\bar{v}^{\boldsymbol{f}}\ {}^{1}u^{\boldsymbol{k}} &= -A_{-}^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} + B^{\boldsymbol{f}*}\ B^{\boldsymbol{k}} + A_{+}^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} - B^{\boldsymbol{f}*}\ B^{\boldsymbol{k}} && \text{(A.150i)}\\
{}^{1}\bar{v}^{\boldsymbol{f}}\ {}^{2}u^{\boldsymbol{k}} &= -A_{-}^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} + B^{\boldsymbol{f}*}\ A_{-}^{\boldsymbol{k}} - A_{+}^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} + B^{\boldsymbol{f}*}\ A_{+}^{\boldsymbol{k}} && \text{(A.150j)}\\
{}^{2}\bar{v}^{\boldsymbol{f}}\ {}^{1}u^{\boldsymbol{k}} &= B^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} - A_{+}^{\boldsymbol{f}}\ B^{\boldsymbol{k}} + B^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} - A_{-}^{\boldsymbol{f}}\ B^{\boldsymbol{k}} && \text{(A.150k)}\\
{}^{2}\bar{v}^{\boldsymbol{f}}\ {}^{2}u^{\boldsymbol{k}} &= B^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - A_{+}^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} - B^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} + A_{-}^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} && \text{(A.150l)}\\
{}^{1}\bar{v}^{\boldsymbol{f}}\ {}^{1}v^{\boldsymbol{k}} &= -A_{-}^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} + B^{\boldsymbol{f}*}\ B^{\boldsymbol{k}} - A_{+}^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} + B^{\boldsymbol{f}*}\ B^{\boldsymbol{k}} && \text{(A.150m)}\\
{}^{1}\bar{v}^{\boldsymbol{f}}\ {}^{2}v^{\boldsymbol{k}} &= -A_{-}^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} + B^{\boldsymbol{f}*}\ A_{-}^{\boldsymbol{k}} + A_{+}^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - B^{\boldsymbol{f}*}\ A_{+}^{\boldsymbol{k}} && \text{(A.150n)}\\
{}^{2}\bar{v}^{\boldsymbol{f}}\ {}^{1}v^{\boldsymbol{k}} &= B^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}} - A_{+}^{\boldsymbol{f}}\ B^{\boldsymbol{k}} - B^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} + A_{-}^{\boldsymbol{f}}\ B^{\boldsymbol{k}} && \text{(A.150o)}\\
{}^{2}\bar{v}^{\boldsymbol{f}}\ {}^{2}v^{\boldsymbol{k}} &= B^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - A_{+}^{\boldsymbol{f}}\ A_{-}^{\boldsymbol{k}} + B^{\boldsymbol{f}}\ B^{\boldsymbol{k}*} - A_{-}^{\boldsymbol{f}}\ A_{+}^{\boldsymbol{k}}\ . && \text{(A.150p)}
\end{aligned}
$$

Using

$$
\begin{pmatrix} B^{\boldsymbol{k}} \\ A_{-}^{\boldsymbol{k}} \\ A_{+}^{\boldsymbol{k}} \end{pmatrix} \equiv \sqrt{\frac{1}{2(E+mc^2)}} \begin{pmatrix} c\hbar k_1 + ic\hbar k_2 \\ E + mc^2 - c\hbar k_3 \\ E + mc^2 + c\hbar k_3 \end{pmatrix} \overset{|c\hbar\boldsymbol{k}| \ll mc^2}{\approx} \begin{pmatrix} 0 \\ \sqrt{mc^2} \\ \sqrt{mc^2} \end{pmatrix}, \tag{A.151}
$$

one gets in the non-relativistic limit the simple expressions

$$
{}^{1}u^{\boldsymbol{f}} \approx \begin{pmatrix} \sqrt{mc^2} \\ 0 \\ \sqrt{mc^2} \\ 0 \end{pmatrix} \qquad {}^{2}u^{\boldsymbol{f}} \approx \begin{pmatrix} 0 \\ \sqrt{mc^2} \\ 0 \\ \sqrt{mc^2} \end{pmatrix} \tag{A.152a}
$$

$$
{}^{1}v^{\boldsymbol{f}} \approx \begin{pmatrix} \sqrt{mc^2} \\ 0 \\ -\sqrt{mc^2} \\ 0 \end{pmatrix} \qquad {}^{2}v^{\boldsymbol{f}} \approx \begin{pmatrix} 0 \\ \sqrt{mc^2} \\ 0 \\ -\sqrt{mc^2} \end{pmatrix} \tag{A.152b}
$$

$$
\text{if } |c\hbar\boldsymbol{f}| \ll mc^2\ .
$$

From these expressions, the following approximations may be concluded in the non-relativistic limit:

$$
\begin{aligned}
{}^{r}\bar{u}^{\boldsymbol{f}}\ {}^{s}u^{\boldsymbol{k}} &\approx 2mc^2\,\delta_{rs} && \text{(A.153a)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\ {}^{s}v^{\boldsymbol{k}} &\approx 0 && \text{(A.153b)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\ {}^{s}u^{\boldsymbol{k}} &\approx 0 && \text{(A.153c)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\ {}^{s}v^{\boldsymbol{k}} &\approx -2mc^2\,\delta_{rs} && \text{(A.153d)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^0\ {}^{s}u^{\boldsymbol{k}} &\approx 2mc^2\,\delta_{rs} && \text{(A.153e)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\gamma^0\ {}^{s}v^{\boldsymbol{k}} &\approx 0 && \text{(A.153f)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^0\ {}^{s}u^{\boldsymbol{k}} &\approx 0 && \text{(A.153g)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^0\ {}^{s}v^{\boldsymbol{k}} &\approx 2mc^2\,\delta_{rs} && \text{(A.153h)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^1\ {}^{s}u^{\boldsymbol{k}} &\approx 0 && \text{(A.153i)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^1\ {}^{s}v^{\boldsymbol{k}} &\approx -2mc^2(1-\delta_{rs}) && \text{(A.153j)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^1\ {}^{s}u^{\boldsymbol{k}} &\approx -2mc^2(1-\delta_{rs}) && \text{(A.153k)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^1\ {}^{s}v^{\boldsymbol{k}} &\approx 0 && \text{(A.153l)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^2\ {}^{s}u^{\boldsymbol{k}} &\approx 0 && \text{(A.153m)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^2\ {}^{s}v^{\boldsymbol{k}} &\approx i2mc^2(1-\delta_{rs})(\delta_{r1}-\delta_{r2}) && \text{(A.153n)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^2\ {}^{s}u^{\boldsymbol{k}} &\approx i2mc^2(1-\delta_{rs})(\delta_{r1}-\delta_{r2}) && \text{(A.153o)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^2\ {}^{s}v^{\boldsymbol{k}} &\approx 0 && \text{(A.153p)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^3\ {}^{s}u^{\boldsymbol{k}} &\approx 0 && \text{(A.153q)}\\
{}^{r}\bar{u}^{\boldsymbol{f}}\,\gamma^3\ {}^{s}v^{\boldsymbol{k}} &\approx 2mc^2\delta_{rs}(\delta_{r2}-\delta_{r1}) && \text{(A.153r)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^3\ {}^{s}u^{\boldsymbol{k}} &\approx 2mc^2\delta_{rs}(\delta_{r2}-\delta_{r1}) && \text{(A.153s)}\\
{}^{r}\bar{v}^{\boldsymbol{f}}\,\gamma^3\ {}^{s}v^{\boldsymbol{k}} &\approx 0 && \text{(A.153t)}
\end{aligned}
$$

if $|c\hbar\boldsymbol{f}| \ll mc^2$ and $|c\hbar\boldsymbol{k}| \ll mc^2$

These expressions can easily be proved due to insertion of (A.152) and (8.15).

A.22 Derivation of Equation (24.33)

In the center-of-mass system ($\boldsymbol{k}_4 = -\boldsymbol{k}_3$)

$$\begin{aligned} E_{\mathrm{CM}}^2 &= \Big(\sqrt{\hbar^2 c^2 \boldsymbol{k}_3^2 + m_3^2 c^4} + \sqrt{\hbar^2 c^2 \boldsymbol{k}_3^2 + m_4^2 c^4} \Big)^2 \\ E_{\mathrm{CM}}^2 - 2\hbar^2 c^2 \boldsymbol{k}_3^2 - (m_3^2 + m_4^2) c^4 \; &= \\ = 2 \sqrt{\hbar^4 c^4 \boldsymbol{k}_3^4 + \hbar^2 c^6 \boldsymbol{k}_3^2 (m_3^2 + m_4^2) + m_3^2 m_4^2 c^8} \; . & \end{aligned}$$

Both sides are squared once more:

$$\begin{aligned} 4\hbar^4 c^4 \boldsymbol{k}_3^4 + \Big(- 4E_{\mathrm{CM}}^2 \hbar^2 c^2 + 4\hbar^2 c^6 (m_3^2 + m_4^2) \Big) \boldsymbol{k}_3^2 + & \\ + E_{\mathrm{CM}}^4 - 2E_{\mathrm{CM}}^2 (m_3^2 + m_4^2) c^4 + (m_3^2 + m_4^2)^2 c^8 \; = & \\ = 4\hbar^4 c^4 \boldsymbol{k}_3^4 + 4\hbar^2 c^6 (m_3^2 + m_4^2) \boldsymbol{k}_3^2 + 4 m_3^2 m_4^2 c^8 & \\ \boldsymbol{k}_3^2 = \frac{4 m_3^2 m_4^2 c^8 - E_{\mathrm{CM}}^4 + 2E_{\mathrm{CM}}^2 (m_3^2 + m_4^2) c^4 - (m_3^2 + m_4^2)^2 c^8}{-4E_{\mathrm{CM}}^2 \hbar^2 c^2} & \\ = \frac{E_{\mathrm{CM}}^4 - 2E_{\mathrm{CM}}^2 (m_3^2 + m_4^2) c^4 + (m_3^2 - m_4^2)^2 c^8}{4E_{\mathrm{CM}}^2 \hbar^2 c^2} & \end{aligned}$$

This results into

$$|\boldsymbol{k}_3| = \frac{S_{34}}{2E_{\mathrm{CM}} \hbar c} \qquad \text{with} \tag{A.154}$$

$$S_{34} \equiv \sqrt{(E_{\mathrm{CM}}^2 - (m_3 + m_4)^2 c^4)(E_{\mathrm{CM}}^2 - (m_3 - m_4)^2 c^4)} \; .$$

A.23 Gauß' Integral Formula

We state the formula for Gauß' integral with real exponent and with n-dimensional vectors without proof:

$$x \equiv \{x^1, \ldots, x^n\} \; , \; b \equiv \{b^1, \ldots, b^n\} \; , \; a > 0 \; , \; x^j, b^j, a \in \mathbb{R}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}^n x \, \exp\Big\{ - a x^2 + b x \Big\} = \Big(\frac{\pi}{a} \Big)^{n/2} \exp\Big\{ \frac{b^2}{4a} \Big\} \tag{A.155a}$$

The integral's convergence is guaranteed by the condition $a > 0$. Again without proof, we state that this formula is valid as well if both x^j and a are imaginary:

$$x^j \to ix^j \ , \ bx \to ibx \ , \ a \to ia \ , \ x^j, b^j, a \in \mathbb{R}$$

Inserting these substitutions, the integral-formula becomes

$$\begin{aligned} & x \equiv \{x^1, \ldots, x^n\} \ , \ b \equiv \{b^1, \ldots, b^n\} \ , \ x^j, b^j, a \in \mathbb{R} \\ & \int\limits_{-\infty}^{+\infty} i\mathrm{d}^n x \, \exp\Big\{iax^2 + ibx\Big\} = \Big(\frac{\pi}{ia}\Big)^{n/2} \exp\Big\{\frac{b^2}{4ia}\Big\} \\ & \int\limits_{-\infty}^{+\infty} \mathrm{d}^n x \, \exp\Big\{iax^2 + ibx\Big\} = (-i)^{1+n/2}\Big(\frac{\pi}{a}\Big)^{n/2} \exp\Big\{\frac{-ib^2}{4a}\Big\} \ . \end{aligned} \tag{A.155b}$$

The restriction $a > 0$ would be meaningless, if the exponential function's argument is purely imaginary. It may be skipped, because the integral's convergence is secured by the fact, that the exponential function is oscillating with increasing speed for increasing $|x|$.

