Zero-Point Energy and Casimir-Effect

The essential arguments for and against the assumption of a physically effective zero-point energy

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Overview

Since the invention of zero-point energy in 1911, there is some dispute whether it is really detectable in experiments, or merely a strange artifact of the theory. The discussion is traced in a historical perspective, and the essential arguments are sketched, which have been cited in support or in refusal of the assumption of zero-point energy. The article in particular is focused on the Casimir-effect, which often is considered to be the most convincing evidence for the measurable existence of zero-point energy.

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1. Planck invents the zero-point energy

By interpolation, Planck found in 1900 the formula [1]

$$\frac{\text{energy}}{\text{volume} \cdot \text{frequency interval}} = \frac{8\pi h\nu^3/c^3}{e^{\frac{h\nu}{k_B T}} - 1} \tag{1}$$

for the energy per volume in the frequency interval $(\nu, \nu + \mathrm{d}\nu)$ of an electromagnetic field, which is at the temperature T in thermodynamical equilibrium with matter. c is the speed of light, $k_B = 1.38 \cdot 10^{-23}\mathrm{J/K}$ is the Boltzmann constant, and $h = 6.63 \cdot 10^{-34}\mathrm{Js}$ is Planck's quantum of action, which here was first time introduced into physics.

Maxwell-Boltzmann statistics assume, that the energy per degree of freedom in thermodynamic equilibrium is $\frac{1}{2}k_B T$. In contrast, from Planck's radiation law (1) follows (see e.g. [2])

$$\frac{\text{energy}}{\text{degree of freedom}} = \frac{h\nu/2}{e^{\frac{h\nu}{k_B T}} - 1} \ . \tag{2}$$

Thus the energy per degree of freedom does not only depend on temperature — as assumed by classical thermodynamics — , but also on frequency. As the exponential function in the denominator increases much faster with increasing frequency than the linear factor in the numerator, degrees of freedom with high frequency get less energy than degrees of freedom with low frequency. Only in the limit $\frac{h\nu}{k_B T} \ll 1$, the energy per degree of freedom is

$$\begin{aligned} \frac{h\nu/2}{e^{\frac{h\nu}{k_B T}} - 1} = \frac{h\nu/2}{1 + \frac{h\nu}{k_B T} + \frac{1}{2}(\frac{h\nu}{k_B T})^2 + \ldots - 1} = \frac{k_B T/2}{1 + \frac{1}{2}\frac{h\nu}{k_B T} + \ldots} = \\ \approx \frac{1}{2}\Big(k_B T - \frac{h\nu}{2}\Big) \approx \frac{1}{2}k_B T \quad \text{if } \frac{h\nu}{k_B T} \ll 1 \ . \end{aligned} \tag{3}$$

Planck imagined the walls of the furnace, with which the electromagnetic radiation field is in thermodynamic equilibrium, to consist of a huge amount of one-dimensional Hertz dipoles. As a one-dimensional oscillator has two degrees of freedom (kinetic and potential energy must be considered as one degree of freedom each), the mean energy U of an oscillator with eigenfrequency ν at temperature T is

$$U = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \ . \tag{4a}$$

One decade later, by February-03-1911, Planck presented the Physikalische Gesellschaft[2] in Berlin with the hypothesis [3], that the oscillating dipoles might have in addition to the energy (4a) a further energy of average value $h\nu/2$, which should be independent of temperature:

$$U = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} + \frac{h\nu}{2} \tag{4b}$$

While the first term vanishes in the limit $T \to 0$, because the exponential function diverges, the second term is describing a zero-point energy of average value $h\nu/2$, which does not disappear even at absolute temperature zero. Note, that the value $h\nu/2$, same as the value U, is the mean value of a large ensemble of oscillators. A single oscillator should have, according to Planck's new hypothesis, the energy

$$\hat{U} = n \cdot h\nu + R \qquad \text{with } 0 \le R < h\nu \ , \ n = 0, 1, 2, \ldots \tag{5}$$

Planck was guided to this hypothesis by the wish, to keep Maxwell's electrodynamics without changes. This does include

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in particular the assumption, that the energy is continuously distributed in the electromagnetic radiation field, but not lumped into quanta (which later-on were named photons). Under this assumption, a material oscillator cannot extract out of the radiation field within infinitely short time (in form of a "quantum jump") the energy $h\nu$. Instead the accumulation of radiation energy within the material oscillator must be a continuous process. Therefore Planck assigned to each oscillator at any time an amount of energy as specified in (5). Only the emission of energy from the material oscillator to the radiation field should take place in packets of size $h\nu$. Planck demonstrated, that this assumption is sufficient for the derivation of the radiation law (1). If the temperature was lowered to $T = 0$, then — according to Planck's new theory — the oscillator could only emit it's n energy quanta of size $h\nu$, but the residual vibration energy of $0 \leq R < h\nu$ remained even at $T = 0$ within the oscillator.

Planck of course was aware of the strong arguments for the quantized nature of electromagnetic radiation, which had been pointed out by Einstein [4] already in 1905. In his speech, Planck in particular mentioned the properties of cathode rays (light-electric effect) as a profound argument, which casted doubt on the complete correctness of Maxwell's theory. But at that time he would rather shift such problems aside than consider seriously fundamental modifications of electrodynamics.

While such concerns are obsolete by today, Planck's February-03-1911 presentation is still worth mentioning, because on that day first time, long before the discovery of quantum mechanics, the possibility of a zero-point energy of oscillators was clearly formulated, and because Planck demonstrated that his radiation law is compatible with the zero-point energy which indeed was postulated 15 years later by quantum mechanics.

In the following years, most physicists by and by abandoned

their resistance to the electromagnetic field's quantization, and thus the motive, which had led Planck to the idea of the zero-point energy of material oscillators, is irrelevant by now. The zero-point energy did not disappear from the scene, however, but became an integral part of quantum physics. Why? In the next section, the main arguments will be illustrated.

2. Arguments in support of zero-point energy

In the presentation of his new hypothesis (5) in February 1911, Planck clearly stated that it would not be easy to confirm or disprove the existence of zero-point energy by experiments. The older formula (4a) had been tested successfully in the previous years due to the measurement of the specific heat of material systems. But the additional term in (4b) has no impact on those results, because it vanishes in the derivative $\mathrm{d}U/\,\mathrm{d}T$. As an indication for a finite zero-point energy, Planck could at least refer to the fact that the decay rate of radioactive matter is not significantly reduced, if it is cooled down to arbitrarily low temperature.

In the first days of 1913, Einstein and Stern published two further arguments [5] for the existence of zero-point energy. Firstly they pointed out, that the last step in the approximation (3) is overly imprecise. If the exponential function in the denominator of Planck's radiation law is expanded in a series of powers of $h\nu/(k_B T)$, then this expansion should be continued to the quadratic term. From the expression (4b) for the energy of an oscillator then follows in the case $h\nu \ll k_B T$ this approximation:

$$U \overset{(4b)}{\approx} \frac{h\nu}{\frac{h\nu}{k_B T}\left(1+\frac{h\nu}{2k_B T}\right)} + \frac{h\nu}{2} \approx$$

$$\approx \left(k_B T - \frac{h\nu}{2}\right) + \frac{h\nu}{2} = k_B T \text{ if } \frac{h\nu}{k_B T} \ll 1 \qquad (6)$$

This result conforms at high temperature with the expectation of classical physics, but deviates from classical physics at $T \to 0$ by the amount of $h\nu/2$. On the other hand, formula (4a) with no zero-point energy gives at $T \to 0$ the result expected by classical physics, but predicts at high temperature a result, which is deviating by the amount of $-h\nu/2$ from the value predicted by classical physics. As the classical theory was derived in the 19. century from observations made at high Temperature, while the research in the range $T \to 0$ in 1913 was still in it's infancy (Helium — boiling point 4.2 K — was first time successfully liquefied by Kamerlingh Onnes only in 1908), Einsteins and Stern concluded, that most likely that one of the two formulae should be correct, which complies with classical physics at high temperature, i.e. (4b) with the zero-point energy $h\nu/2$.

The second argument, presented by Einstein and Stern in the same publication, makes reference to the measurement results of Arnold Eucken [6], who had explored the specific heat of hydrogen. Eucken's data could be readily explained by the specific heat as computed from (4b), but differed significantly from the specific heat as computed from (4a). From today's point of view, this second argument is not completely correct in the form, in which it was used by Einstein and Stern in 1913. Apparently Einstein and Stern assumed a finite zero-point energy not only for the vibration, but also for the rotation of the hydrogen molecule. According to quantum mechanics, which was detected in 1925, the angular momentum L of rotating molecules is quantized, and L^2 can assume only the values

$$L^2 = \hbar^2 J(J+1) \quad \text{with } J = 0, 1, 2, 3, \ldots \tag{7}$$

Thus the possible rotation energy of a molecule is

$$E_{\text{rot}} = \frac{\hbar^2}{2I} J(J+1) \quad \text{with } J = 0, 1, 2, 3, \ldots \quad , \tag{8}$$

Here I is the molecule's moment of inertia. In case of $J = 0$, the molecule's energy of rotation is zero. A zero-point energy of finite value $h\nu/2$ exists only for the vibration, but not for the rotation of the hydrogen molecule.