A.24 Derivation of (26.12)

For the computation of $F_{\mathrm{P}}^{\alpha\beta}$ = (26.5), several tricky manipulations[1] will be applied. As the $N + 1$ terms all have the same form, we perform the manipulations explicitly only for one term. It is understood, that the same manipulations are applied to all $N + 1$ terms.

Remember that the factors under the trace may be cyclically permuted. Comparing the trace in (26.5) with the trace in (24.61), one therefore may conclude:

[1] The quite original method for the computation of vacuum polarization, which is described in the sequel, can be found in the book of Greiner and Reinhardt [6, section 5.2].

$$
\begin{aligned}
&\mathrm{Sp}\Big\{\Big(\gamma^\sigma(k_\gamma+k)_\sigma + m\tfrac{c}{\hbar}\Big)\gamma^\beta\Big(\gamma^\tau k_\tau + m\tfrac{c}{\hbar}\Big)\gamma^\alpha\Big\} \;= \\
&= 4\Big((k_\gamma+k)^\beta k^\alpha - (k_\gamma+k)k g^{\beta\alpha} + (k_\gamma+k)^\alpha k^\beta + (m\tfrac{c}{\hbar})^2 g^{\beta\alpha}\Big) \\
&= -4\Big(\frac{\mathrm{d}}{\mathrm{d}\,r_{1\beta}}\frac{\mathrm{d}}{\mathrm{d}\,r_{2\alpha}} - \frac{\mathrm{d}}{\mathrm{d}\,r_1}\frac{\mathrm{d}}{\mathrm{d}\,r_2}g^{\beta\alpha} + \frac{\mathrm{d}}{\mathrm{d}\,r_{1\alpha}}\frac{\mathrm{d}}{\mathrm{d}\,r_{2\beta}} - (m\tfrac{c}{\hbar})^2 g^{\beta\alpha}\Big) \\
&\qquad \exp\{ir_1(k_\gamma+k) + ir_2 k\}\Big|_{r_1=r_2=0}
\end{aligned}
\tag{A.156}
$$

The denominators of the integrands in (26.5) are modified using

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}s\, \exp\{is(K+i\epsilon)\} = \frac{\exp\{isK - s\epsilon\}}{i(K+i\epsilon)}\Big|_0^\infty = \\
&\quad = \frac{0-1}{i(K+i\epsilon)} = \frac{i}{K+i\epsilon} \quad \text{with } K,\epsilon \in \mathbb{R} \text{ and } \epsilon > 0\ .
\end{aligned}
$$

This results into

$$
\begin{aligned}
&\frac{i}{\Big((k_\gamma+k)^2 - m^2\tfrac{c^2}{\hbar^2} + i\epsilon'\Big)\Big(k^2 - m^2\tfrac{c^2}{\hbar^2} + i\epsilon'\Big)} = \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \\
&\quad \cdot \exp\Big\{is_1\Big((k_\gamma+k)^2 - m^2\tfrac{c^2}{\hbar^2}\Big) + is_2\Big(k^2 - m^2\tfrac{c^2}{\hbar^2}\Big)\Big\}\ .
\end{aligned}
\tag{A.157}
$$

The small terms $i\epsilon'$ have been inserted before, to avoid the poles at $(k_\gamma + k)^2 = m^2c^2/\hbar^2$ and at $k^2 = m^2c^2/\hbar^2$. In the exponential function they are not needed any more. The dimensions of the four new inserted parameters are

$$
[r_j] = \text{length} \qquad\qquad [s_j] = \text{length}^2\ . \tag{A.158}
$$

After these manipulations, the first term in (26.5) is looking like this:

$$
\begin{aligned}
&- i\,\frac{4q^2}{\hbar^2}\int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2\, \frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \\
&\cdot \Big(\frac{\mathrm{d}}{\mathrm{d}r_{1\beta}}\frac{\mathrm{d}}{\mathrm{d}r_{2\alpha}} - \frac{\mathrm{d}}{\mathrm{d}r_1}\frac{\mathrm{d}}{\mathrm{d}r_2}g^{\beta\alpha} + \frac{\mathrm{d}}{\mathrm{d}r_{1\alpha}}\frac{\mathrm{d}}{\mathrm{d}r_{2\beta}} - (m\tfrac{c}{\hbar})^2 g^{\beta\alpha}\Big) \\
&\cdot \exp\Big\{i(s_1+s_2)k^2 + i(r_1+r_2+s_1 2k_\gamma)k\Big\}\Big|_{r_1=r_2=0} \\
&\cdot \exp\Big\{i(r_1 k_\gamma + s_1 k_\gamma^2)\Big\}\Big|_{r_1=r_2=0}
\end{aligned}
\tag{A.159}
$$

Only the first exponential function does depend on k. The sum and the integral over k can be computed — respecting (7.5) — by means of the formula for the Gauß' integral with imaginary exponent and 4-dimensional vectors, which is printed in appendix A.23:

$$
\begin{aligned}
&\frac{1}{\Omega}\sum_{\boldsymbol{k}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^0}{2\pi} \exp\Big\{i(s_1+s_2)k^2 + i(r_1+r_2+s_1 2k_\gamma)k\Big\} \\
&\overset{\text{(A.155b)}}{=} \frac{-i}{16\pi^2(s_1+s_2)^2} \exp\Big\{\frac{-i(r_1+r_2+s_1 2k_\gamma)^2}{4(s_1+s_2)}\Big\}\,.
\end{aligned}
\tag{A.160}
$$

This result is inserted into (A.159):

$$
\begin{aligned}
(\text{A.159}) = &\int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2\, \frac{-q^2}{4\pi^2\hbar^2(s_1+s_2)^2} \\
&\cdot \Big(\frac{\mathrm{d}}{\mathrm{d}r_{1\beta}}\frac{\mathrm{d}}{\mathrm{d}r_{2\alpha}} - \frac{\mathrm{d}}{\mathrm{d}r_1}\frac{\mathrm{d}}{\mathrm{d}r_2}g^{\beta\alpha} + \frac{\mathrm{d}}{\mathrm{d}r_{1\alpha}}\frac{\mathrm{d}}{\mathrm{d}r_{2\beta}} - (m\tfrac{c}{\hbar})^2 g^{\beta\alpha}\Big) \\
&\cdot \exp\Big\{\frac{-i(r_1^2+r_2^2+4s_1^2k_\gamma^2+2r_1r_2+4r_1s_1k_\gamma+4r_2s_1k_\gamma)}{4(s_1+s_2)} + \\
&\quad + i(r_1k_\gamma + s_1k_\gamma^2) - i(s_1+s_2)m^2\tfrac{c^2}{\hbar^2}\Big\}\Big|_{r_1=r_2=0}
\end{aligned}
\tag{A.161}
$$

Now the derivatives of the exponential function must be computed.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r_{1\beta}}\frac{\mathrm{d}}{\mathrm{d}r_{2\alpha}}\exp\Big\{\dots\Big\}\Big|_{r_1=r_2=0} = \\
&= \frac{\mathrm{d}}{\mathrm{d}r_{1\beta}}\frac{-i(2r_2^\alpha + 2r_1^\alpha + 4s_1k_\gamma^\alpha)}{4(s_1+s_2)}\exp\Big\{\dots\Big\}\Big|_{r_1=r_2=0} \\
&= \Big(\frac{-i2g^{\alpha\beta}}{4(s_1+s_2)} - \frac{i(2r_2^\alpha + 2r_1^\alpha + 4s_1k_\gamma^\alpha)}{4(s_1+s_2)}\cdot \\
&\quad\cdot\Big(\frac{-i(2r_1^\beta + 2r_2^\beta + 4s_1k_\gamma^\beta}{4(s_1+s_2)} + ik_\gamma^\beta\Big)\Big)\exp\Big\{\dots\Big\}\Big|_{r_1=r_2=0} \\
&= \Big(\frac{-ig^{\alpha\beta}}{2(s_1+s_2)} + \frac{s_1s_2k_\gamma^\alpha k_\gamma^\beta}{(s_1+s_2)^2}\Big)\exp\Big\{\frac{is_1s_2k_\gamma^2}{s_1+s_2} - i(s_1+s_2)m^2\tfrac{c^2}{\hbar^2}\Big\} \qquad \text{(A.162)}
\end{aligned}$$

In the third term of the derivative in (A.159) there are just the indices α and β exchanged. This does not change the derivative's value. The second term of the derivative in (A.159) can be found by multiplication of (A.162) with $\sum_\alpha \sum_\beta g_{\alpha\beta}$:

$$\frac{\mathrm{d}}{\mathrm{d}r_1}\frac{\mathrm{d}}{\mathrm{d}r_2}\exp\Big\{\dots\Big\}\Big|_{r_1=r_2=0} = \Big(\frac{-2i}{s_1+s_2} + \frac{s_1s_2k_\gamma^2}{(s_1+s_2)^2}\Big)\exp\Big\{\dots\Big\}$$

Thus one gets

$$\begin{aligned}
\text{(A.159)} = \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \, \frac{-q^2}{4\pi^2\hbar^2(s_1+s_2)^2}\Big(&\frac{2s_1s_2(k_\gamma^\alpha k_\gamma^\beta - k_\gamma^2 g^{\alpha\beta})}{(s_1+s_2)^2} + \\
&+ \frac{ig^{\alpha\beta}}{s_1+s_2} - \frac{s_1s_2(1-2)k_\gamma^2 g^{\alpha\beta}}{(s_1+s_2)^2} - (m\tfrac{c}{\hbar})^2 g^{\alpha\beta}\Big) \\
\cdot \exp\Big\{&\frac{is_1s_2k_\gamma^2}{s_1+s_2} - i(s_1+s_2)m^2\tfrac{c^2}{\hbar^2}\Big\}\,. \qquad \text{(A.163)}
\end{aligned}$$

The both green colored terms, which add to zero, have been inserted in addition. Note that the the divergence, which in the original integral (26.5) turned up for $k \to \pm\infty$, now due to the manipulations has been moved to $s_1, s_2 \to 0$. There is no divergence at $s_1, s_2 \to \infty$, because the argument of

the exponential function is imaginary.

We demonstrated the conversion at the example of the first term of (26.5). But we also have emphasized, that the same manipulations have to be done with all of the $N+1$ terms. To achieve a compact notation, we define

$$C_0 \equiv 1 \qquad M_0 \equiv m \qquad s \equiv s_1 + s_2 \ . \tag{A.164}$$

Using (26.6), one gets

$$\sum_{j=0}^{N} C_j = 0 \ . \tag{A.165}$$

Now (26.5) can be written in the form

$$\begin{aligned} F_{\mathrm{P}}^{\alpha\beta} &\overset{(26.5)}{=} \lim_{M_j\to\infty} \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \, \frac{-q^2}{4\pi^2\hbar^2 s^2} \sum_{j=0}^{N} C_j \Big(\\ &\frac{2s_1 s_2 (k_\gamma^\alpha k_\gamma^\beta - k_\gamma^2 g^{\alpha\beta})}{s^2} + g^{\alpha\beta}\Big(\frac{i}{s} + \frac{s_1 s_2 k_\gamma^2}{s^2} - (M_j \tfrac{c}{\hbar})^2\Big)\Big) \\ &\cdot \exp\Big\{\frac{i s_1 s_2 k_\gamma^2}{s} - i s M_j^2 \tfrac{c^2}{\hbar^2}\Big\} \ . \end{aligned} \tag{A.166}$$

Note, that $\lim_{M_j\to\infty}$ refers to $j \geq 1$ only, but not to $M_0 = m$. Now we want to demonstrate, that only the first term in the green bracket survives, and that the integral over the rest is zero. Thus we assert

$$\begin{aligned} &\int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \, \frac{1}{s^2} \sum_{j=0}^{N} C_j \Big(\frac{i}{s} + \frac{s_1 s_2 k_\gamma^2}{s^2} - (M_j \tfrac{c}{\hbar})^2\Big) \\ &\qquad \cdot \exp\Big\{\frac{i s_1 s_2 k_\gamma^2}{s} - i s M_j^2 \tfrac{c^2}{\hbar^2}\Big\} \overset{?}{=} 0 \ . \end{aligned} \tag{A.167}$$

To prove this assertion, we again write this expression with a derivation-parameter r:

$$
\begin{aligned}
(\text{A.167}) &= \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \, \frac{(-i)}{s^3} \, \frac{\mathrm{d}}{\mathrm{d}r} \, \frac{1}{r} \sum_{j=0}^{N} C_j \\
&\qquad \cdot \exp\Big\{ ir\Big(\frac{s_1 s_2 k_\gamma^2}{s} - s M_j^2 \tfrac{c^2}{\hbar^2} \Big) \Big\} \Big|_{r=1} \\
&= \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \, \frac{(-i)}{s^3} \sum_{j=0}^{N} C_j \Big(i \, \frac{s_1 s_2 k_\gamma^2}{s} - i s M_j^2 \tfrac{c^2}{\hbar^2} - 1 \Big) \\
&\qquad \cdot \exp\Big\{ i\Big(\frac{s_1 s_2 k_\gamma^2}{s} - s M_j^2 \tfrac{c^2}{\hbar^2} \Big) \Big\}
\end{aligned}
$$

It is important for the next step of the proof, that the derivative with respect to r may be factored out from the integrals over s_1 and s_2. This is only correct if the integrals converge, which at first sight is not obvious for $s_1, s_2 \to 0$. There could be a divergence, only if *both* integration variables would approach zero at the same time. In that case, the integrand

$$
\begin{aligned}
&\lim_{s_1, s_2 \to 0} \sum_{j=0}^{N} C_j \Big(i \, \frac{s_1 s_2 k_\gamma^2}{(s_1+s_2)^4} - \frac{i}{(s_1+s_2)^2 M_j^2 \frac{c^2}{\hbar^2}} - \frac{1}{(s_1+s_2)^3} \Big) \\
&\qquad \cdot \exp\Big\{ i\Big(\frac{s_1 s_2 k_\gamma^2}{s_1+s_2} - (s_1+s_2) M_j^2 \tfrac{c^2}{\hbar^2} \Big) \Big\} \; = \\
&= \lim_{s_1, s_2 \to 0} \sum_{j=0}^{N} C_j \, \frac{1}{(s_1+s_2)^3} \, \exp\{0\} \; , \qquad (\text{A.168})
\end{aligned}
$$

would become independent of M_j, because the exponential function converges to $\exp\{0\} = 1$, while the term $(s_1 + s_2)^{-3}$ is growing faster than the both others in the same bracket. Therefore the sum over j is zero for $s_1, s_2 \to 0$ because of (A.165), and the integral does converge.

With the substitution $s_i \to \widetilde{s}_i \equiv r s_i$, the expression therefore can be written as follows:

$$(\text{A.167}) = \frac{\mathrm{d}}{\mathrm{d}\,r} \int_0^\infty \mathrm{d}\widetilde{s}_1 \int_0^\infty \mathrm{d}\widetilde{s}_2 \, \frac{(-i)}{\widetilde{s}^3} \sum_{j=0}^{N} C_j$$
$$\cdot \exp\Big\{ i\Big(\frac{\widetilde{s}_1 \widetilde{s}_2 k_\gamma^2}{\widetilde{s}} - \widetilde{s} M_j^2 \tfrac{c^2}{\hbar^2} - \frac{\widetilde{s}}{i}\,\epsilon' \Big) \Big\} \Big|_{r=1} = 0 \tag{A.169}$$

As the integral does not depend on r, the derivative is zero. This proves (A.167), and therefore

$$F_{\mathrm{P}}^{\alpha\beta} \overset{(26.5)}{=} \lim_{M_j \to \infty} (k_\gamma^\alpha k_\gamma^\beta - k_\gamma^2 g^{\alpha\beta})\, \Pi(k_\gamma) \tag{A.170a}$$

$$\Pi(k_\gamma) \equiv \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \, \frac{-q^2 s_1 s_2}{2\pi^2 \hbar^2 s^4} \sum_{j=0}^{N} C_j \exp\Big\{ \frac{i s_1 s_2 k_\gamma^2}{s} - i s M_j^2 \tfrac{c^2}{\hbar^2} \Big\} \,. \tag{A.170b}$$

$\Pi(k_\gamma)$ is called polarization function, and $F_{\mathrm{P}}^{\alpha\beta}$ is called polarization tensor. Now a factor

$$1 = \int_0^\infty \mathrm{d}s \, \delta(s - s_1 - s_2)$$

is inserted into the equation. Due to this factor, from now on s is to be considered an independent variable, and the definition $s \equiv s_1 + s_2$ from (A.164) isn't valid any more. Subsequently, the substitution $s_i \to s_i s$ is applied. The variable s — being independent of the s_i — is not touched by that substitution:

$$\Pi(k_\gamma) = \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \int_0^\infty \mathrm{d}s \, \delta(s - s_1 s - s_2 s) \, \frac{-q^2 s_1 s_2}{2\pi^2 \hbar^2} \sum_{j=0}^{N} C_j$$
$$\cdot \exp\Big\{ i s_1 s_2 s k_\gamma^2 - i s M_j^2 \tfrac{c^2}{\hbar^2} \Big\} \tag{A.171}$$

Note, that the s_j have become dimension-less due to this substitution. In contrast, the dimension of s still is $[s] = \text{length}^2$. Eventually s is factored out of the delta-function:

$$\Pi(k_\gamma) = \int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \, \delta(1-s_1-s_2) \, \frac{-q^2 s_1 s_2}{2\pi^2\hbar^2} \cdot B \cdot V \tag{A.172a}$$

$$V \equiv \begin{cases} 1 & \text{if } s_1 s_2 k_\gamma^2 < m^2c^2/\hbar^2 \\ 0 & \text{if } s_1 s_2 k_\gamma^2 \geq m^2c^2/\hbar^2 \end{cases} \tag{A.172b}$$

$$B \equiv \sum_{j=0}^{N} C_j \int_0^\infty \frac{\mathrm{d}s}{s} \, \exp\left\{ i s_1 s_2 s k_\gamma^2 - i s M_j^2 \tfrac{c^2}{\hbar^2} \right\} . \tag{A.172c}$$

This expression is zero for $s_1 + s_2 > 1$ due to the delta-function. Therefore the upper limit of the integrals over s_1 and s_2 could be changed to 1. Only the integral over s still is running to $+\infty$. Now it becomes visible, that $\Pi(k_\gamma)$ (and therefore also $F_{\mathrm{P}}^{\alpha\beta} = (\text{A.170})$) would diverge at $s \to 0$ without regularization due to the counter-terms "only" logarithmically, but against the first appearance of (26.4) not quadratically.

The function V has been inserted for the following reason: Our computation is valid for the modification of the free photon propagator ($k_\gamma^2 = 0$), as well as for the modification of space-like virtual photons in t- or u-channel scattering ($k_\gamma^2 < 0$), as well as for the modification of time-like virtual photons in s-channel scattering ($k_\gamma^2 > m^2c^2/\hbar^2$). At $k_\gamma^2 \geq 4m^2c^2/\hbar^2$, the photon's energy is sufficient for the creation of two real particles. That opens a competing channel, which extracts probability amplitude from the photon's self-energy graph. The result of this extraction is, that some terms in the formulas, which are real at $k_\gamma^2 < 4m^2c^2/\hbar^2$, become imaginary.

As these imaginary terms (which sometimes mutate to negative arguments of logarithms) don't contribute to the probability amplitudes which we really want to compute, they are blanked out due to the function V. The product $s_1 s_2$ can be maximum 1/4 due to the delta-function. Because of this factor, the boundary in (A.172b) has been set to $m^2c^2/\hbar^2$, but not to $4m^2c^2/\hbar^2$.

We define two dimension-less real parameters

$$0 < \epsilon \in \mathbb{R} \, , \; \epsilon \neq \epsilon' \, , \; [\epsilon] = [A_j] = 1$$

$$A_j \equiv \left(s_1 s_2 \, \frac{k_\gamma^2 \hbar^2}{m^2c^2} - \frac{M_j^2}{m^2} \right) , \tag{A.173}$$

and substitute s by the dimension-less parameters r_j:

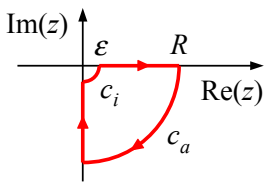
$$s \to r_j = -sA_j \frac{m^2c^2}{\hbar^2}$$

Thus $B =$ (A.172c) can be written as

$$B = \lim_{\epsilon\to 0} \sum_{j=0}^{N} C_j \int_{\epsilon\hbar^2/(mc)^2}^{\infty} \frac{\mathrm{d}s}{s} \exp\Big\{ \underbrace{is_1 s_2 s k_\gamma^2 - isM_j^2 \tfrac{c^2}{\hbar^2}}_{isA_j m^2c^2/\hbar^2} \Big\} =$$

$$= \lim_{R\to\infty} \lim_{\epsilon\to 0} \sum_{j=0}^{N} C_j \int_{-\epsilon A_j}^{R} \frac{\mathrm{d}r_j}{r_j} \exp\{-ir_j\} \; . \tag{A.174}$$

Due to the function V, there is a contribution to $\Pi(k_\gamma)$ only if the lower integration limit is > 0. For the evaluation of B, we compute the contour-integral



$$I \equiv \lim_{R\to\infty} \lim_{\epsilon\to 0} \oint_{\circlearrowleft} \frac{\mathrm{d}z}{z} \exp\{-iz\} = 0 \; , \tag{A.175}$$

over the sketched path in the complex plane. The integral's segment on the real axis is $B =$ (A.174). According to Cauchy's integral theorem[2], the closed contour-integral is zero, because the integrand is free of singularities within the red drawn contour. The integration variable is in polar coordinates (ρ, φ)

$$z = \rho \exp\{i\varphi\} = \rho(\cos\varphi + i \sin\varphi)$$
$$\frac{\mathrm{d}z}{z} = \frac{\mathrm{d}\left(\rho \exp\{i\varphi\}\right)}{\rho \exp\{i\varphi\}} = \mathrm{d} \, \ln(\exp\{i\varphi\}) = i\mathrm{d}\varphi \; .$$

[2] In [37], a short explanation, which is tailored to the needs of physicists, of this important mathematical tool can be found.