Convincing arguments for the existence of zero-point energy resulted in the following years from the precise analysis of the vibration-spectra of molecules. The observed ware numbers $\widetilde{\nu}$ of absorption could be accurately fitted by formulae of the type

$$\widetilde{\nu} = \frac{1}{\lambda} = A + \left(B_1 n' - B_2 (n')^2\right) - \left(C_1 n'' - C_2 (n'')^2\right)$$
$$\text{with } n' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots \text{ and } n'' = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots \qquad (9)$$

with empirically determined constants A, B_1, B_2, C_1, C_2. A is interpreted as the wave number of electronic excitation. n' is the quantum number of vibration in the excited state, and n'' is the quantum number of vibration in the ground state. If the molecules would be harmonic oscillators, then only the integer differences $n' - n''$ could be extracted from the spectra of vibration, but not the absolute values of n' and n''. As in reality the vibration-potentials of molecules are not harmonic, but have a measurable unharmonic distortion, the coefficients B_2 and C_2 of the non-linear terms are different from zero, and it becomes possible to find out the absolute values of n' and n'' due to analysis of the absorption spectra.

Mulliken, who proved the correctness of the halfinteger quantum numbers (9) by means of the vibration-rotation-spectrum of boron-oxide [7], commented: "This result cannot be accomplished by any other assignment of minimum values of n' and n'', if we restrict ourselves to the possibilities $0, +1/2, +1$. Since half-integral values of n' and n'' with a minimum value $+1/2$ are by far more probable than a large electronic isotope effect or other serious failure in the theory, they may be quite definitely accepted — unless one wishes to entertain the possibility of fractional values other than $1/2$."

Further experimental evidence for the physical existence of zero-point energy came in the following years for example from the scattering of X-rays or neutrons by crystals at very low temperature (the reduced intensity of the structures makes directly visible, that the crystal atoms don't rest without motion at one point), as well as from the fact, that ^{4}He at normal pressure does not crystallize even at lowest temperature (the zero-point energy of the Helium atoms in a crystal would exceed the binding energy between the atoms due to van der Waals interaction, which causes crystallization of the heavier rare gases at low temperatures).

When Heisenberg [8] in summer 1925 invented quantum mechanics, harmonic and unharmonic oscillators were the first systems, to which he applied his new formalism. In Equation (23) of his paper, he found

$$W = \left(n + \frac{1}{2}\right) \frac{h\omega_0}{2\pi} \tag{10}$$

with $n = 0, 1, 2, 3, \ldots$ as energy of a harmonic oscillator in it's nth quantum state. Thus the zero-point energy $h\omega_0/(4\pi) = h\nu/2$ of harmonic oscillators was firmly established by mid of the twentieth both in experimental and in theoretical physics — however only with regard to systems with a finite number of vibrational degrees of freedom, like for example molecules or solids (which may be considered as huge, but on no account infinitely large molecules). In contrast, there were (and still are) severe arguments against the assumption, that continuous fields, like for example the electromagnetic field (whose's quanta are the photons) or the electron/positron field (whose's quanta are the electrons and positrons) should have a zero-point energy. We will occupy ourselves with these counter-arguments in the next section.

3. Collision with GRT

The canonical quantization of continuous fields regularly leads to an infinitely large zero-point energy. This happens, because each single one of the infinitely many oscillation modes, which are in principle accessible to the continuous field, is represented in quantum field theory by a harmonic oscillator, which according to (10) has the zero-point energy $\pm h\nu/2$. The plus sign is valid for boson fields, the minus sign for fermion fields. See for example [9, chapters 14 - 17] or any other textbook on quantum field theory.

As an example, let's consider an evacuated volume of size $V = X \cdot Y \cdot Z$ with material walls. At temperature T, there is an electromagnetic field inside this volume, which is in thermal equilibrium with the walls. The oscillation modes of the radiation field, which are geometrically possible (that is to say the standing waves), have the wave numbers

$$k_{lmn} = \frac{2\pi\nu_{lmn}}{c} = \sqrt{\left(\frac{l\pi}{X}\right)^2 + \left(\frac{m\pi}{Y}\right)^2 + \left(\frac{n\pi}{Z}\right)^2} \qquad \text{(11a)}$$

with $l, m, n = 0, 1, 2, 3, \ldots$, but maximum one index per mode can be zero.

k is the wave number, and ν is the frequency of the radiation modes. At temperature $T = 0$, there are exclusively zero-point oscillations in the volume, which's energy is $h\nu_{lmn}/2$ respectively. The total zero-point energy U_0 within the volume V is

$$U_0 = 2\sum_{l,m,n=0}{}' \frac{\hbar c k_{lmn}}{2} = \sum_{l,m,n=0}{}' \hbar c \sqrt{\left(\frac{l\pi}{X}\right)^2 + \left(\frac{m\pi}{Y}\right)^2 + \left(\frac{n\pi}{Z}\right)^2} = \infty \,. \qquad \text{(11b)}$$

The factor 2 is caused by the electromagnetic field's two polarization degrees of freedom. The prime$'$ is a reminder, that radiation modes

with one index zero only exist with one polarization, and therefore get a factor 1/2 in (11b). As there are infinitely many oscillation modes, the zero-point energy is infinitely large.

The explorers of the young quantum mechanics were from the outset very well aware of the vexing fact, that the zero-point energy of quantum fields diverges. In November 1925, Born, Heisenberg, and Jordan submitted a paper [10] for publication to the Zeitschrift für Physik, in which they evaluated the quantization of a one-dimensional scalar field. In their equation (36′), they found the zero-point energy $C = \frac{1}{2}h\sum_k \nu_k$, and commented: Die Nullpunktsenergie „wäre insbesondere im Grenzfall unendlich vieler Freiheitsgrade unendlich groß.“ [3] The wording “would be infinitely large . . . in the limit” reveals, that the authors had substantial doubts whether this result should be taken seriously, or whether it might be merely a strange artifact of the theory. There is no further comment in their paper on this disturbing issue.

In contrast, Jordan and Pauli two years later made very clear remarks regarding the diverging zero-point energy, when they published a paper [11] on the relativistically invariant quantization of the electromagnetic field: „Verschiedene Erwägungen scheinen uns dafür zu sprechen, daß im Gegensatz zu den Eigenschwingungen im Kristallgitter (wo sowohl theoretische als auch empirische Gründe für das Vorhandensein einer Nullpunktsenergie sprechen) bei den Eigenschwingungen der Strahlung jener „Nullpunktsenergie“ $h\nu/2$ pro Freiheitsgrad keine physikalische Realität zukommt. Da man es nämlich bei dieser mit streng harmonischen Oszillatoren zu tun hat und da jene „Nullpunktsstrahlung“ weder absorbiert noch zerstreut oder reflektiert werden kann, scheint sie sich, einschließlich ihrer Energie oder Masse, jeder Möglichkeit eines Nachweises zu entziehen. Es ist deshalb wohl die einfachere und befriedigendere

[3] the zero-point energy “would be infinitely large in particular in the limit of infinitely many degrees of freedom.”

Auffassung, daß beim elektromagnetischen Felde jene Nullpunktsstrahlung überhaupt nicht existiert.“ [4] [11, page 154]

Pauli rephrased the next-to-last sentence even more pronounced in his article [12] on wave mechanics in the Handbuch der Physik: Die Nullpunktsenergie ist „prinzipiell unbeobachtbar, da sie weder emittiert, absorbiert oder gestreut wird, also nicht in Wände eingeschlossen werden kann, und da sie, wie aus der Erfahrung evident ist, auch kein Gravitationsfeld erzeugt.“ [5] [12, page 250]

When Pauli wrote this sentence, a subtle point slipped his attention: While zero-point energy indeed “cannot be enclosed in walls”, it still can be pushed out of certain volumina by walls, and thus may become observable. That is the Casimir-effect, which will be discussed thoroughly in the next sections. But prior to that we will occupy ourselved with Pauli’s further argument, that zero-point energy, “as is evident from experience, also does not produce a gravitational field.” This is a most important and severe argument, which is making reference to a discrepancy between QFT = quantum field theory and GRT = general relativity theory:

In 1915, Einstein had published the basic equations of general relativity theory [13, 14]. The field equation of GRT describes, how

[4] “It seems to us, that several considerations are indicating, that — in contrast to the eigen-oscillations in the crystal grid (where both theoretical and empirical reasons are indicating the existence of a zero-point energy) — no reality can be assigned to that “zero-point energy” $h\nu/2$ per degree of freedom in case of the eigen-oscillations of the radiation. As one is dealing with regard to the latter with strictly harmonic oscillators, and as that “zero-point radiation” can neither be absorbed nor scattered nor reflected, it seems to elude, including it’s energy or mass, any method of detection. Therefore it may be the simplest and most satisfactory conception, that in case of the electromagnetic field that zero-point radiation does not exist at all.”