In the limit $R \to \infty$ the outer quarter circle c_a does not contribute to the value of I:

$$I_{c_a} = \lim_{R\to\infty} i \int_{2\pi}^{3\pi/4} \mathrm{d}\varphi \, \exp\{-iR\cos\varphi\} \, \exp\{R\sin\varphi\} = 0 \tag{A.176}$$

With the exception of the point $\varphi = 2\pi$, whose measure is zero, the argument of the second exponential function is negative and real. This factor goes to zero exponentially with R, while the first exponential function does oscillate for increasing R between -1 and 1. Now we consider the path integral over the inner quarter circle:

$$I_{c_i} = \lim_{\eta\to 0} i \int_{3\pi/4}^{2\pi} \mathrm{d}\varphi \, \exp\{-i\eta(\cos\varphi + i\sin\varphi)\} = i\varphi\Big|_{3\pi/4}^{2\pi} = i\,\frac{\pi}{4}$$

The exponential function is always 1 for $\eta \to 0$. Because of (A.175) we therefore have

$$\begin{aligned} B &\overset{(A.174)}{=} \lim_{R\to\infty} \lim_{\epsilon\to 0} \sum_{j=0}^{N} C_j \Big(-i\,\frac{\pi}{4} - \int_{-iR}^{+i\epsilon A_j} \frac{\mathrm{d}(ir_j)}{ir_j} \, \exp\{-ir_j\} \Big) \\ &= \lim_{R\to\infty} \lim_{\epsilon\to 0} \sum_{j=0}^{N} C_j \Big(-i\,\frac{\pi}{4} - \int_{+R}^{-\epsilon A_j} \frac{\mathrm{d}r}{r} \, \exp\{-r\} \Big) \, . \end{aligned} \tag{A.177}$$

In the last step, $ir_j \to r$ has been substituted. The integration parameter r is a real number, which does not depend on M_j. As the first term does not depend on M_j, it vanishes because of (A.165). In the second term, we exchange the integration boundaries, take the limit $R \to \infty$, and integrate by parts:

$$
\begin{aligned}
B &= \lim_{\epsilon\to 0} \sum_{j=0}^{N} C_j \Big(\ln(r)\, \exp\{-r\}\Big|_{-\epsilon A_j}^{\infty} - \int_{-\epsilon A_j}^{\infty} \mathrm{d}\ln(r)\, \exp\{-r\}\Big) \\
&= V \lim_{\epsilon\to 0} \sum_{j=0}^{N} C_j \left(0 - \Big(\ln(\epsilon) + \ln(-A_j)\Big)\, \exp\{+\epsilon A_j\} \right. \\
&\quad \left. - \int_{-\epsilon A_j}^{\infty} \mathrm{d}\ln(r)\, \exp\{-r\}\right) . \qquad \text{(A.178)}
\end{aligned}
$$

The case $-A_j \leq 0$ can not be excluded. The function V, which has been defined in (A.172), makes sure that B in this case will not contribute to $\Pi(k_\gamma)$. Thus the logarithm will not explode. The terms in

$$
\sum_{j=0}^{N} C_j \lim_{\epsilon\to 0} \left(-\ln(\epsilon)\, \exp\{+\epsilon A_j\} - \int_{-\epsilon A_j}^{\infty} \mathrm{d}\ln(r)\, \exp\{-r\}\right) = 0
$$

are due to

$$
\lim_{\epsilon\to 0} (-\epsilon A_j) = 0 \quad , \quad \lim_{\epsilon\to 0} \exp\{\epsilon A_j\} = 1
$$

not depending on M_j. Therefore, and because of (A.165), this sum is zero as well. Thus only one term remains:

$$
\begin{aligned}
B &= -\sum_{j=0}^{N} C_j\, \ln(-A_j) \underbrace{\lim_{\epsilon\to 0} \exp\{+\epsilon A_j\}}_{1} \\
&= -\sum_{j=0}^{N} C_j\, \ln\left((-s_1 s_2 k_\gamma^2 + M_j^2 \tfrac{c^2}{\hbar^2})\, \frac{\hbar^2}{m^2 c^2}\right) . \qquad \text{(A.179)}
\end{aligned}
$$

Compare this to (A.172c)! Starting from a logarithmically divergent integral, we have arrived due to Pauli-Villars-regularization at this finite expression. In the long course of the derivation, we have repeatedly used

$$\sum_{j=0}^{N} C_j \overset{(A.165)}{=} 0 \tag{A.180a}$$

$$C_0 \overset{(A.164)}{=} 1 \qquad M_0 \overset{(A.164)}{=} m \ . \tag{A.180b}$$

We never needed to specify the exact number N of counterterms, and nowhere $N > 1$ was needed. One single counterterm is sufficient. Thus for simplicity we now fix

$$N = 1 \qquad C_1 = -1 \qquad M \equiv M_1 \ . \tag{A.180c}$$

Furthermore we choose M sufficiently large, such that

$$M^2 \gg -s_1 s_2 k_\gamma^2 \tfrac{\hbar^2}{c^2} \geq 0 \ .$$

Thus we get

$$B = -\ln\Big(1 - \frac{s_1 s_2 k_\gamma^2 \hbar^2}{m^2 c^2}\Big) + \ln\Big(\frac{M^2}{m^2}\Big) \ . \tag{A.181}$$

This is inserted into the polarization function, and the variable s_2 is eliminated due to integration over the delta function:

$$F_{\mathrm{P}}^{\alpha\beta} \overset{(A.170)}{=} (k_\gamma^\alpha k_\gamma^\beta - k_\gamma^2 g^{\alpha\beta})\, \Pi(k_\gamma) \tag{A.182a}$$

$$\Pi(k_\gamma) \overset{(A.172)}{=} \lim_{M\to\infty} \int_0^1 \mathrm{d}s_1 \, \frac{-q^2 s_1(1-s_1)}{2\pi^2\hbar^2} \cdot V \cdot$$
$$\cdot \Big(-\ln\Big(1 - \frac{s_1(1-s_1)k_\gamma^2\hbar^2}{m^2c^2}\Big) + \ln\Big(\frac{M^2}{m^2}\Big)\Big) \tag{A.182b}$$

$$V = \begin{cases} 1 & \text{if } s_1(1-s_1)k_\gamma^2 < m^2c^2/\hbar^2 \\ 0 & \text{if } s_1(1-s_1)k_\gamma^2 \geq m^2c^2/\hbar^2 \end{cases} \tag{A.182c}$$

A.25 Proof of (27.33) resp. (28.19)

$$\tilde{F}'_{\sigma\tau} \overset{(27.31b)}{=} \mathrm{D}'_\sigma \tilde{W}'_\tau - \mathrm{D}'_\tau \tilde{W}'_\sigma$$

$$\overset{(27.29b)}{=} \mathrm{D}'_\sigma\left(U\tilde{W}_\tau U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\tau U)U^\dagger\right) - \\ - \mathrm{D}'_\tau\left(U\tilde{W}_\sigma U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\sigma U)U^\dagger\right)$$

$$\tilde{F}'_{\sigma\tau} \overset{(27.23)}{=} U\,\mathrm{D}_\sigma\,\tilde{W}_\tau U^\dagger - U\,\mathrm{D}_\tau\,\tilde{W}_\sigma U^\dagger + \\ + \left(\mathrm{d}_\sigma + \frac{i}{\hbar}\,g\Big[U\tilde{W}_\sigma U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\sigma U)U^\dagger\Big]\right)\frac{i\hbar}{g}(\mathrm{d}_\tau U)U^\dagger + \\ - \left(\mathrm{d}_\tau + \frac{i}{\hbar}\,g\Big[U\tilde{W}_\tau U^\dagger + \frac{i\hbar}{g}(\mathrm{d}_\tau U)U^\dagger\Big]\right)\frac{i\hbar}{g}(\mathrm{d}_\sigma U)U^\dagger$$

Using $\mathrm{d}_\sigma\mathrm{d}_\tau U = \mathrm{d}_\tau\mathrm{d}_\sigma U$, one gets

$$\tilde{F}'_{\sigma\tau} \overset{(27.31b)}{=} U\tilde{F}_{\sigma\tau}U^\dagger + U\tilde{W}_\tau\mathrm{d}_\sigma U^\dagger - U\tilde{W}_\sigma\mathrm{d}_\tau U^\dagger + \\ + \frac{i\hbar}{g}(\mathrm{d}_\tau U)\mathrm{d}_\sigma U^\dagger - U\tilde{W}_\sigma\underbrace{U^\dagger(\mathrm{d}_\tau U)U^\dagger}_{-\mathrm{d}_\tau U^\dagger} - \frac{i\hbar}{g}(\mathrm{d}_\sigma U)\underbrace{U^\dagger(\mathrm{d}_\tau U)U^\dagger}_{-\mathrm{d}_\tau U^\dagger} - \\ - \frac{i\hbar}{g}(\mathrm{d}_\sigma U)\mathrm{d}_\tau U^\dagger + U\tilde{W}_\tau\underbrace{U^\dagger(\mathrm{d}_\sigma U)U^\dagger}_{-\mathrm{d}_\sigma U^\dagger} + \frac{i\hbar}{g}(\mathrm{d}_\tau U)\underbrace{U^\dagger(\mathrm{d}_\sigma U)U^\dagger}_{-\mathrm{d}_\sigma U^\dagger} \\ = U\tilde{F}_{\sigma\tau}U^\dagger\ . \tag{A.183}$$

Here the relation

$$\mathrm{d}_\sigma UU^\dagger = 0 = (\mathrm{d}_\sigma U)U^\dagger + U\mathrm{d}_\sigma U^\dagger \quad \Longrightarrow \\ \Longrightarrow \quad U^\dagger(\mathrm{d}_\sigma U)U^\dagger = -\mathrm{d}_\sigma U^\dagger \tag{A.184}$$

was used, and mutually compensating terms have been marked by colors.