[5] The zero-point energy is “under no circumstances observable, as it is not emitted nor absorbed nor scattered, and thus cannot be enclosed in walls, and because it, as is evident from experience, also does not produce a gravitational field.”

the geometry of four-dimensional space-time is deformed due to it's energy content. The larger the energy density is within a certain volume of space-time, the more space-time is curved in that volume. As the volume V of equation (11b) is containing the infinitely large energy of zero-point oscillations, the energy density — and therefore the curvature of space-time in this volume — is infinitely large.

This obviously is absurd. Searching for the cause of the wrong result, one might consider a possible limitation of quantum electrodynamics. QED might be merely the low-energy limit of a more general, yet unknown theory. Given that case, it might be correct not to extend the summation in (11b) to infinitely large wave numbers, but only up to a maximum wave number K, whose's value is not yet known by today. The only information we have regarding K is, that this wave number must be larger than those wave numbers, for which quantum electrodynamics until now (2013) has been checked and confirmed, i.e. that $K > 10^{18}\mathrm{m}^{-1}$ must hold. This information is inserted into (11b):

$$\frac{U_0}{V} > \sum_{|k_{l,m,n}| \leq K} \hbar c k_{lmn} \approx 4 \cdot 10^{47}\mathrm{Jm}^{-3} \quad \text{with } K = 10^{18}\mathrm{m}^{-1} \tag{12}$$

This is the minimum zero-point energy of the electromagnetic field, which is a boson field with two polarization degrees of freedom. The elementary fermion fields (negative zero-point energy) known by today in total have 90 degrees of freedom. The elementary boson fields (positive zero-point energy) in total have about 30 degrees of freedom, depending on our assumptions on the number of Higgs-fields. For the following estimate we will assume, that the total zero-point energy of all fields is about -30 times as large as

(12), i.e. approximately

$$\frac{U_0}{V} < -1.2 \cdot 10^{49}\mathrm{Jm}^{-3} \quad \text{with } K = 10^{18}\mathrm{m}^{-1} \ . \tag{13a}$$

When astronomers interpret their observations by means of general relativity theory, then they conclude from the smallness of the mean curvature (which even might be exactly zero) of the intergalactic space, that the energy density of the vacuum cannot significantly exceed $10^{-9}\,\mathrm{Jm}^{-3}$, with the most likely value being

$$\frac{U_{\mathrm{vacuum}}}{V} \approx 6.8 \cdot 10^{-10}\,\mathrm{Jm}^{-3} \ . \tag{13b}$$

From the both equations (13) we get the ratio

$$\frac{\text{theory}}{\text{observation}} = \frac{(13a)}{(13b)} < -2 \cdot 10^{58} \ . \tag{14}$$

Really an astronomic discrepancy! Still there is a loophole in GRT, by which zero-point energy can be saved: The cosmological constant [15]. The cosmological constant is an additive term in the field equation of GRT, which may be chosen arbitrarily. The discrepancy between theory and observation is purged away, once the cosmological constant is chosen to be proportional to $(U_0/V - 6.8 \cdot 10^{-10}\mathrm{Jm}^{-3})$.

But nobody is feeling well with this crude "solution" of the problem. To introduce arbitrarily and without physical justification a constant, which must be adjusted with a precision of more than 58 decimal digits would be a brute intervention, which is without parallel in modern physics. The baffling fine-tuning up to more than 58 decimal digits was dubbed the "problem of the cosmological constant" in the literature. A well-known review of this issue was published in 1989 by Weinberg [16]. For a recent review, see [17].

It's not surprising that Pauli, though he for sure was aware[6] of the loophole of the cosmological constant, did not exercise that option, but stated without further discussion that zero-point energy, "as is evident from experience, also does not produce a gravitational field."

Not all scientists perceived the infinitely large zero-point energy as a problem. For Walther Nernst, it was instead a most welcome idea. Nernst wasn't ready to accept, that the world was inexorably approaching the "heat death". This was the fate, which Boltzmann had prophesied to the universe, because — according to the second law of thermodynamics — the entropy of a closed systems does steadily increase. Therefore all chemical processes will eventually come to an end, and the world thus will reach a death state.

Things are different for open systems. In particular the huge afflux of solar energy secures, that the biological processes of life, which are characterized by a decrease of entropy, can take place on earth. But as the energy of the sun is finite, it can only retard the earth's heat death, but cannot impede it forever.

That could be achieved only by an infinitely large energy source. Exactly this infinitely large, inexhaustible reservoir of energy, for which Nernst was looking to fight the heat death of the universe, had Planck provided to him in the form of zero-point energy! Zero-point energy is the basic constituent of a cosmological model without heat death, which Nernst presented by January-18-1916 to the amazed Physikalische Gesellschaft[7] in Berlin [18]. Nernst's model was assessed quite sceptically by most of his colleagues, and got only few support. By today, it is reduced to a feeble existence in esoteric circles. In an article by Kragh [19], a brief account of

[6] Pauli was a renowned expert of general relativity theory. His article on GRT, which he had written at the age of 19 years for the Handbuch der Physik, had been emphatically praised by Einstein.

[7] physical society

the reception history of Nernst's cosmology can be found, and furthermore plenty of worth reading informations on the evolution of the idea of "zero-point energy".

4. The Casimir-Effect

In the previous section, Pauli was cited with the words, that zero-point energy is — besides other reasons — not detectable, because it "cannot be enclosed in walls". But by May-29-1948 Hendrik Casimir gave a presentation [20] to the Koninklijke Nederlandse Akademie van Wetenschappen, in which he exposed the stupefying idea, to block the zero-point energy (more precisely: small parts of zero-point energy) out of certain volumina by means of walls, and thus make it observable after all.

To explain the basic idea behind Casimir's proposal, let's consider the rectangular resonator with electrically conductive walls, which is sketched in figure 1. It's size is $X \times Y \times Z$. In the resonator's interior there is a plate of thickness P, which as well is electrically conductive. The plate is aligned parallel to the resonator's XY outer walls, and is free to move along the resonator's Z-axis. The plate is dividing the resonator in two resonators of

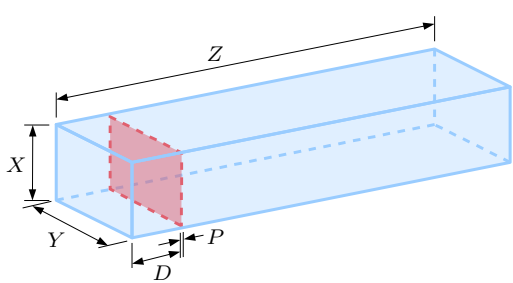

Fig. 1 : Resonator with movable intermediate wall

sizes $X \times Y \times D$ and $X \times Y \times (Z - D - P)$. Zero-point oscillations, whose's half wavelength in Z-direction is larger than D, but smaller than $Z - D - P$, can according to equation (11) develop in the right cavity, but not in the left one. If the mobile plate is slightly moved in Z-direction, then the spectrum of zero-point oscillations of both cavities will change, and consequently their content of zero-point energy U_0. Therefore onto the mobile plate a force

$$F_{\text{Casimir}} = -\frac{\mathrm{d}U_{0,\text{left cavity}}}{\mathrm{d}D} + \frac{\mathrm{d}U_{0,\text{right cavity}}}{\mathrm{d}D} \tag{15}$$

to the right is exerted. The detailed quantitative evaluation in the sequel will show, that the Casimir-force F_{Casimir} is negative, and therefore the movable plate is pulled to the left, i.e. to the nearer outer wall. As the derivation is only sketched quite shortly in the printed text of Casimir's presentation, it proved helpful to consult the elaborate delineation in [21, chapter 5].

First we compute the electromagnetic field in the left cavity. At temperature $T = 0$, it is consisting exclusively of zero-point oscillations, and the total zero-point energy is

$$U_0 \overset{(11b)}{=} {\sum_{l,m,n=0}^{\infty}}' \hbar c k_{lmn} = {\sum_{l,m,n=0}^{\infty}}' \hbar c \sqrt{\left(\frac{l\pi}{D}\right)^2 + \left(\frac{m\pi}{X}\right)^2 + \left(\frac{n\pi}{Y}\right)^2} = \infty \ . \tag{16}$$

To make the following modifications feasible, we introduce a factor

$$\exp\{-k_{lmn}/k_{\text{M}}\} \tag{17}$$

with a finite wavenumber k_{M}, to enforce the convergence of the series. Later we will consider the limit $k_{\text{M}} \to \infty$, to return to (16). In his speech, Casimir pointed out that the cut-off term $\exp\{-k_{lmn}/k_{\text{M}}\}$ isn't merely a mathematical trick, but can as well

be justified physically, because for very high frequencies all metals become transparent (on pages 25 through 30 we will quantify this effect). Therefore the walls of the cavity anyway can only influence radiation of finite frequency.