A.26 Derivation of (19.45)

We write the S-matrix in the form

$$S_{\boldsymbol{f}_1\ldots\boldsymbol{f}_n\boldsymbol{k}_1\ldots\boldsymbol{k}_m} \overset{(19.43a)}{=} \langle t_a \boldsymbol{f}_1 \ldots \boldsymbol{f}_n \,|\, t_e \boldsymbol{k}_1 \ldots \boldsymbol{k}_m \rangle$$
$$= \langle 0 |\, a_{\boldsymbol{f}_1 a} \ldots a_{\boldsymbol{f}_n a} a^\dagger_{\boldsymbol{k}_1 e} \ldots a^\dagger_{\boldsymbol{k}_m e} \,| 0 \rangle \qquad \text{(A.185)}$$

with creation- and annihilation-operators, which are related to the times t_e and t_a. The Fourier-operators, which are (and will remain) time-independent, here get a time-index for the following reason: We are looking for a formulation of the S-matrix, which is not based on the Fourier-operators, but on the field-operators $\psi_{(W)}$ of the interacting particles[3]. These field-operators are solutions of equation (19.4). We do not know those solutions, and they can not be expanded in a series with respect to the Fourier-operators. But that expansion is possible for the field-operators in the interaction picture, see (19.11). Therefore we try to set up a relation between the Fourier-operators and the field-operators $\psi_{(W)}$ via detour over the field-operators

$$\psi(x) \overset{(19.11)}{=} \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{k}}\Omega}} \Big(a_{\boldsymbol{k}} \exp\{-ikx\} + a^\dagger_{\boldsymbol{k}} \exp\{+ikx\} \Big) \qquad \text{(A.186)}$$

in the interaction picture. We assert, that these field-operators can be solved with respect to the Fourier-operators in the form

$$a_{\boldsymbol{k}} = ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int_\Omega \mathrm{d}^3x \exp\{ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \psi(x) \qquad \text{(A.187a)}$$

$$a^\dagger_{\boldsymbol{k}} = -ic \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}} \int_\Omega \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \psi(x) \qquad \text{(A.187b)}$$

with

$$F \overset{\leftrightarrow}{\mathrm{d}_\mu} G \equiv F \mathrm{d}_\mu G - (\mathrm{d}_\mu F) G \,. \qquad \text{(A.188)}$$

As $\psi^\dagger = \psi$ for the uncharged Klein-Gordon field, (A.187b) is the adjoint equation of (A.187a). Therefore (A.187b) is correct, if (A.187a) is correct.

[3] Remember the re-naming (19.21) of indices!

(A.187a) can be checked due to insertion of (A.186):

$$\begin{aligned} a_{\boldsymbol{k}} &= ic\sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}\Omega}}\int\limits_{\Omega}\mathrm{d}^3x\exp\{ikx\}\sum_{\boldsymbol{f}}\sqrt{\frac{1}{2\hbar\omega_{\boldsymbol{f}}\Omega}}\cdot \\ &\quad\cdot\Big(-if^0a_{\boldsymbol{f}}\exp\{-ifx\}+if^0a^\dagger_{\boldsymbol{f}}\exp\{+ifx\} \\ &\quad -ik^0(a_{\boldsymbol{f}}\exp\{-ifx\}+a^\dagger_{\boldsymbol{f}}\exp\{+ifx\})\Big) \\ &= -\frac{c}{2\Omega}\int\limits_{\Omega}\mathrm{d}^3x\sum_{\boldsymbol{f}}\sqrt{\frac{1}{\omega_{\boldsymbol{k}}\,\omega_{\boldsymbol{f}}}}\Big(-(f^0+k^0)a_{\boldsymbol{f}}\exp\{i(k-f)x\}+ \\ &\quad +(f^0-k^0)a^\dagger_{\boldsymbol{f}}\exp\{+i(k+f)x\}\Big) \end{aligned} \tag{A.189}$$

Because of

$$\frac{1}{\Omega}\int\limits_{\Omega}\mathrm{d}^3x\,\exp\{-i(\boldsymbol{k}-\boldsymbol{f})\boldsymbol{x}\}\overset{(7.12)}{=}\delta_{\boldsymbol{kf}} \tag{A.190}$$

we get

$$\begin{aligned} a_{\boldsymbol{k}} &= \frac{c}{2}\sum_{\boldsymbol{f}}\sqrt{\frac{1}{\omega_{\boldsymbol{k}}\,\omega_{\boldsymbol{f}}}}\Big((f^0+k^0)a_{\boldsymbol{f}}\exp\{i(k^0-f^0)x_0\}\delta_{\boldsymbol{kf}} \\ &\quad -(f^0-k^0)a^\dagger_{\boldsymbol{f}}\exp\{+i(k^0+f^0)x_0\}\delta_{\boldsymbol{k},-\boldsymbol{f}}\Big)\,. \end{aligned} \tag{A.191}$$

The second line is zero because of (7.18), while the first term is identical to the equation's left side because of $k^0=\omega_{\boldsymbol{k}}/c$. This proves (A.187a).

Using the abbreviation

$$N_{\boldsymbol{k}}\equiv\sqrt{2\hbar\omega_{\boldsymbol{k}}\Omega}\,, \tag{A.192}$$

the creation-operators of in- and outgoing particles are

$$a^\dagger_{\boldsymbol{k}e} = -\frac{i\hbar c}{N_{\boldsymbol{k}}}\int\limits_\Omega \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \psi(t_e, \boldsymbol{x}) \tag{A.193a}$$

$$a^\dagger_{\boldsymbol{k}a} = -\frac{i\hbar c}{N_{\boldsymbol{k}}}\int\limits_\Omega \mathrm{d}^3x \exp\{-ikx\} \overset{\leftrightarrow}{\mathrm{d}_0} \psi(t_a, \boldsymbol{x}) \ . \tag{A.193b}$$

The adjoints of these operators are the annihilation-operators $a_{\boldsymbol{k}e}$ of an incoming and $a_{\boldsymbol{k}a}$ of an outgoing particle. These equations are valid at time t_e resp. t_a, but not during the interaction time $t_e < t < t_a$. Only at times $t \leq t_e$ and $t \geq t_a$, the field-operators $\psi_{(W)}$ of the particles are identical to the field-operators ψ in the interaction picture, and can be expanded with respect to the Fourier-operators, as done in (A.186).

Using the time-order operator (15.44), we try to write the S-Matrix as follows:

$$S_{\boldsymbol{f}\boldsymbol{k}} = \langle 0|\, a_{\boldsymbol{f}a}\, a^\dagger_{\boldsymbol{k}e}\, |0\rangle \overset{??}{=} \langle 0|\, T(a_{\boldsymbol{f}a} - a_{\boldsymbol{f}e})(a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a})\, |0\rangle \tag{A.194}$$

To check the correctness of this equation, we compute

$$\begin{aligned}
&\langle 0|\, T(a_{\boldsymbol{f}a} - a_{\boldsymbol{f}e})(a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a})\, |0\rangle \ = \\
&= \langle 0|\, T(a_{\boldsymbol{f}a}a^\dagger_{\boldsymbol{k}e} - a_{\boldsymbol{f}a}a^\dagger_{\boldsymbol{k}a} - a_{\boldsymbol{f}e}a^\dagger_{\boldsymbol{k}e} + a_{\boldsymbol{f}e}a^\dagger_{\boldsymbol{k}a})\, |0\rangle \\
&= \langle 0|\, a_{\boldsymbol{f}a}a^\dagger_{\boldsymbol{k}e}\, |0\rangle - \langle 0|\, a_{\boldsymbol{f}a}a^\dagger_{\boldsymbol{k}a}\, |0\rangle - \langle 0|\, a_{\boldsymbol{f}e}a^\dagger_{\boldsymbol{k}e}\, |0\rangle + \langle 0|\, a^\dagger_{\boldsymbol{k}a}a_{\boldsymbol{f}e}\, |0\rangle \ .
\end{aligned}$$

The last three terms should be zero, as is obvious by comparison with the left side of (A.194). If $\boldsymbol{f}_i \neq \boldsymbol{k}_j$ for all $i = 1 \ldots n$ and for all $j = 1 \ldots m$, then these terms indeed are zero because of $a|0\rangle = \langle 0|a^\dagger = 0$. But in case of one ore several $\boldsymbol{f}_i = \boldsymbol{k}_j$ (which is to be expected, if only self-interactions take place), only the last term is zero. The second term then is the probability amplitude, that one or several outgoing particles with wavenumbers $\boldsymbol{k}_j$ are created and annihilated again. The third term then is the probability amplitude, that one or several incident particles are created and annihilated again. These two probability amplitudes are different from zero, though they should be zero. Therefore (A.194) is not yet correct and needs modification.

The deficiencies of (A.194) can be remedied by the definition of time-dependent vacua, onto which the different creation- and annihilation-operators

are acting differently:

$$\begin{aligned} |0_e\rangle &\equiv |0\rangle\Big|_{t\le t_e} & \langle 0_a| &\equiv \langle 0|\Big|_{t\ge t_a} \\ a^\dagger_{\boldsymbol{k}e}|0_e\rangle &\equiv |\boldsymbol{k}\rangle & \langle 0_a|a_{\boldsymbol{k}e} &\equiv 0 \\ a^\dagger_{\boldsymbol{k}a}|0_e\rangle &\equiv 0 & \langle 0_a|a_{\boldsymbol{k}a} &\equiv \langle \boldsymbol{k}| \end{aligned} \tag{A.195}$$

Using these definitions,

$$\begin{aligned} S_{\boldsymbol{f}\boldsymbol{k}} &= \langle 0_a|\, a_{\boldsymbol{f}a}\, a^\dagger_{\boldsymbol{k}e}\, |0_e\rangle = \langle 0_a|\, T(a_{\boldsymbol{f}a} - a_{\boldsymbol{f}e})(a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a})\, |0_e\rangle \\ S_{\boldsymbol{f}\boldsymbol{k}} &= \langle 0_a|\, a_{\boldsymbol{f}_1 a} \dots a_{\boldsymbol{f}_n a}\, a^\dagger_{\boldsymbol{k}_1 e} \dots a^\dagger_{\boldsymbol{k}_m e}\, |0_e\rangle = \\ &= \langle 0_a|\, T(a_{\boldsymbol{f}_1 a} \dots a_{\boldsymbol{f}_n a} - a_{\boldsymbol{f}_1 e} \dots a_{\boldsymbol{f}_m e}) \cdot \\ &\qquad \cdot (a^\dagger_{\boldsymbol{f}_1 e} \dots a^\dagger_{\boldsymbol{f}_m e} - a^\dagger_{\boldsymbol{f}_1 a} \dots a^\dagger_{\boldsymbol{f}_n a})\, |0_e\rangle \end{aligned} \tag{A.196}$$

is a correct equation.

The differences of the Fourier-operators are inserted into the S-matrix, because this expression can be written as a function of $\psi_{(W)}(x)$:

$$a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a} = + \int_{t_e}^{t_a} \mathrm{d}x^0\, \mathrm{d}_0\, \frac{i\hbar c}{N_{\boldsymbol{k}}} \int_\Omega \mathrm{d}^3x \exp\{-ikx\} \overleftrightarrow{\mathrm{d}_0}\, \psi_{(W)}(x)$$

$\psi_{(W)}(t, \boldsymbol{x})$ is known only for $t \le t_e$ and for $t \ge t_a$. At those times $\psi_{(W)}(t, \boldsymbol{x}) = \psi(t, \boldsymbol{x})$. But that knowledge is sufficient for the formulation of this artful integral over $\psi_{(W)}(x)$ in the time interval $t_e \le t \le t_a$. We apply another transformation:

$$a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a} = \frac{i\hbar c}{N_{\boldsymbol{k}}} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4x \, \mathrm{d}_0 \Big(\exp\{-ikx\} \mathrm{d}_0 \psi_{(W)}(x) + \\ + ik^0 \exp\{-ikx\} \psi_{(W)}(x) \Big)$$

$$= \frac{i\hbar c}{N_{\boldsymbol{k}}} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4x \Big(\\ - ik^0 \exp\{-ikx\} \mathrm{d}_0 \psi_{(W)}(x) + \exp\{-ikx\} \mathrm{d}_0 \mathrm{d}_0 \psi_{(W)}(x) + \\ + (k^0)^2 \exp\{-ikx\} \psi_{(W)}(x) + ik^0 \exp\{-ikx\} \mathrm{d}_0 \psi_{(W)}(x) \Big) \qquad \text{(A.197)}$$

Two terms compensate mutually. Furthermore we consider

$$k^0 k_0 = k^j k_j + \frac{m^2 c^2}{\hbar^2}$$

$$\Longrightarrow \quad (k^0)^2 \exp\{-ikx\} = \Big(\frac{m^2 c^2}{\hbar^2} - \boldsymbol{\nabla}^2 \Big) \exp\{-ikx\} \; . \qquad \text{(A.198)}$$

The amplitude of $\psi_{(W)}(x)$ certainly is zero at the boundaries of the normalization volume Ω during the integration time interval $t_e \leq t \leq t_a$. Therefore one term in the third summand of (A.197) can be reshaped by integrating it two-times partially:

$$- \int_{\Omega} \mathrm{d}^3x \, \psi_{(W)}(x) \boldsymbol{\nabla}^2 \exp\{-ikx\} \; =$$

$$= \underbrace{-\psi_{(W)}(x) \boldsymbol{\nabla} \exp\{-ikx\} \Big|_{\Omega}}_{0} + \int_{\Omega} \mathrm{d}^3x \Big(\boldsymbol{\nabla} \psi_{(W)}(x) \Big) \boldsymbol{\nabla} \exp\{-ikx\}$$

$$= \underbrace{\Big(\boldsymbol{\nabla} \psi_{(W)}(x) \Big) \exp\{-ikx\} \Big|_{\Omega}}_{0} - \int_{\Omega} \mathrm{d}^3x \Big(\boldsymbol{\nabla}^2 \psi_{(W)}(x) \Big) \exp\{-ikx\}$$

The mark $\Big|_{\Omega}$ is indicating, that the value at the boundary of the normalization volume shall be inserted. Thus one finds

$$a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a} = \frac{i\hbar c}{N_{\boldsymbol{k}}} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4x \Big(\exp\{-ikx\} \mathrm{d}_0 \mathrm{d}_0 \psi_{(W)}(x)$$

$$- \exp\{-ikx\} \boldsymbol{\nabla}^2 \psi_{(W)}(x) + \exp\{-ikx\} \frac{m^2c^2}{\hbar^2} \psi_{(W)}(x) \Big)$$

$$= \frac{-i\hbar c}{N_{\boldsymbol{k}}} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4x \, \exp\{-ikx\} \Big(- \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\mathrm{d}}{\mathrm{d}x_\mu} - \frac{m^2c^2}{\hbar^2} \Big) \psi_{(W)}(x) \; .$$

Because of $-(a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a})^\dagger \overset{\boldsymbol{k}\to\boldsymbol{f}}{=} a_{\boldsymbol{f}a} - a_{\boldsymbol{f}e}$, one finds the difference $a_{\boldsymbol{f}a} - a_{\boldsymbol{f}e}$ by taking the negative adjoint of $a^\dagger_{\boldsymbol{k}e} - a^\dagger_{\boldsymbol{k}a}$, and replacing everywhere $\boldsymbol{k}$ by $\boldsymbol{f}$:

$$a_{\boldsymbol{f}a} - a_{\boldsymbol{f}e} = \frac{-i\hbar c}{N_{\boldsymbol{f}}} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4y \, \exp\{+ify\} \cdot$$

$$\cdot \Big(- \frac{\mathrm{d}}{\mathrm{d}y^\mu} \frac{\mathrm{d}}{\mathrm{d}y_\mu} - \frac{m^2c^2}{\hbar^2} \Big) \psi_{(W)}(y) \qquad \text{(A.199)}$$

Thus the matrix element becomes

$$S_{\boldsymbol{f}\boldsymbol{k}} = \langle \boldsymbol{f} | \, S \, | \boldsymbol{k} \rangle \overset{\text{(A.196)}}{=}$$

$$= \frac{(-i\hbar c)}{N_{\boldsymbol{k}}} \frac{(-i\hbar c)}{N_{\boldsymbol{f}}} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4y \, \exp\{+ify\} \int_{t_e}^{t_a} \int_{\Omega} \mathrm{d}^4x \, \exp\{-ikx\}$$

$$\cdot \Big(- \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\mathrm{d}}{\mathrm{d}x_\mu} - \frac{m^2c^2}{\hbar^2} \Big) \Big(- \frac{\mathrm{d}}{\mathrm{d}y^\mu} \frac{\mathrm{d}}{\mathrm{d}y_\mu} - \frac{m^2c^2}{\hbar^2} \Big) \, G(y,x)$$

$$\text{with} \quad G(y,x) \equiv \langle 0 | \, T \psi_{(W)}(y) \, \psi_{(W)}(x) \, | 0 \rangle \; . \qquad \text{(A.200)}$$

The function $G(y,x) \overset{\text{(A.196)}}{=} G(y_1, \ldots, y_n, x_1, \ldots, x_m)$ is called a multi-point Greens-function. If $G(y,x)$ would be a solution of the free Klein-Gordon-equation, then this equation would be zero. But $G(y,x)$ is a solution of the inhomogeneous equation (12.4), and therefore (A.200) is different from zero. The essential trick now is, not to apply the differential operators immediately onto $G(y,x)$, but instead first to perform a Fourier-transformation. That

results into

$$\Big(-\frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\mathrm{d}}{\mathrm{d}x_\mu}-\frac{m^2c^2}{\hbar^2}\Big)\Big(-\frac{\mathrm{d}}{\mathrm{d}y^\mu}\frac{\mathrm{d}}{\mathrm{d}y_\mu}-\frac{m^2c^2}{\hbar^2}\Big)\,G(y,x)\;=$$

$$\overset{(7.15)}{=}\Big(-\frac{\mathrm{d}}{\mathrm{d}x^\mu}\frac{\mathrm{d}}{\mathrm{d}x_\mu}-\frac{m^2c^2}{\hbar^2}\Big)\Big(-\frac{\mathrm{d}}{\mathrm{d}y^\mu}\frac{\mathrm{d}}{\mathrm{d}y_\mu}-\frac{m^2c^2}{\hbar^2}\Big)$$

$$\cdot\,\frac{1}{\Omega}\sum_{\boldsymbol{k}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k^0}{2\pi}\frac{1}{\Omega}\sum_{\boldsymbol{f}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}f^0}{2\pi}\widetilde{G}(k,f)\exp\{-i(kx+fy)\}\;=$$

$$=\Big(+k^2-\frac{m^2c^2}{\hbar^2}\Big)\Big(+f^2-\frac{m^2c^2}{\hbar^2}\Big)\,G(y,x)$$

$$\overset{(12.7)}{=}\frac{i}{\hbar c}\,\widetilde{G}^{-1}(k)\,\frac{i}{\hbar c}\,\widetilde{G}^{-1}(f)\,G(y,x)\;\neq\;0\;. \tag{A.201}$$

Now we display explicitly again, that $\boldsymbol{k}$ and $\boldsymbol{f}$ are products, and insert $N_{\boldsymbol{k}} =$ (A.192):

$$S_{\boldsymbol{f}_1\ldots\boldsymbol{f}_n\boldsymbol{k}_1\ldots\boldsymbol{k}_m}=\langle\boldsymbol{f}_1\ldots\boldsymbol{f}_n|\,S\,|\boldsymbol{k}_1\ldots\boldsymbol{k}_m\rangle\;=$$

$$=\prod_{j=1}^{n}\frac{\widetilde{G}^{-1}(f_j)}{\sqrt{2\hbar\omega_{\boldsymbol{f}_j}\Omega}}\int_{t_e}^{t_a}\int_{\Omega}\mathrm{d}^4y_j\,\exp\{+if_jy_j\}\cdot$$

$$\cdot\prod_{l=1}^{m}\frac{\widetilde{G}^{-1}(k_l)}{\sqrt{2\hbar\omega_{\boldsymbol{k}_l}\Omega}}\int_{t_e}^{t_a}\int_{\Omega}\mathrm{d}^4x_l\,\exp\{-ik_lx_l\}\cdot$$

$$\cdot\,\langle0|\,T\psi_{(W)}(y_1)\ldots\psi_{(W)}(y_n)\,\psi_{(W)}(x_1)\ldots\psi_{(W)}(x_m)\,|0\rangle \tag{A.202}$$

Note, that the integrals over x and y are including the matrix element in the last line. Due to the time-order operator T, the operators contained in $\psi_{(W)}(x_j)$ in case of $y^0 > x^0$ are acting towards the right side. The creation-operators $a^\dagger_{\boldsymbol{k}_j}$ create particles $|\,\rangle$, while the annihilation-operators $a_{\boldsymbol{k}_j}$ result into terms of value zero. The operators $a_{\boldsymbol{f}_i}$ contained within $\psi_{(W)}(y_i)$ are acting as creation-operators towards the left side, and create particles $\langle\,|$. The operators $a^\dagger_{\boldsymbol{f}_i}$ are acting as annihilation-operators towards the left side, resulting into terms of value zero. For $x^0 < y^0$, the field-operators are

permuted. In this case, the operators $a^\dagger_{\boldsymbol{f}_i}$ are creating incoming particles $|\,\rangle$, and the operators $a_{\boldsymbol{k}_j}$ are creating outgoing particles $\langle\,|$. In any case, incoming particles are created in the state $|\,\rangle$, and outgoing particles are created in the state $\langle\,|$. Thus the discrimination between $|0_e\rangle$ and $|0_a\rangle$ isn't needed any more.