Furthermore we assume $X \to \infty$ and $Y \to \infty$. Then the sums over the discrete numbers m and n may be replaced by integrals. Only the sum over l will still be considered discrete. In this approximation, the restrictions, defined at (11b) and indicated by the prime$'$, with regard to modes with indices $m = 0$ and $n = 0$ may be neglected. Only the sum over l keeps the prime$'$ as a reminder, that the term with $l = 0$ gets a factor $1/2$. The integration variables and their integrals are

$$k_x \equiv \frac{m\pi}{X} \quad , \quad k_y \equiv \frac{n\pi}{Y} \tag{18}$$
$$\frac{1}{2X}\sum_{m=0}^{\infty}\frac{1}{2Y}\sum_{n=0}^{\infty} \xrightarrow{X,Y\to\infty} \int_0^\infty \frac{\mathrm{d}k_x}{2\pi}\int_0^\infty \frac{\mathrm{d}k_y}{2\pi} \ .$$

Thus one gets

$$U_0 = \frac{\hbar c XY}{\pi^2}\sum_{l=0}^{\infty}{}' \int_0^\infty \mathrm{d}k_x \int_0^\infty \mathrm{d}k_y \sqrt{\Big(\frac{l\pi}{D}\Big)^2 + k_x^2 + k_y^2}\ \cdot$$
$$\cdot \exp\Big\{-\frac{1}{k_{\mathrm{M}}}\sqrt{\Big(\frac{l\pi}{D}\Big)^2 + k_x^2 + k_y^2}\Big\} \ . \tag{19}$$

As k_x and k_y only show up in the form $k_r^2 \equiv k_x^2 + k_y^2$, it's advantageous to change to polar coordinates with radial coordinate k_r and angular coordinate ϕ. As the integration is only over positive k_x and k_y (i.e. only over the first quadrant of the k_xk_y-plane), ϕ is

to be integrated only over 1/4 circle.

$$\int_0^\infty \mathrm{d}k_x \int_0^\infty \mathrm{d}k_y = \int_0^\infty \mathrm{d}k_r \int_0^{\pi/2} k_r \,\mathrm{d}\phi = \frac{\pi}{2} \int_0^\infty \mathrm{d}k_r \, k_r \tag{20}$$

The term $l = 0$ can immediately be determined:

$$\begin{aligned} C &\equiv \frac{1}{2} \frac{\hbar c XY}{\pi^2} \frac{\pi}{2} \int_0^\infty \mathrm{d}k_r \, k_r \sqrt{k_r^2} \, \exp\Big\{ - \frac{1}{k_{\mathrm{M}}} \sqrt{k_r^2} \Big\} = \\ &= \frac{\hbar c XY}{4\pi} \Big[\Big(- k_{\mathrm{M}} k_r^2 - 2 k_{\mathrm{M}}^2 k_r - 2 k_{\mathrm{M}}^3 \Big) \exp\Big\{ - \frac{k_r}{k_{\mathrm{M}}} \Big\} \Big]_0^\infty = \\ &= \frac{\hbar c XY k_{\mathrm{M}}^3}{2\pi} \end{aligned} \tag{21}$$

Therefore

$$\begin{aligned} U_0 = C + \frac{\hbar c XY}{2D} \sum_{l=1}^\infty l \int_0^\infty \mathrm{d}k_r \, k_r \sqrt{1 + \Big(\frac{D k_r}{l\pi} \Big)^2} \cdot \\ \cdot \exp\Big\{ - \frac{l\pi}{D k_{\mathrm{M}}} \sqrt{1 + \Big(\frac{D k_r}{l\pi} \Big)^2} \Big\} \, . \end{aligned} \tag{22}$$

It will be convenient to define the length

$$\eta \equiv \frac{\pi}{k_{\mathrm{M}}} \, . \tag{23}$$

By means of the substitution

$$u \equiv \sqrt{\Big(1 + \frac{D k_r}{l\pi} \Big)^2} \implies \int_0^\infty \mathrm{d}k_r \, k_r = \Big(\frac{l\pi}{D} \Big)^2 \int_1^\infty \mathrm{d}u \, u \tag{24}$$

we get

$$U_0 = C + \frac{\pi^2 \hbar c XY}{2D^3} \sum_{l=1}^{\infty} l^3 \int_1^{\infty} \mathrm{d}u\, u^2 \, \exp\Big\{ -\eta \frac{lu}{D} \Big\}. \qquad (25)$$

The third derivative of the exponential function with respect to η is

$$\frac{\mathrm{d}^3}{\mathrm{d}\eta^3} \exp\Big\{ -\eta \frac{lu}{D} \Big\} = -\frac{l^3 u^3}{D^3} \exp\Big\{ -\eta \frac{lu}{D} \Big\} \, . \qquad (26)$$

Thereby U_0 can be written as

$$U_0 = C - \frac{\pi^2 \hbar c XY}{2} \int_1^{\infty} \mathrm{d}u \, \frac{1}{u} \, \frac{\mathrm{d}^3}{\mathrm{d}\eta^3} \sum_{l=1}^{\infty} \exp\Big\{ -\eta \frac{lu}{D} \Big\} \, . \qquad (27)$$

As the integral is converging, the sequence of integration and summation could be changed. Now the sum over l can be computed by means of the formula

$$\sum_{j=0}^{\infty} A^j = \frac{1}{1-A} \quad \text{if } |A| < 1 \qquad (28)$$

for the infinite geometric series:

$$\begin{aligned} U_0 &= C - \frac{\pi^2 \hbar c XY}{2} \int_1^{\infty} \mathrm{d}u \, \frac{1}{u} \, \frac{\mathrm{d}^3}{\mathrm{d}\eta^3} \Big(\frac{1}{1 - \exp\{-\eta u/D\}} - 1 \Big) \\ &= C - \frac{\pi^2 \hbar c XY}{2} \, \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \int_1^{\infty} \mathrm{d}u \, \frac{1}{u} \, \underbrace{\frac{\mathrm{d}}{\mathrm{d}\eta} \Big(\frac{1}{\exp\{+\eta u/D\} - 1} \Big)}_{\frac{-(u/D)\exp\{\eta u/D\}}{(\exp\{\eta u/D\} - 1)^2}} \end{aligned} \qquad (29)$$

With the further substitution

$$w \equiv \exp\{\eta u/D\} - 1 \implies \int_1^\infty \mathrm{d}u = \int_{\exp\{\eta/D\}-1}^\infty \mathrm{d}w \, \frac{D}{\eta(w+1)}$$

one gets

$$\begin{aligned} U_0 &= C + \frac{\pi^2 \hbar c XY}{2} \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \frac{1}{\eta} \int_{\exp\{\eta/D\}-1}^\infty \mathrm{d}w \, \frac{1}{w^2} \\ &= C - \frac{\pi^2 \hbar c XY}{2} \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \frac{1}{\eta} \left[\frac{1}{w}\right]_{\exp\{\eta/D\}-1}^\infty \\ &= C + \frac{\pi^2 \hbar c XY}{2} \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \frac{D}{\eta^2} \underbrace{\frac{\eta}{D} \frac{1}{\exp\{\eta/D\} - 1}}_{\displaystyle\sum_{j=0}^\infty \frac{B_j}{j!} \left(\frac{\eta}{D}\right)^j} . \end{aligned} \tag{30}$$

The coefficients B_j in the series expansion are the Bernulli-numbers. They are defined implicitly by the generating function

$$\frac{A}{\exp\{A\} - 1} = \sum_{j=0}^\infty B_j \frac{A^j}{j!} \quad \text{with } A \in \mathbb{R},\ A \neq 0,\ |A| < 2\pi,\ j \in \mathbb{N} .$$

From this definition, one gets the Bernulli-numbers

$$\begin{aligned} B_0 &= 1 & B_1 &= -\frac{1}{2} \\ B_2 &= \frac{1}{6} & B_3 &= 0 \\ B_4 &= -\frac{1}{30} & B_5 &= 0 \\ B_j &= -\sum_{n=0}^{j-1} \frac{j!}{n!(j-n)!} \frac{B_n}{j-n+1} & & \text{for } j > 2 . \end{aligned} \tag{31}$$

Performing the second derivative with respect to η, then returning to $k_{\mathrm{M}} \overset{(23)}{=} \pi/\eta$, and inserting $C = (21)$, one gets

$$U_0 = \frac{\pi^2 \hbar c XY}{2} \Bigg(\frac{k_{\mathrm{M}}^3}{\pi^3} + \frac{6 D k_{\mathrm{M}}^4}{\pi^4} - \frac{k_{\mathrm{M}}^3}{\pi^3} - \frac{1}{360\, D^3} + \\ + \sum_{j=6}^{\infty} \frac{B_j}{j!} \frac{(j^2 - 5j + 6)\, \pi^{j-4}}{k_{\mathrm{M}}^{j-4} D^{j-1}} \Bigg) \,. \tag{32}$$

Until now, we only considered the zero-point energy in the left cavity of figure 1 . The energy in the right cavity can be found by inserting everywhere $Z - D - P$ instead of D. Thus the total zero-point energy in the resonator's both cavities is

$$U_{0,\mathrm{total}} = \frac{\pi^2 \hbar c XY}{2} \Bigg(\frac{6(Z-P) k_{\mathrm{M}}^4}{\pi^4} - \frac{1}{360\, (Z - D - P)^3} - \\ - \frac{1}{360\, D^3} + \sum_{j=6}^{\infty} \frac{B_j}{j!} \frac{(j^2 - 5j + 6)}{k_{\mathrm{M}}^{j-4}\, \pi^{4-j}} \Big(\frac{1}{D^{j-1}} + \frac{1}{(Z - D - P)^{j-1}} \Big) \Bigg) \,. \tag{33}$$

Due to the term $\sim k_{\mathrm{M}}^4$, the energy is still diverging in the limit $k_{\mathrm{M}} \to \infty$. That has (of course) not been changed by the conversions.

The derivative of $U_{0,\mathrm{total}}$ with respect to D equals the force, which is acting onto the movable plate in-between the both cavities:

$$- \frac{\mathrm{d} U_{0,\mathrm{total}}}{\mathrm{d} D} = - \frac{\pi^2 \hbar c XY}{2} \Bigg(- \frac{1}{120\, (Z - D - P)^4} + \frac{1}{120\, D^4} + \\ + \sum_{j=6}^{\infty} \frac{B_j}{j!} \frac{(j^2 - 5j + 6)}{k_{\mathrm{M}}^{j-4}\, \pi^{4-j}} \Big(\frac{1 - j}{D^j} + \frac{j - 1}{(Z - D - P)^j} \Big) \Bigg) \,. \tag{34}$$

Now Casimir restricted his investigation to the case

$$(Z - D - P)^4 \gg D^4 \tag{35a}$$

$$\text{and} \quad k_{\mathrm{M}} \to \infty \,, \tag{35b}$$

and got the simple formula:

$$\boxed{F_{\text{Casimir}} = -\frac{\mathrm{d}\,U_{0,\text{total}}}{\mathrm{d}\,D} = -\frac{\pi^2\hbar c}{240}\,\frac{XY}{D^4}} \tag{36}$$

The Casimir-force is pulling the movable wall to the resonator's left outer wall. The limit $k_{\text{M}} \to \infty$ is equivalent to the assumption of infinitly large conductivity of the metal. Due to this assumption, the material of the walls does not show up in formula (36) of the Casimir-force. The Casimir-force is proportional to the area XY of the movable plate, and inversely proportional to the fourth power of the distance D inbetween the movable plate and the left outer wall of the resonator sketched in figure 1 .

In the last sentence of his publication, Casimir remarked: "Although the effect is small, an experimental confirmation seems not unfeasable". To confirm this estimation, we compute the value of the Casimir-force for the parameters

$$\begin{aligned} X \cdot Y = 1\,\text{mm}^2\ ,\ D = 1\,\mu\text{m} \implies \\ \implies F_{\text{Casimir}} \overset{(36)}{=} -1.3 \cdot 10^{-9}\text{N}\ . \end{aligned} \tag{37}$$

If one wants to achieve a precision of 1 %, then with these parameters the measurement error must be smaller than 10 pico-Newton. Actually in most experiments the area is smaller than $1\,\text{mm}^2$. Therefore forces of 1 pN and less must be resolved. The measurement of forces of that size indeed is not impossible, but it is extremely challenging, and demands extraordinary virtuosity from the experimentalists. In the simple form as sketched in figure 1 and computed in (36), the Casimir-force was never measured. Several modifications are necessary to make the effect accessible to experiments. In the sequel we will go through the modifications one by one.

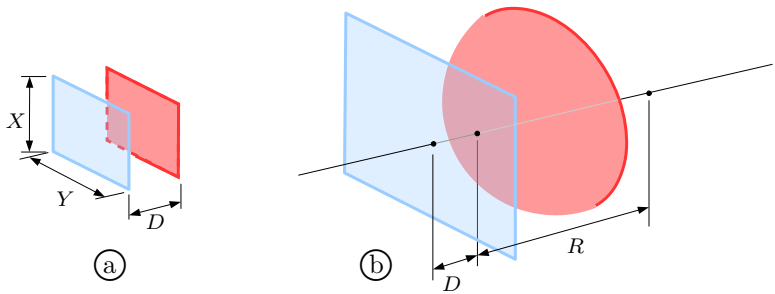

Fig. 2 : Two more realistic resonator geometries

Simplified resonator: As we assumed $Z \gg D$ for the geometry displayed in figure 1, the limit $Z \to \infty$ may be taken, i.e. the resonator's right outer wall may be removed completely. If in addition $X \gg D$ and $Y \gg D$ is valid, then also the resonator's side walls may be removed without significantly changing the Casimir-force. Thus only the two plates remain, which are sketched in figure 2 ⓐ. This is the geometry, which was applied in the first experimental attempts for the measurement of the Casimir-force.

Modified geometry: It's extremely difficult to align two macroscopic plates parallel, if the gap between them is only $D \approx 1\,\mu$m (remember that one plate must be movable). Because of $F_{\text{Casimir}} \sim D^{-4}$, a very small tilt will already cause a huge measurement error. Therefore in many experiments a geometry is preferred, which is displayed in figure 2 ⓑ. One of the plates is replaced by a spherical surface with radius R. For the sphere-plate geometry, the force is [22, (3)]

$$F_{KP} = F_{PP} \cdot \frac{D}{3} \cdot \frac{2\pi R}{X^2} = -\frac{\pi^3 \hbar c}{360} \frac{R}{D^3} \quad \text{if } R \gg D \,. \tag{38}$$

Finite temperature: Most experiments for the evaluation of the Casimir-effect are performed at room temperature, while our derivation of (36) was based on the assumption $T = 0$. At finite temperature, formula (16) changes as follows:

$$U_0 = U_{T=0} \overset{(16)}{=} 2\sum_{l,m,n=1}^{\infty} \frac{\hbar c k_{lmn}}{2} \xrightarrow{T\neq 0}$$

$$\xrightarrow{T\neq 0} U_{T\neq 0} = 2\sum_{l,m,n=1}^{\infty} \hbar c k_{lmn}\Big(\frac{1}{2} + \frac{1}{\exp\{\hbar c k_{lmn}/(k_B T)\} - 1}\Big) \quad (39)$$

k_B is the Boltzmann-constant. The second summand is representing the probability of the thermal excitation of an oscillation mode with wavenumber k_{lmn} at temperature T according to Bose-Einstein statistics.

It's no surprise, that for a long time no significant influence of the additional term could be observed in any experiment at $T \approx 300\,\mathrm{K}$, because the Casimir-force is dominated by wavelengths, which just fit into the resonator with gap width D, but don't fit any more if the gap is infinitesimally reduced to $D - \mathrm{d}D$. These are waves with Z-components $n \cdot \lambda_Z/2 \approx D$, $n = 1, 2, 3, \ldots$ Because at 300 K the Planck radiation spectrum has it's maximum at wave length $17\,\mu\mathrm{m}$, a significant effect of thermal radiation is to be expected only at a gap width D of several micrometers. Most of the older experiments have been performed with much smaller D (typically $0.5\,\mu\mathrm{m} \ldots 1\,\mu\mathrm{m}$), because the Casimir-force according to (36) is scaling like D^{-4}, and the experiments were not sufficiently sensitive to measure a significant force at larger D.

In the year 2010, experimental results were published [23], which cover the range $D = 0.7\,\mu\mathrm{m} \ldots D = 7\,\mu\mathrm{m}$. The experimentalists are convinced to have verified the impact of the finite temperature. With regard to the appreciable correction factors, which had to be applied in data postprocessing, it may be advisable to consider

these findings with some reservation, and wait for further results.

Finite surface conductivity: In the derivation of (36) we have assumed, that the penetration depth of the radiation into the resonator's walls is zero, and that 100 % of the radiation is reflected. This assumption has turned out to be an overly rough approximation. To achieve satisfying accordance of theory and experiments, one must account for the finite and frequency-dependent reflectivity of the surfaces of plates and spheres, which are used to measure the Casimir-force. With this correction, the illusory universality of the simple formula (36) vanishes: The Casimir-force is *not* independent of the resonator's material.

The model of electrical conductivity of metals, which Drude published as early as 1900, is in spite of it's simplicity suitable for the analysis of the Casimir-effect. In particular for metals, whose's atoms have just one valence-electron in the s-shell, i.e. for the alkali-metals, and for the noble metals copper, silver, and gold, this model produces surprisingly good results. The mentioned noble metals are preferred surface materials of the resonators in the experiments.

The following derivation of those formulas of the Drude-model, which are important for the Casimir-effect, follows by and large the delineations in [24, chapter 1] and [25, Kapitel 11.2]. There are exactly four free parameters in these formulae, which must be determined experimentally: The effective mass m of the conduction electrons, the relaxation time τ, the plasma frequency ω_P, and the frequency-dependent relative dielect constant ϵ_ω.