A.27 Some spinor products

We want to compute products of the spinors

$$
{}^1u^{\boldsymbol{k}} = N\begin{pmatrix} A_+ \\ B \\ A_- \\ -B \end{pmatrix}\ ,\ {}^2u^{\boldsymbol{k}} = N\begin{pmatrix} B^* \\ A_- \\ -B^* \\ A_+ \end{pmatrix}\ ,\ {}^1v^{\boldsymbol{k}} = N\begin{pmatrix} A_+ \\ B \\ -A_- \\ B \end{pmatrix}
$$

$$
{}^2v^{\boldsymbol{k}} = N\begin{pmatrix} B^* \\ A_- \\ B^* \\ -A_+ \end{pmatrix}\ ,\quad \begin{aligned} {}^1\bar{u}^{\boldsymbol{k}} &= N\begin{pmatrix} A_- & -B^* & A_+ & B^* \end{pmatrix} \\ {}^2\bar{u}^{\boldsymbol{k}} &= N\begin{pmatrix} -B & A_+ & B & A_- \end{pmatrix} \\ {}^1\bar{v}^{\boldsymbol{k}} &= N\begin{pmatrix} -A_- & B^* & A_+ & B^* \end{pmatrix} \\ {}^2\bar{v}^{\boldsymbol{k}} &= N\begin{pmatrix} B & -A_+ & B & A_- \end{pmatrix} \end{aligned} \tag{A.203a}
$$

According to (A.65), the components are

$$
N \equiv \sqrt{\frac{1}{2(\hbar\omega_{\boldsymbol{k}} + mc^2)}} \tag{A.203b}
$$

$$
A_+ \equiv \hbar\omega_{\boldsymbol{k}} + mc^2 + c\hbar k_3 \tag{A.203c}
$$

$$
B \equiv c\hbar k_1 + ic\hbar k_2 \tag{A.203d}
$$

$$
A_- \equiv \hbar\omega_{\boldsymbol{k}} + mc^2 - c\hbar k_3\ . \tag{A.203e}
$$

Upfront we compute the products $X\gamma^\rho Y$ with

$$
X \equiv \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix}\ ,\ Y \equiv \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}\ ,\ \gamma^\rho = (8.15)\ . \tag{A.204a}
$$

$$X\gamma^0 Y = X_3Y_1 + X_4Y_2 + X_1Y_3 + X_2Y_4 \tag{A.204b}$$
$$X\gamma^1 Y = -X_4Y_1 - X_3Y_2 + X_2Y_3 + X_1Y_4 \tag{A.204c}$$
$$X\gamma^2 Y = i(-X_4Y_1 + X_3Y_2 + X_2Y_3 - X_1Y_4) \tag{A.204d}$$
$$X\gamma^3 Y = -X_3Y_1 + X_4Y_2 + X_1Y_3 - X_2Y_4 \tag{A.204e}$$

Combining (A.203) and (A.204), we get

$$\begin{aligned}
{}^1\bar{u}^{\boldsymbol{k}}\gamma^0\,{}^1u^{\boldsymbol{k}} &= N^2(A_+^2 + |B|^2 + A_-^2 + |B|^2) = 2\hbar\omega_{\boldsymbol{k}} = 2c\hbar k^0\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^1\,{}^1u^{\boldsymbol{k}} &= N^2(-B^*A_+ - A_+B - B^*A_- - A_-B) = 2c\hbar k^1\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^2\,{}^1u^{\boldsymbol{k}} &= iN^2(-B^*A_+ + A_+B - B^*A_- + A_-B) = 2c\hbar k^2\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^3\,{}^1u^{\boldsymbol{k}} &= N^2(-A_+^2 + |B|^2 + A_-^2 - |B|^2) = 2c\hbar k^3\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^0\,{}^2u^{\boldsymbol{k}} &= N^2(A_+B^* + B^*A_- - A_-B^* - B^*A_+) = 0\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^1\,{}^2u^{\boldsymbol{k}} &= N^2(-B^*B^* - A_+A_- + B^*B^* + A_-A_+) = 0\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^2\,{}^2u^{\boldsymbol{k}} &= iN^2(-B^*B^* + A_+A_- + B^*B^* - A_-A_+) = 0\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^3\,{}^2u^{\boldsymbol{k}} &= N^2(-A_+B^* + B^*A_- - A_-B^* + B^*A_+) = 0\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^0\,{}^1v^{\boldsymbol{k}} &= N^2(A_+A_+ + B^*B - A_-A_- - B^*B)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^1\,{}^1v^{\boldsymbol{k}} &= N^2(-B^*A_+ - A_+B + B^*A_- + A_-B)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^2\,{}^1v^{\boldsymbol{k}} &= iN^2(-B^*A_+ + A_+B + B^*A_- - A_-B)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^3\,{}^1v^{\boldsymbol{k}} &= N^2(-A_+A_+ + B^*B - A_-A_- + B^*B)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^0\,{}^2v^{\boldsymbol{k}} &= N^2(A_+B^* + B^*A_- + A_-B^* + B^*A_+)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^1\,{}^2v^{\boldsymbol{k}} &= N^2(-B^*B^* - A_+A_- - B^*B^* - A_-A_+)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^2\,{}^2v^{\boldsymbol{k}} &= iN^2(-B^*B^* + A_+A_- - B^*B^* + A_-A_+)\\
{}^1\bar{u}^{\boldsymbol{k}}\gamma^3\,{}^2v^{\boldsymbol{k}} &= N^2(-A_+B^* + B^*A_- + A_-B^* - B^*A_+)\\
{}^2\bar{u}^{\boldsymbol{k}}\gamma^0\,{}^1u^{\boldsymbol{k}} &= N^2(BA_+ + A_-B - BA_- - A_+B) = 0\\
{}^2\bar{u}^{\boldsymbol{k}}\gamma^1\,{}^1u^{\boldsymbol{k}} &= N^2(-A_-A_+ - BB + A_+A_- + BB) = 0\\
{}^2\bar{u}^{\boldsymbol{k}}\gamma^2\,{}^1u^{\boldsymbol{k}} &= iN^2(-A_-A_+ + BB + A_+A_- - BB) = 0\\
{}^2\bar{u}^{\boldsymbol{k}}\gamma^3\,{}^1u^{\boldsymbol{k}} &= N^2(-BA_+ + A_-B - BA_- + A_+B) = 0
\end{aligned}$$

$$
\begin{aligned}
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{0}\,{}^{2}u^{\boldsymbol{k}} &= N^2(|B|^2 + A_-^2 + |B|^2 + A_+^2) = 2\hbar\omega_{\boldsymbol{k}} = 2c\hbar k^0\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{1}\,{}^{2}u^{\boldsymbol{k}} &= N^2(-A_-B^* - BA_- - A_+B^* - BA_+) = 2c\hbar k^1\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{2}\,{}^{2}u^{\boldsymbol{k}} &= iN^2(-A_-B^* + BA_- - A_+B^* + BA_+) = 2c\hbar k^2\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{3}\,{}^{2}u^{\boldsymbol{k}} &= N^2(-|B|^2 + A_-^2 + |B|^2 - A_+^2) = 2c\hbar k^3\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{0}\,{}^{1}v^{\boldsymbol{k}} &= N^2(BA_+ + A_-B + BA_- + A_+B)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{1}\,{}^{1}v^{\boldsymbol{k}} &= N^2(-A_-A_+ - BB - A_+A_- - BB)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{2}\,{}^{1}v^{\boldsymbol{k}} &= iN^2(-A_-A_+ + BB - A_+A_- + BB)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{3}\,{}^{1}v^{\boldsymbol{k}} &= N^2(-BA_+ + A_-B + BA_- - A_+B)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{0}\,{}^{2}v^{\boldsymbol{k}} &= N^2(BB^* + A_-A_- - BB^* - A_+A_+)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{1}\,{}^{2}v^{\boldsymbol{k}} &= N^2(-A_-B^* - BA_- + A_-B^* - BA_+)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{2}\,{}^{2}v^{\boldsymbol{k}} &= iN^2(-A_-B^* + BA_- + A_+B^* - BA_+)\\
{}^{2}\bar{u}^{\boldsymbol{k}}\gamma^{3}\,{}^{2}v^{\boldsymbol{k}} &= N^2(-BB^* + A_-A_- - BB^* + A_+A_+)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{1}u^{\boldsymbol{k}} &= N^2(A_+A_+ + B^*B - A_-A_- - B^*B)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{1}u^{\boldsymbol{k}} &= N^2(-B^*A_+ - A_+B + B^*A_- + A_-B)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{1}u^{\boldsymbol{k}} &= iN^2(-B^*A_+ + A_+B + B^*A_- - A_-B)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{1}u^{\boldsymbol{k}} &= N^2(-A_+A_+ + B^*B - A_-A_- + B^*B)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{2}u^{\boldsymbol{k}} &= N^2(A_+B^* + B^*A_- + A_-B^* + B^*A_+)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{2}u^{\boldsymbol{k}} &= N^2(-B^*B^* - A_+A_- - B^*B^* - A_-A_+)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{2}u^{\boldsymbol{k}} &= iN^2(-B^*B^* + A_+A_- - B^*B^* + A_-A_+)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{2}u^{\boldsymbol{k}} &= N^2(-A_+B^* + B^*A_- + A_-B^* - B^*A_+)\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{1}v^{\boldsymbol{k}} &= N^2(A_+^2 + |B|^2 + A_-^2 + |B|^2) = 2\hbar\omega_{\boldsymbol{k}} = 2c\hbar k^0\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{1}v^{\boldsymbol{k}} &= N^2(-B^*A_+ - A_+B - B^*A_- - A_-B) = 2c\hbar k^1\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{1}v^{\boldsymbol{k}} &= iN^2(-B^*A_+ + A_+B - B^*A_- + A_-B) = 2c\hbar k^2\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{1}v^{\boldsymbol{k}} &= N^2(-A_+^2 + |B|^2 + A_-^2 - |B|^2) = 2c\hbar k^3\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{2}v^{\boldsymbol{k}} &= N^2(A_+B^* + B^*A_- - A_-B^* - B^*A_+) = 0\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{2}v^{\boldsymbol{k}} &= N^2(-B^*B^* - A_+A_- + B^*B^* + A_-A_+) = 0
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{2}v^{\boldsymbol{k}} &= iN^2(-B^*B^* + A_+A_- + B^*B^* - A_-A_+) = 0\\
{}^{1}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{2}v^{\boldsymbol{k}} &= N^2(-A_+B^* + B^*A_- - A_-B^* + B^*A_+) = 0\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{1}u^{\boldsymbol{k}} &= N^2(BA_+ + A_-B + BA_- + A_+B)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{1}u^{\boldsymbol{k}} &= N^2(-A_-A_+ - BB - A_+A_- - BB)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{1}u^{\boldsymbol{k}} &= iN^2(-A_-A_+ + BB - A_+A_- + BB)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{1}u^{\boldsymbol{k}} &= N^2(-BA_+ + A_-B + BA_- - A_+B)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{2}u^{\boldsymbol{k}} &= N^2(BB^* + A_-A_- - BB^* - A_+A_+)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{2}u^{\boldsymbol{k}} &= N^2(-A_-B^* - BA_- + A_+B^* + BA_+)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{2}u^{\boldsymbol{k}} &= iN^2(-A_-B^* + BA_- + A_+B^* - BA_+)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{2}u^{\boldsymbol{k}} &= N^2(-BB^* + A_-A_- - BB^* + A_+A_+)\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{1}v^{\boldsymbol{k}} &= N^2(BA_+ + A_-B - BA_- - A_+B) = 0\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{1}v^{\boldsymbol{k}} &= N^2(-A_-A_+ - BB + A_+A_- + BB) = 0\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{1}v^{\boldsymbol{k}} &= iN^2(-A_-A_+ + BB + A_+A_- - BB) = 0\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{1}v^{\boldsymbol{k}} &= N^2(-BA_+ + A_-B - BA_- + A_+B) = 0\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{0}\,{}^{2}v^{\boldsymbol{k}} &= N^2(|B|^2 + A_-^2 + |B|^2 + A_+^2) = 2\hbar\omega_{\boldsymbol{k}} = 2c\hbar k^0\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{1}\,{}^{2}v^{\boldsymbol{k}} &= N^2(-A_-B^* - BA_- - A_+B^* - BA_+) = 2c\hbar k^1\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{2}\,{}^{2}v^{\boldsymbol{k}} &= iN^2(-A_-B^* + BA_- - A_+B^* + BA_+) = 2c\hbar k^2\\
{}^{2}\bar{v}^{\boldsymbol{k}}\gamma^{3}\,{}^{2}v^{\boldsymbol{k}} &= N^2(-|B|^2 + A_-^2 + |B|^2 - A_+^2) = 2c\hbar k^3\ .
\end{aligned}
$$