If a static electric field $\boldsymbol{E}_0$ is applied to a metal, it causes within the metal the current density

$$\boldsymbol{j} = -ne\boldsymbol{v} = \sigma_0 \boldsymbol{E}_0 \ , \tag{40}$$

which equals the product of the number n of conduction electrons

per volume, their charge $-e$, and their mean velocity $\boldsymbol{v}$. σ_0 is the metal's DC-conductivity. On their way through the metal, the conduction electrons are again and again scattered by the ions, which are constituting the crystal grid. The mean time interval between two collisions of the same electron is the relaxation time τ. The conduction electron is scattered into an arbitrary direction. Therefore it's *mean* velocity immediately after the collision is zero. Then it is accelerated by the force $-e\boldsymbol{E}$. If the collision happened at time $t = 0$, then the electron's *mean* velocity at time τ will be

$$\langle \boldsymbol{v}_{\text{single}} \rangle_{t=\tau} = \int_0^\tau \mathrm{d}t \, \frac{\mathrm{d}\,\boldsymbol{v}_{\text{single}}}{\mathrm{d}\,t} = -\int_0^\tau \mathrm{d}t \, \frac{e\boldsymbol{E}_0}{m} = -\frac{e\boldsymbol{E}_0}{m}\,\tau \ . \tag{41}$$

The mass m is not the mass of a free electron, but it's so-called effective mass, i.e. one of the four free parameters of the Drude-model. From (41) follows, that the mean velocity of all electrons is[8]

$$\boldsymbol{v} = -\frac{e\boldsymbol{E}_0}{m}\,\tau \ . \tag{42}$$

This velocity is inserted into (40):

$$\boldsymbol{j} = +\frac{ne^2\tau}{m}\,\boldsymbol{E}_0 = \sigma_0 \boldsymbol{E}_0 \tag{43}$$

[8] Intuitively, one would assume $\boldsymbol{v} = \langle \boldsymbol{v}_{\text{single}} \rangle_{t=\tau}/2$ in the first moment. A failure, which also crept into Drude's publication. To make the correct value (42) plausible, lets consider the extreme cases of one electron, which is scattered exactly backwards, and one electron, which is scattered with almost no deflection into forward direction. If both electrons had the velocity $\boldsymbol{v}$ before the collision, then the velocity of the backwards scattered electron immediately after the collision is $-\boldsymbol{v}$, and after the further time τ it is 0. The velocity of the forward scattered electron immediately after the collision is $\boldsymbol{v}$, and after the further time τ it is $\boldsymbol{v} - e\boldsymbol{E}/m = 2\boldsymbol{v}$. The mean value of the both extreme cases is $\boldsymbol{v}$, in accordance with (42).

The conductivity σ_0 is a constant. Thus the collisions have the effect, that the conduction electrons are not infinitely accelerated by the force $-e\boldsymbol{E}$, but assume the constant, finite *mean* velocity $\boldsymbol{v}$. The effect of the collisions may be modeled as a friction force $-\boldsymbol{v}/\tau$, which is proportional to the mean electron velocity. This force is inserted in addition to the force $-e\boldsymbol{E}$ into the equation of motion of the *mean* velocity $\boldsymbol{v}$ of the conduction electrons:

$$\frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}\,t} = -\frac{\boldsymbol{v}}{\tau} - \frac{e\boldsymbol{E}_0}{m} \overset{(42)}{=} 0 \tag{44}$$

If instead of the static field an oscillating field $\boldsymbol{E}_\omega = \hat{\boldsymbol{E}}\exp\{-i\omega t\}$ is applied to the metal, then the mean velocity of the conduction electrons will be $\boldsymbol{v}_\omega = \hat{\boldsymbol{v}}\exp\{-i\omega t\}$. As $\boldsymbol{v}_\omega$ in contrast to $\boldsymbol{v}$ is variable, equation (44) isn't zero any more, but becomes

$$\begin{aligned} \frac{\mathrm{d}\,\boldsymbol{v}_\omega}{\mathrm{d}\,t} &= -i\omega\boldsymbol{v}_\omega = -\frac{\boldsymbol{v}_\omega}{\tau} - \frac{e\boldsymbol{E}_\omega}{m} \\ \boldsymbol{v}_\omega &= -\frac{e\boldsymbol{E}_\omega}{m(-i\omega + 1/\tau)}\ . \end{aligned} \tag{45}$$

The current density and the conductivity then will be

$$\begin{aligned} \boldsymbol{j}_\omega &\overset{(40)}{=} +\frac{ne^2\boldsymbol{E}_\omega}{m(-i\omega + 1/\tau)} = \sigma_\omega\boldsymbol{E}_\omega \\ \sigma_\omega &= \frac{ne^2\tau}{m(1 - i\omega\tau)} \overset{(43)}{=} \frac{\sigma_0}{1 - i\omega\tau}\ . \end{aligned} \tag{46}$$

The conductivity σ_ω, and consequently the metal's reflectivity, decreases according to (46) with increasing frequency. The phase factor i in the damping term $i\omega\tau$ reveals, that this is a dissipative process. That part of the radiation, which is not reflected, is converted into heat in the metal.

The DC-conductivity σ_0 and the AC-conductivity σ_ω can easily be measured. Thereby the relaxation time τ can be computed

from the right side of (46), and then from the middle term of this equation (as n and e are known) the effective mass m. Thus two of the four free parameters of the Drude-model are experimentally determined.

The formulae of the Drude-model, which we derived so far, are sufficient to describe the conductivity of metals in the frequency range 0 (DC) up to approximately 10^{12}Hz. At higher frequencies, i.e. in the range of infrared light up to approximately $4 \cdot 10^{14}$Hz, and even more distinctly in the frequency range of visible light of approximately $(4 \ldots 7) \cdot 10^{14}$Hz and higher, a change happens. For the investigation of the material properties in this frequency range we make use of the Maxwell equations

$$\boldsymbol{\nabla} \times \boldsymbol{E}_\omega = -\frac{\partial \boldsymbol{B}_\omega}{\partial t} \tag{47a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}_\omega = \mu_0 \mu_\omega \boldsymbol{j}_\omega + \frac{\mu_\omega \epsilon_\omega}{c^2} \frac{\partial \boldsymbol{E}_\omega}{\partial t} \ . \tag{47b}$$

The index $_\omega$ is indicating, that we still are looking for solutions with the same time dependence as in (46), i.e. we make the ansatz

$$\boldsymbol{E}_\omega = \hat{\boldsymbol{E}} \exp\{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\} \ . \tag{48}$$

ϵ_0 is the dielectric constant of the vacuum, ϵ_ω is the relative dielectric constant of the metal at the frequency $\omega = 2\pi\nu$. μ_0 is the magnetic permeability of the vacuum, μ_ω is the relative permeability of the metal at the frequency $\omega = 2\pi\nu$. We here are interested exclusively in non-magnetic metals, and therefore set $\mu_\omega = 1$. $c = (\epsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum. We compute the rotation of (47a), insert (47b), and insert the expression (46) for $\boldsymbol{j}_\omega$:

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E}_\omega = -\mu_0\sigma_\omega\frac{\partial\boldsymbol{E}_\omega}{\partial t} - \frac{\epsilon_\omega}{c^2}\frac{\partial^2\boldsymbol{E}_\omega}{\partial t^2} = \\ = \underbrace{\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}_\omega)}_{0} - \boldsymbol{\Delta}\boldsymbol{E}_\omega \tag{49}$$

We are assuming that the wavelength of $\boldsymbol{E}_\omega$ is much larger than the grid constant of the metal. Therefore the metal will be electrically neutral on this scale, and $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{E}_\omega)$ may be set to zero. Because of (48),

$$\frac{\partial\boldsymbol{E}_\omega}{\partial t} = \frac{+i}{\omega}\frac{\partial^2\boldsymbol{E}_\omega}{\partial t^2} \tag{50}$$

holds. Thus one gets the wave equation

$$\boldsymbol{\Delta}\boldsymbol{E}_\omega = \underbrace{\left(\epsilon_\omega + \frac{i\sigma_\omega}{\omega\epsilon_0}\right)}_{\text{complex relative dielectric constant}}\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}_\omega}{\partial t^2}\ . \tag{51}$$

Apart from the complex relative dielectric constant, this is a normal wave-equation. Inserting (48), one finds the dispersion relation

$$k = \frac{\omega}{c}\sqrt{\epsilon_\omega + \frac{i\sigma_\omega}{\omega\epsilon_0}} \quad \text{with } k = \sqrt{\boldsymbol{k}^2}\ . \tag{52}$$

The plasma frequency is defined by

$$\omega_P \equiv \sqrt{\frac{ne^2}{m\epsilon_0}}\ . \tag{53}$$

If ω is of same order of magnitude as ω_P, then in good approximation

$$\left.\begin{aligned} \sigma_\omega &\overset{(46)}{=} \frac{ne^2\tau}{-i\omega\tau m(1/(-i\omega\tau)+1)} \approx +\frac{ine^2}{m\omega} \\ k &\overset{(52)}{=} \frac{\omega}{c}\sqrt{\epsilon_\omega - \frac{ne^2}{m\omega^2\epsilon_0}} = \frac{\omega}{c}\sqrt{\epsilon_\omega - \frac{\omega_P^2}{\omega^2}} \end{aligned}\right\} \begin{array}{l}\text{if } \omega \text{ is of}\\ \text{same order}\\ \text{of magnitude}\\ \text{as } \omega_P\ .\end{array} \tag{54}$$

In case $\omega < \omega_P/\sqrt{\epsilon_\omega}$, the wave number k is imaginary, and the amplitudes of the solutions (48) of the wave equation (51) decrease exponentially towards zero within the metal. The penetration depth is finite, but not negligibly small. If in contrast the frequency ω becomes so large that $\omega > \omega_P/\sqrt{\epsilon_\omega}$, then k becomes real, and waves can propagate within the metal. The metal becomes transparent, and the reflectivity decreases towards zero.

Only the density n of conduction electrons is considered in the plasma frequency's definition $\omega_P =$ (53), while the core electrons, which are bound to the ions of the crystal grid, are ignored. That's a poor approximation at high frequencies. Also the frequency-dependent relative dielectric constant ϵ_ω is strongly influenced by the core electrons. Therefore it's better, not to compute these two parameters by the Drude model, but to determine them experimentally. They are found due to the frequency-dependent measurement of the reflection and the absorption of electromagnetic radiation. The plasma frequencies of the alkali metals are in the near UV from approximately $\nu_P = \omega_P/2\pi \approx 7 \cdot 10^{14}$Hz up to $\nu_P \approx 15 \cdot 10^{14}$Hz [24, Table 1.5] close above the visible spectrum. The plasma frequency of gold is not much higher, namely $\nu_P \approx 22 \cdot 10^{14}$Hz [26].

Electrostatic patches: Even if a surface is geometrically flat, it usually is still not a perfect equipotential plane, because it is made of micro-crystallites with different work functions. The micro-crystals thus carry different electrostatic charges, resulting in a "voltage-roughness" of the surface. This effect is reduced by adsorbed contaminations on the surface. Literature references regarding this issue can be found in [27]. As it is very difficult to control the electrostatic forces caused by the patches, and as these forces furthermore are of same order of magnitude as the Casimir-force, the electrostatic patches are a serious problem for

the interpretation and the reliability of the measured data, and are a persistent source of systematic errors.

In a recent experiment [28] it was first time tried to measure the patches directly, and to consider their influence in detail, when the Casimir-force was extracted from the measured data. The criticism of other authors [29] regarding these results is demonstrating, how difficult it is to identify and to quantify the systematic errors in measurements of the Casimir-force.

5. How Casimir discovered the effect

In the next section we will discuss, whether the Casimir-effect is supplying conclusive evidence for the physical existence of zero-point energy. The way, on which Casimir arrived at his formula (36), sheds an indicative light on this question.

In 1930, Fritz London published the very influential article „Zur Theorie und Systematik der Molekularkräfte“[9] [30]. London investigated atoms and molecules, which are interacting due to van der Waals-forces, by quantum-mechanical perturbation computation of second order. He considered the interaction of pairs of molecules, which don't have a permanent electrical dipole- or quadrupole-moment. If one of the molecules gets an instantaneous dipole moment due to a temporary fluctuation (such fluctuations exist only in quantum theory, but not in classical theory), then it will induce a dipole moment in the other molecule as well. London arrived at the result, that the interaction energy between the two molecules is $\sim R^{-6}$, if their distance R is considerably larger than the diameter of the molecules according to Bohr's old model of atoms. In-between the both molecules an attractive force $\sim R^{-7}$ is acting, which London called dispersion force.

[9] “On the theory and systematics of molecular forces”

In 1947, Casimir and Polder were consulted by an experimentalist, who was researching the force in-between the molecules of colloidal films as a function of their distance R. It was assumed that London's theory of dispersion forces should fit to this system. But the observed forces seemed to depend on distance rather $\sim R^{-8}$ than $\sim R^{-7}$ (which is the value expected by London's theory). In the search for a possible error in London's theory, suspicion soon concentrated on the fact, that London had computed only the static interaction between the molecules, and had neglected the finite propagation velocity of electromagnetic interaction.

This approximation indeed was too rough. The computation by Casimir and Polder [31], in which they considered the retardation of the interaction, resulted into the following van der Waals-force between two atoms without permanent electrical dipole moments at large distance R:

$$F = -\frac{161\hbar c\beta_1\beta_2}{4\pi R^8} \tag{55}$$

β_1 and β_2 are the static polarizabilities of the two atoms.

The result is looking simple, but it's derivation turned out to be quite tough and complicated. To approach the solution systematically, Casimir and Polder therefore firstly investigated a simpler setup, in which a single polarizable atom is placed in distance R from a metal plane with infinite conductivity. According to classical theory, an interaction between the atom and it's mirror picture was expected, and this was by and large the case. But what surprised the theoreticians was the simplicity, with which the result

$$F = -\frac{3\hbar c\beta}{2\pi R^5} \tag{56}$$

could be computed. β is the atom's static polarizability. The derivation of this result is almost identical to that, which Casimir

later-on used when he analyzed the force between two metal plates. The only — however physically most important — difference is, that in this case the atom is providing the electromagnetic field, while in case of the Casimir-effect (apparently, see the following section) no electrical charges are involved, and therefore the zero-point oscillations must contribute the necessary electromagnetic field.

Casimir got the essential hint, that for the setup with two metal plates a possible action of zero-point energy should be considered, when he was chatting in those days with Bohr about his actual activities. "Bohr mumbled something about zero-point energy", remembered Casimir many years later [32]. This tip was sufficient for Casimir, to compute — after the preparatory work with Polder, in which he only needed to replace the atom by the movable plate — within shortest time the Casimir-effect.

6. What is proved by the Casimir-effect?

Six decades after Casimir's seminal publication, one certainly can state, that the Casimir-force is definitely existing, that it has been demonstrated experimentally beyond doubt, and that it is described approximately correct by (36), at least if the gap width between the plates is in the range $D \approx 0.2\,\mu\mathrm{m} \ldots 5\,\mu\mathrm{m}$.

But does the existence of the Casimir-force conclusively prove a measurable effect of the electromagnetic field's zero-point oscillations? Remarkably not. Casimir has demonstrated, that formula (36) can be derived from the assumption of zero-point oscillations, but he did not prove that this is this only possible interpretation of the measured results.

The computation of the attractive force between a metal plate and a polarizable atom, which he had performed in cooperation with Polder, is already an indication for a better interpretation of

the observations. The baffling formal similarity of the derivations of the both formulae (56) and (36) is suggesting the assumption, that also the cause of the attractive force should be the same in both cases, i.e. that the Casimir-force should be generated due to long-range van der Waals-interactions between those atoms, which are constituting the surfaces of the two metal plates.

Lifshitz [33] derived already in 1956 a theory of the retarded van der Waals-force between two parallel plates with relative dielectric constants ϵ_1 and ϵ_3, with the gap between the plates filled by a material with relative dielectric constant ϵ_2:

$$\epsilon_1 \;\Big|\; \epsilon_2 \;\Big|\; \epsilon_3 \tag{57a}$$

Casimir's setup would be represented in this notation by

$$\epsilon_1 \to \infty \;\Big|\; \epsilon_2 \to 1 \;\Big|\; \epsilon_3 \to \infty \;. \tag{57b}$$

Few years later, Dzyaloshinskii, Lifshitz, and Pitaevskii [34] improved this theory, and corroborated it by the methods of quantum electrodynamics. Schwinger, DeRaad, and Milton [35] in 1978 reproduced the result of Lifshitz et al. with a more elegant (but not simpler) mathematical method. While the derivation is quite complicated, the result is looking relatively simple. Consider a setup with three plates of size $X \times Y$ with relative dielectric constants $\epsilon_1, \epsilon_2, \epsilon_3$ according to (57a). If the thickness D of the middle plate is sufficiently small versus X and Y, such that edge effects are negligible, then according to Lifshitz's theory the following force is acting between the outer plates:

$$
\begin{aligned}
F_{\text{Lifshitz}} = &- \frac{\hbar c X Y}{2\pi^2} \int\limits_0^\infty \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}k\, \kappa_2 k \cdot \\
&\cdot \left(\left(\frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1} \cdot \frac{\kappa_2 + \kappa_3}{\kappa_2 - \kappa_3} \cdot \exp\{2\kappa_2 D\} - 1 \right)^{-1} + \right. \\
&\left. + \left(\frac{\kappa_2' + \kappa_1'}{\kappa_2' - \kappa_1'} \cdot \frac{\kappa_2' + \kappa_3'}{\kappa_2' - \kappa_3'} \cdot \exp\{2\kappa_2 D\} - 1 \right)^{-1} \right) \\
&\kappa_j^2 = k^2 + \epsilon_j \xi^2 \ , \ \kappa_j' = \frac{\kappa_j}{\epsilon_j} \ , \ k, \xi \in \mathbb{R} \ , \ [k] = [\xi] = \mathrm{m}^{-1}
\end{aligned}
\tag{58}
$$

The integration variables ξ and k both have the dimensions of wave numbers. Note, that κ_2 is unprimed in both exponential functions.