Obviously the general result is:

$$
{}^{r}\bar{u}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}u^{\boldsymbol{k}} = {}^{r}\bar{v}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}v^{\boldsymbol{k}} = 2c\hbar k^{\rho}\delta_{rs} \tag{A.205a}
$$

$$
{}^{r}\bar{u}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}v^{\boldsymbol{k}} \neq 0 \qquad , \qquad {}^{r}\bar{v}^{\boldsymbol{k}}\gamma^{\rho}\,{}^{s}u^{\boldsymbol{k}} \neq 0 \tag{A.205b}
$$

A.28 Some Integrals

$$[64, \text{integral } 187]: \quad \int \mathrm{d}x\, x^2 \sqrt{x^2+a^2} = \tag{A.206}$$
$$= \frac{x(x^2+a^2)^{3/2}}{4} - \frac{a^2}{8}\left[x\sqrt{x^2+a^2} + a^2 \ln(x+\sqrt{x^2+a^2})\right]$$

$$[64, \text{integral } 194]: \quad \int \mathrm{d}x\, \frac{x^2}{\sqrt{x^2+a^2}} =$$
$$= \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) \tag{A.207}$$

References

[1] Edward G. Harris : *A pedestrian Approach to Quantum Field Theory* (John Wiley & Sons, New York, 1972)

[2] James D. Bjorken, Sidney D. Drell : *Relativistic Quantum Fields* (Mc-Graw-Hill, New York, 1965)

[3] Michael E. Peskin, Daniel V. Schroeder : *An Introduction to Quantum Field Theory* (Westview Press, Boulder 1995)

[4] Michele Maggiore : *A modern Introduction to Quantum Field Theory* (Oxford University Press, Oxford, UK, 2005)

[5] Franz Mandl, Graham Shaw : *Quantenfeldtheorie* (Aula Verlag, Wiesbaden 1993)

[6] Walter Greiner, Joachim Reinhard : *Quantenelektrodynamik* (Harri Deutsch, Thun, 2.Auflage 1995)

[7] Walter Greiner, Joachim Reinhard : *Feldquantisierung* (Harri Deutsch, Thun, 1993)

[8] Walter Greiner : *Quantentheorie (Spezielle Kapitel)* (Harri Deutsch, Thun, 4.Auflage 1993)

[9] Franz Gross : *Relativistic Quantum Mechanics and Field Theory* (John Wiley, New York, 1993)

[10] Lewis H. Ryder : *Quantum Field Theory* (Cambridge University Press, Cambridge, UK, 2nd ed. 1996)

[11] Ashok Das : *Lectures on Quantum Field Theory* (World Scientific, New Jersey, USA, 2008)

[12] Claude Itzykson, Jean-Bernhard Zuber : *Quantum Field Theory* (McGraw-Hill, New York 1980)

[13] Kerson Huang : *Quantum Field Theory* (John Wiley & Sons, New York 1998)

[14] Taichiro Kugo : *Eichtheorie* (Springer, Berlin 1997)

[15] Ashok Das : *Field Theory – A Path Integral Approach* (World Scientific, Singapore, 2nd ed. 2006)

[16] Morton Hammermesh : *Group Theory and its Application to Physical Problems* (Addison-Wesley, Reading, MA, USA, 1962)

[17] A. W. Joshi : *Elements of Group Theory for Physicists* (Wiley Eastern Private, New Delhi 1973)

[18] Michael Tinkham : *Group Theory and Quantum Mechanics* (McGraw-Hill, New York 1964)

[19] H. Bacry : *Lectures on Group Theory and Particle Theory* (Gordon & Breach, London, 1977)

[20] Y. S. Kim, M. E. Noz : *Theory and Applications of the Poincaré Group* (D.Reidel, Dordrecht, Netherlands, 1986)

[21] Hagen Kleinert : *Pfadintegrale in Quantenmechanik, Statistik und Polymerphysik* (Spektrum Akademischer Verlag, Heidelberg, 1993) http://users.physik.fu-berlin.de/~kleinert/kleiner_reb4/psfiles/

[22] Christoph Berger : *Teilchenphysik* (Springer, Berlin, 1992)

[23] Stephen Weinberg : *The Quantum Theory of Fields, Vol.I* (Cambridge University Press, Cambridge, UK, 1995)

[24] J. Zinn-Justin : *Renormalization and Renormalization Group* , (11 pp, Les Houches 1998) http://ipht.cea.fr/Docspht//articles/t98/118/public/publi.pdf much overlap, but some notable differences (17pp): http://fds.oup.com/www.oup.co.uk/pdf/0-19-922719-5.pdf

[25] B. Delamotte : *A hint of renormalization* , Am. J. Phys. **72**, 170–184 (2004) arXiv:0212049 http://arxiv.org/abs/hep-th/0212049v3

[26] Kenneth G. Wilson : *Problems in Physics with Many Scales of Length*, Sci. Am., p. 158–179 (Aug. 1979) http://www.physics.ohio-state.edu/~jay/846/Wilson.pdf

[27] J. M. Jauch, F. Rohrlich : *The Theory of Photons and Electrons* (Springer, Berlin, 2nd, 1980)

[28] I. J. R. Aitchison, A. J. G. Hey : *Gauge Theories in Particle Physics* (A. Hilger LTD., Bristol, UK, 2nd 1989)

[29] P. Becher, M. Böhm, H. Joos : *Eichtheorien der starken und elektroschwachen Wechselwirkung* (B. G. Teubner, Stuttgart, 1981)

[30] S. F. Novaes : *Standard Model: An Introduction* , arXiv [hep-ph] 0001283 , 101 pp. (2000), http://arxiv.org/abs/hep-ph/0001283v1

[31] *The Partial Derivative*, APIN Circular se77211 (2010) http://www.astrophys-neunhof.de/mtlg/se77211.pdf

[32] *Anmerkungen zur Speziellen Relativitätstheorie* APIN Mitteilung sd05211 (2010) http://www.astrophys-neunhof.de/mtlg/sd05211.pdf

[33] James D. Bjorken, Sidney D. Drell : *Relativistic Quantum Mechanics* (Mc-Graw-Hill, New York, 1964)

[34] E. Schrödinger: *Über die kräftefreie Bewegung in der relativistischen Quantenmechanik*,
Sitz. Ber. Preuß. Ak. Wiss., Phys.-Math. **24**, 418–428 (1930)

[35] P. A. M. Dirac: *The Quantum Theory of the Electron*,
Proc. Roy. Soc. (London) A **117**, 610–624 (1928)
http://rspa.royalsocietypublishing.org/content/117/778/610.short

[36] L. H. Thomas: *The Motion of the Spinning Electron*,
Nature **117**, 514 (1926), http://doi.org/10.1038/117514a0

[37] *Elements of Complex Analysis*, APIN Circular se98027 (2010)
http://www.astrophys-neunhof.de/mtlg/se98027.pdf

[38] *Quantensprünge*, (APIN, Nürnberg, 2010)
http://www.astrophys-neunhof.de/mtlg/sd67113.pdf

[39] P. A. M. Dirac: *The Principles of Quantum Mechanics*
(Clarendon Press, Oxford, 4th ed. 1958)

[40] W. Pauli: *The Connection Between Spin and Statistics*,
Phys. Rev. **58**, 716–22 (1940)
http://doi.org/10.1103/PhysRev.58.716

[41] Carl Friedrich von Weizsäcker: *Aufbau der Physik*
(Carl Hanser Verlag, München 1985)

[42] G. Gründler: *The zero-point energy of elementary quantum fields*,
APIN Circular se86311 (2017)
https://www.astrophys-neunhof.de/mtlg/se86311.pdf
arXiv 1711.03877 (2017), https://arxiv.org/abs/1711.03877

[43] P. A. M. Dirac: *The Lagrangian in Quantum Mechanics*,
Phys. Zeitschr. d. Sowjetunion **3**, 64–72 (1933)
reprinted in *Selected Papers on Quantum Electrodynamics* (ed. by J. Schwinger, Dover Publications, New York, 1958)

[44] R. Feynman: *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys. **20**, 367–87 (1948)

[45] A. Houriet, A. Kind: *Classification invariante des termes de la Matrice* $\boldsymbol{S}$, Helv. Phys. Acta **22**, 319–330 (1949) http://dx.doi.org/10.5169/seals-112007

[46] G. C. Wick: *The evaluation of the Collision Matrix*, Phys. Rev. **80**, 268–272 (1950)

[47] W. Heisenberg: *Über die in der Theorie der Elementarteilchen auftretende universelle Länge*, Ann. d. Phys. (5) **32**, 20–33 (1938)

[48] L. D. Landau, E. M. Lifschitz: *Lehrbuch der Theoretischen Physik, Band II* (Harri Deutsch, Thun, 12. Auflage 1997)

[49] B. Rossi, D. B. Hall: *Variation of the Rate of Decay of Mesotrons with Momentum*, Phys. Rev. **59**, 223–228 (1941) http://dx.doi.org/10.1103/PhysRev.59.223

[50] Christian Møller: *Zur Theorie des Durchgangs schneller Elektronen durch Materie*, Ann. d. Phys. **406**, 531–585 (1932)

[51] H. J. Bhabha: *The Scattering of Positrons by Electrons . . .*, Proc. Roy. Soc. Lond. **A 154**, 195–206 (1936) http://rspa.royalsocietypublishing.org/content/154/881/195.full.pdf+html

[52] Arthur H. Compton: *A Quantum Theory of the Scattering of X-rays by Light Elements*, Phys. Rev. **21**, 483–502 (1923)

[53] D. R. Yennie, S. Frautschi, H. Suura: *The infrared divergence phenomena and high-energy processes*, Annals of Physics (NY) **13**, 379–452 (1961)

[54] Wolfgang Bietenholz, Lilian Prado: *40 Years of Calculus in* $4+\epsilon$ *Dimensions*, arXiv:1211.1741 (2012) http://arxiv.org/pdf/1211.1741.pdf

[55] D. Hanneke, S. Fogwell Hoogerheide, G. Gabrielse: *Cavity Control of a Single-Electron Quantum Cyclotron: Measuring the Electron Magnetic Moment*, Phys. Rev. A **83**, 052122 (2010) http://pra.aps.org/pdf/PRA/v83/i5/e052122

[56] Takahashi Y.: *On the Generalized Ward Identity*, Nuovo Cimento **6**, 371–375 (1957)

[57] J. C. Ward: *An Identity in Quantum Electrodynamics*, Phys. Rev. **78**, 182 (1950)

[58] R. Karplus, M. Neuman: *Non-Linear Interactions between Electromagnetic Fields*, Phys. Rev. **80**, 380–385 (1950)

[59] W. Heisenberg: *Über den Bau der Atomkerne* Zeits. Phys. **77**, 1–11 (1932)

[60] C. N. Yang, R. L. Mills: *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, Phys. Rev. **96**, 191–195 (1954)

[61] Murray Gell-Mann: *Symmetries of Barions and Mesons*, Phys. Rev. **125**, 1067–1084 (1962)

[62] Torsten Fließbach: *Allgemeine Relativitätstheorie* (Spektrum Akademischer Verlag, Heidelberg, $^{(4)}$2004)

[63] C. L. Bennet et al.: *Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results*, Astrophys. J. Suppl. Ser. **208**:20, 54pp (2013) http://dx.doi.org/10.1088/0067-0049/208/2/20

[64] I. N. Bronstein, K. A. Semendjajew: *Taschenbuch der Mathematik* (Harry Deutsch, Zürich und Frankfurt/Main, 1973)