Let's first check the result of the Lifshitz-formula for two metal plates with infinite conductivity, which are separated by a vacuum gap of width D. Because of

$$
\epsilon_1 = \epsilon_3 = \infty \ , \epsilon_2 = 1 \implies \kappa_1 = \kappa_3 = \infty \ , \ \kappa_1' = \kappa_3' = 0 \ , \tag{59}
$$

the formula in this case simplifies appreciably:

$$
F_{\text{Lifshitz}} = - \frac{\hbar c X Y}{2\pi^2} \int\limits_0^\infty \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}k\, \frac{2\kappa_2 k}{\exp\{2\kappa_2 D\} - 1} \tag{60}
$$

We define polar coordinates r, ϕ by

$$
r \cos\phi \equiv k \quad , \quad r \sin\phi \equiv \xi \ .
$$

The integral over the first quadrant of the $k\xi$-plane is

$$
\int\limits_0^\infty \mathrm{d}k \int\limits_0^\infty \mathrm{d}\xi = \int\limits_0^\infty \mathrm{d}r \int\limits_0^{\pi/2} r \ \mathrm{d}\phi \ . \tag{61}
$$

Furthermore $\kappa_2 = \sqrt{k^2 + \xi^2} = r$ is inserted into (60):

$$F_{\text{Lifshitz}} = -\frac{\hbar c XY}{8\pi^2 D^3} \underbrace{\int_0^{\pi/2} \mathrm{d}\phi \, \cos\phi}_{1} \int_0^\infty \mathrm{d}r \, \frac{2^3 r^3 D^3}{\exp\{2rD\} - 1} \tag{62}$$

Here the numerator and the denominator have been multiplied by $2^2 D^3$. With the further substitution

$$x \equiv 2rD \quad , \quad \mathrm{d}r = \frac{\mathrm{d}x}{2D}$$

follows[10]

$$F_{\text{Lifshitz}} = -\frac{\hbar c XY}{16\pi^2 D^4} \underbrace{\int_0^\infty \mathrm{d}x \, \frac{x^3}{\exp\{x\} - 1}}_{\Gamma(4)\,\zeta(4) = 3!\,\pi^4/90 = \pi^4/15} = -\frac{\pi^2 \hbar c}{240} \frac{XY}{D^4} \, , \tag{63}$$

which is identical to Casimir's formula (36).

Lifshitz's theory has been tested and confirmed by several experiments. In this context, the elegant and impressive experiment of Sabisky and Anderson [37] deserves particular attention. In that experiment, a scenario was realized, which Dzyaloshinskii, Lifshitz, and Pitaevskii had already computed in their work cited above: The evaluated layer-stack consisted of a substrate of CaF or SrF or BaF (correlates in (57a) with ϵ_1), a film of liquid helium with thickness of 1 nm up to 25 nm (correlates in (57a) with ϵ_2), and a cloud of saturated helium gas above (correlates in (57a) with ϵ_3). The measurements confirmed Lifshitz's theory with spectacular precision.

[10] The integral over x is listed in [36] with number 3.411.1

Now the essential point is, that zero-point oscillations are playing no role at all in Lifshitz's theory. Instead that theory is computing the direct interaction, transmitted by an electromagnetic field, between the polarizable matter of the two outer plates.

Jaffe [38] considered Casimir's derivation of his result (36) to be "heuristic", because it is inconsistent: On the one hand, Casimir is making use of the zero-point oscillations, which are a result of quantum field theory. On the other hand, he does not compute the interaction of these oscillations with the resonator's walls by the methods of quantum field theory, but in an idealizing manner: The assumption of infinite conductivity of the metals has the effect, that the fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \ , \tag{64}$$

which's value in quantum electrodynamics is determining the strength of interaction between the electromagnetic field and electrically charged matter, doesn't show up any more in formula (36). Casimir's assumption

$$\sigma_\omega \overset{(46)}{=} \frac{ne^2\tau}{m(1-i\omega\tau)} = \frac{\sigma_0}{1-i\omega\tau} = \infty$$

is because of $\sigma_\omega \sim e^2 \sim \alpha$ equivalent to the approximation $\alpha = \infty$. We stated already in the previous section, that this approximation is overly rough. The finite, frequency-dependent conductivity $\sigma_\omega = (46)$ must be included into the theory, to reproduce the measured results. But that is to say, that the Casimir-force is caused due to the exchange of photons in-between charges in the two metal plates. The fact, that the frequency-dependence of the conductivity σ_ω must be considered in the computations of the Casimir-force does prove, that the photons, which are exerting forces on the metal plates, do not somehow emerge from a see of

zero-point oscillations, but are being emitted from the charges in the opposite plate.

It can be stated like this: If the fine-structure constant α would be zero, then σ_ω would be zero as well, and the Casimir-effect would disappear [38]. The fact, that the Casimir-force can show up only in case $\alpha \neq 0$ does prove, that it is based on the interaction between the electromagnetic field and charged matter, and can not be explained as the effect of a zero-point energy, which is floating in-between the metal plates without interacting with charges.

With the theory of Lifshitz, there exists an alternative theory to that one of Casimir, which – without assuming a physically effective zero-point energy – firstly does reproduce Casimir's formula in the limit (57b), secondly does reproduce all experimental observations with higher accuracy than Casimir's theory, and thirdly is making further amazing predictions: For example, Lifshitz's theory is predicting repulsive forces for certain geometries of the interacting macroscopic objects and for certain combinations of the dielectric constants $\epsilon_1, \epsilon_2, \epsilon_3$ in (57a). This aspect of Lifshitz's theory has been experimentally confirmed as well, see the references in [39]. By the way, the above mentioned experiment of Sabisky and Anderson is belonging to the class with repulsive Casimir-force.

Iver H. Brevik reported the following anecdote: "... in March 1972 [..., after a talk of Hendrik Casimir at the Norwegian Institute of Technology in Trondheim, I] asked: 'Is the Casimir effect due to the quantum fluctuations of the electromagnetic field, or is it due to the van der Waals forces between the molecules in the two media?' Casimir's answer began, 'I have not made up my mind.' 11 years later I met Casimir again [... and ...] put to him the same question as before. And as far as I can remember, his answer and explanations were in effect the same as in 1972." [40, Foreword]

The Casimir-effect does not prove a physical effectiveness of zero-point oscillations. That makes the fact even more surprising, that

Casimir's derivation of (36) led him to a result, which is at least approximately correct. Should this be interpreted as an indication, that the assumption of physically effective zero-point oscillations still might make some sense, which we just don't yet understand?

Boyer, who wrote in 1970 a review [41] on some systems, for which experimentally testable results can be derived relatively easy, if one assumes physically effective zero-point oscillations, commented this situation as follows: "Often in physics, a problem may be attacked from several points of view with the same conclusion. [...] Here we will show that zero-point energy is a useful concept in understanding and calculating some forces arising in quantum electrodynamics, — forces which may often be obtained alternatively by perturbation and dispersion-theoretic methods. We believe that the usefulness of a quantity often indicates that it is worthy a further investigation." [41, page 479]

7. Are zero-point oscillations physically effective?

One evidently must discriminate between the zero-point oscillations of material systems with a finite number of degrees of freedom, and the zero-point oscillations of continuous fields with infinitely many degrees of freedom.

Things are quite obvious with regard to the first group: There are plenty of experimental confirmations for the existence of their zero-point oscillations. These zero-point oscillations don't cause a conflict with GRT, because the energy of the finite number of zero-point oscillations in any case is negligible as compared the the rest energies of these material systems.

With regard to continuous fields, the situation in the first moment seems to be of similar clarity: After the Casimir-effect, being the allegedly strongest argument for the detectability of a physically effective zero-point energy of the electromagnetic field, has been

demolished in the previous section, not any experimental evidence is left, which might indicate it's existence. Furthermore there is the extremely strong counter-argument, which has been discussed in section 3: The tininess of the energy density in intergalactic space, which is by *at least* 58 orders of magnitude smaller than one would expect if the zero-point energies of continuous fields would cause gravitation. We have mentioned the loophole of the cosmological constant. But we agree with Pauli, that this option is not worth to be considered.

Still some doubts are left. The canonical quantization of continuous fields formally is a true copy of the canonical quantization of finite material systems. Therefore the description of the oscillation modes of fields conforms mathematical exactly with the description of the oscillation modes of material systems with a finite number of degrees of freedom. This approach was successful beyond any expectation. The precision of the experimentally confirmed predictions of the quantum field theories, in particular of quantum electrodynamics, is impressive and convincing.

Why then is there suddenly in this single point a basic difference? Why are zero-point oscillations, which have been definitely confirmed experimentally in material systems with finite numbers of degrees of freedom, in case of elementary quantum fields merely an artifact of the theory, which somehow (for example due to "normal order") must be removed? If this difference really exists, then at least a plausible explanation would be appreciated.[11]

[11] Note added in 2017: That "plausible explanation" can be found in [42, sec. 5]. In [42, sec. 9] an additional, very important argument against the assumption of a zero-point energy of elementary fields is outlined.

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